APPLICATIONS AND GENERALIZATIONS OF THE APPROXIMATION THEOREM

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In its basic form, the approximation theorem referred to provides simple combinatorial models for spaces $\Omega^n\Sigma^n X$, where X is a connected based space. The first such result was given by James [26], who showed that $\Omega\Sigma X$ is equivalent to the James construction MX. The unpublished preprint form of Dyer and Lashof's paper [25] gave an approximation to $QX = \lim\limits_{\rightarrow} \Omega^n\Sigma^n X$, and Milgram [41] gave a cellular model for $\Omega^n\Sigma^n X$ for all finite n.

Starting from Boardman and Vogt's spaces $\mathcal{C}_{n,j}$ of j-tuples of little n-cubes [5], Dold and Thom's treatment of the infinite symmetric product NX in terms of quasifibrations [24], and the category theorists' comparison between finitary algebraic theories and monads (as for example in Beck [4]), I gave a new approximation $C_n X$ to $\Omega^n\Sigma^n X$ in [36]. This model has proven most useful for practical calculational purposes when $n > 1$, and it is its applications and generalizations that I wish to discuss here. This will be a survey of work by various people, and I would like to mention that I have also given a survey of other recent developments in iterated loop space theory in [39], updating my 1976 summary [38].

The first section will give background, mention miscellaneous relevant work, and discuss generalizations, notably Caruso and Waner's recent homotopical approximation to $\Omega^n\Sigma^n X$ for general non-connected spaces X [9,11]. The second and third sections will outline the two main lines of applications. Both are based on certain stable splittings of $C_n X$, due originally to Snaith [47]. One line, initiated by Mahowald [33] and with other major contributors Brown and Peterson [7,8] and Ralph Cohen [21], is primarily concerned with a detailed analysis of the pieces in the resulting splitting of $\Omega^2 S^q$ and leads to new infinite families of elements in the stable stems. The other line, primarily due to Fred Cohen, Taylor, and myself [16-19] but also contributed to by Caruso [10] and Koschorke and Sanderson [30], is based on a detailed analysis of the splitting maps and their homotopical implications and leads to an unstable form of the Kahn-Priddy theorem, among various other things. These lines, and thus sections 2 and 3, are essentially independent of each other.

§1. Background and generalizations

The construction of the approximating spaces is naively simple. Suppose given a collection of Σ_j-spaces $\mathcal{C}_j$ with suitable degeneracy operators $\sigma_i : \mathcal{C}_{j-1} \to \mathcal{C}_j$, $1 \le i \le j$. (Here Σ_j is the $j^{\underline{th}}$ symmetric group.) Given a space X with (nondegenerate) basepoint $*$, construct a space $CX = \coprod \mathcal{C}_j \times_{\Sigma_j} X^j/(\approx)$, where the equivalence relation is generated by

$$(c, x_1, \ldots, x_j) \approx (c\sigma_i, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_j) \quad \text{if } x_i = * .$$

See [16, §§1, 2] for details, examples, naturality properties, etc. If each $\mathcal{C}_j = \Sigma_j$, the resulting space is MX. If each $\mathcal{C}_j$ is a point, the resulting space is NX.

We shall largely be concerned with examples $C(Y,X)$ obtained from the configuration spaces $\mathcal{C}_j(Y) = F(Y,j)$ of j-tuples of distinct points of Y, and we let $B(Y,j)$ be the orbit (braid) space $F(Y,j)/\Sigma_j$. $B(R^2,j) = K(B_j,1)$, where B_j is Artin's group of j-stranded braids. Following Koschorke and Sanderson [30], we think of $C(Y,X)$ as the space of pairs (L,λ), where L is a finite (unordered) subset of Y and $\lambda : L \to X$ is a function. Here we impose the equivalence relation generated by

$$(L,\lambda) \approx (L - \{y\}, \lambda \,|\, L - \{y\}) \quad \text{if } \lambda(y) = * .$$

The crucial example $C_n X$ may be described similarly, with the sets L taken to have affine embeddings $I^n \to I^n$ with disjoint interiors as their elements; see [36, §4]. We think of the interior of I^n as R^n, by abuse, and obtain a natural homotopy equivalence

$$g : C_n X \to C(R^n, X)$$

by restriction of little n-cubes to their center points. (The proof that g is an equivalence in [36, 4.8] is not quite right; it is corrected in [14, p. 485] and also in [30].)

Define $\alpha_n : C_n X \to \Omega^n \Sigma^n X$ by letting $\alpha_n(L,\lambda) : S^n \to X \wedge S^n$ be specified on points $s \in S^n = I^n/\partial I^n$ by

$$\alpha_n(L,\lambda)(s) = \begin{cases} \lambda(c) \wedge c^{-1}(s) & \text{if } s \in \operatorname{Im} c \text{ for } c \in L \\ * & \text{if } s \notin \operatorname{Im} c \text{ for } c \in L . \end{cases}$$

The approximation theorem [36, 6.1] asserts that α_n is a weak equivalence if X is connected. Segal [45] and Cohen [14] later proved that α_n is a group completion in the general, non-connected, case. This means that $\pi_0 \Omega^n \Sigma^n X$ is the universal group associated to the monoid $\pi_0 C_n X$ (which is easy [36, 8.14]) and that $H_*\Omega^n\Sigma^n X$ is obtained from the Pontryagin ring $H_* C_n X$ by localizing at its submonoid $\pi_0 C_n X$. Of course, the same conclusions hold for $\alpha_n g^{-1} : C(R^n, X) \to \Omega^n\Sigma^n X$.

In the case $X = S^0$, McDuff [40] gave another proof of the group completion property, viewing it as a special case of a general homological relationship between $C(M, S^0)$ and the space of sections of the tangent sphere bundle of M for suitable manifolds M. She also gave a construction $C^{\pm}(M, S^0)$ of pairs of finite sets in M, with points thought of as positive and negative particles with suitable annihilation properties. While this construction is of some interest and yields a homotopical approximation to the space of sections of a bundle, the bundle in question is not the tangent sphere bundle of M. In particular, $C^{\pm}(R^n, S^0)$ fails to be a homotopical approximation to $\Omega^n S^n$.

Various other people have tried to obtain homotopical (rather than merely homological) approximations to $\Omega^n\Sigma^n X$ for general non-connected spaces X. The problem is quite delicate. It is very easy to give intuitive arguments for plausible candidates but very hard to pin down correct details. Such an approximation has recently been obtained by Caruso and Waner [11]. They construct a model $\tilde{C}_n X$ for $\Omega^n\Sigma^n X$ by use of partial little cubes $c' \times c'' : [a,b] \times I^{n-1} \to I^n$, where $c' : [a,b] \to I$ is a linear map (increasing or decreasing) and $c'' : I^{n-1} \to I^{n-1}$ is an affine little (n-1)-cube. These partial little cubes are required to appear in closed configurations, which may be thought of as piecewise linear maps $I^n \to I^n$ given by a piecewise linear path in the first coordinate and a single affine embedding in the remaining coordinates. The labels in X of the component partial little cubes of a closed configuration are all required to be the same.

Caruso [9] has generalized both this result and the original approximation theorem by proving that $\Omega^n C(Y, \Sigma^n X)$ is a group completion of $C(Y \times R^n, X)$ and obtaining a homotopical approximation $\tilde{C}_n(Y, X)$ to $\Omega^n C(Y, \Sigma^n X)$. The case when Y is a point reduces to the earlier approximations, and the use of general Y allows a quick inductive reduction of the entire result to the case n = 1. Further generalizations, based on a combination of the ideas of Caruso and McDuff, are in the works.

Related to these ideas are Cohen and Taylor's extensive calculations of $H_* C(M, X)$ for certain manifolds M [20]. Their arguments work best when $M = N \times R^n$ for some positive n. In particular, they give complete information on the rational homology of $C(M, X)$ for such M, these calculations having direct implications for Gelfand-Fuks cohomology.

There is also an equivariant generalization of the approximation theorem, the present status of which is discussed in [39, § 5].

However, we shall mainly concentrate on the specific approximations

$$C(R^n, X) \xleftarrow{g} C_n X \xrightarrow{\alpha_n} \Omega^n \Sigma^n X$$

and their applications in the rest of this paper.

Perhaps I should first explain just why these approximations are so useful a tool. One reason is that $C_n X$ has internal structure faithfully reflecting that of iterated loop spaces. Much of this is captured by the assertion that C_n is a monad and α_n is a map of monads [36, 5.2]. Less cryptically, the three displayed spaces all have actions by the little cubes operad $\mathcal{C}_n$ and g and α_n are both $\mathcal{C}_n$-maps [16, 6.2 and 36, 5.2]. In particular, they are H-maps, but this is only a fragment of the full structure preserved. Further, there are smash and composition products

$$C_m X \wedge C_n Y \to C_{m+n}(X \wedge Y) \qquad \text{and} \qquad C_n X \times C_n S^0 \to C_n X$$

which are carried by the α_n to the standard smash and composition products for iterated loop spaces [36, § 8]. There is a similar and compatible smash product on the $C(R^n, X)$, but there is no precise point set level (as opposed to homotopical) composition product on the $C(R^n, X)$.

By specialization to $X = S^0$, $\alpha_n : C_n S^0 \to \Omega^n S^n$ is a map of topological monoids. As Cohen has recently observed [13], this leads to approximations for the localizations of the classifying spaces BSF(n) for S^n-fibrations (with section). Indeed, $C_n S^0 = \coprod_{q \geq 0} \mathcal{C}_{n,q}/\Sigma_q$ and $\Omega^n S^n = \coprod_{q \in Z} \Omega^n_q S^n$, where $\Omega^n_q S^n$ consists of the maps of degree q. If p is any prime, then the map of classifying spaces

$$B\alpha_n : B(\coprod_{i \geq 0} \mathcal{C}_{n,p^i}/\Sigma_{p^i}) \to B(\coprod_{i \geq 0} \Omega^n_{p^i} S^n)$$

is an equivalence, and the universal cover of the target is the localization of BSF(n) away from p. When $n = \infty$, this result is due to Tornehave and is a special case of a general phenomenon [37, VII § 5].

A key reason for the usefulness of the approximation theorem is that spaces of the general form CX come with an evident natural filtration. The successive (and equivalent) quotients of $C(R^n, X)$ and $C_n X$ are

$$D_q(R^n, X) = F(R^n, q)^+ \wedge_{\Sigma_q} X^{[q]} \qquad \text{and} \qquad D_{n,q} X = \mathcal{C}^+_{n,q} \wedge_{\Sigma_q} X^{[q]} ,$$

where $X^{[q]}$ denotes the q-fold smash power of X. Just as the simplicial version of the James construction admits the splitting $\Sigma MX \simeq \bigvee_q \Sigma X^{[q]}$ found by Milnor [42], so Kahn in 1972 proved that Barratt's simplicial model [2] for QX splits stably as the wedge of its filtration quotients; Kahn has just recently published a proof [27], a different argument having been given by Barratt and Eccles [3]. When word of Kahn's splitting reached Cambridge, where I was lecturing on the approximation theorem, Snaith [47] worked out a corresponding stable splitting

$$\Sigma^{\infty} C_n X \simeq \bigvee_{q \geq 1} \Sigma^{\infty} D_{n,q} X ,$$

where Σ^{∞} is the stabilization functor from spaces to spectra (denoted Q_{∞} in all my earlier papers). New proofs of such splittings by Cohen, Taylor, and myself [16] are the starting point of the work discussed in section 3. Incidentally, by work of Kirley [29], these splittings for $n \geq 2$ cannot be realized after any finite number of suspensions (see also [16, 5.10]).

There are two points of view on these splittings. One can either ignore how they were obtained and concentrate on analyzing the pieces or one can concentrate on the splitting maps and see what kind of extra information they yield. These two viewpoints are taken respectively in the following two sections.

The crucial reason for the usefulness of the approximation theorem is that we have very good homological understanding of the filtered spaces $C_n X$. Historical background and complete calculations of $H_* QX$ and $H_* C_{\infty} X$ (including $X = S^0$, when the latter is $\sum_{q \geq 0} H_* B\Sigma_q$) are in [14, I]. Cohen [14, III] has given complete calculations of $H_* \Omega^n \Sigma^n X$ and $H_* C_n X$. (Some minor corrections are in [12, App] and also in [7].) Here "complete" means as Hopf algebras over the Steenrod algebra, with full information on all relevant homology operations. Since these operations are nicely related to the geometric filtration, complete calculations of all $H_* D_{n,q} X$ drop out. Here homology is understood to be taken mod p for some prime p, but we also give complete information on the Bockstein spectral sequences of all spaces in sight.

A major drawback to these calculations is that they give inductive formulae for the Steenrod operations, but not a global picture. One wants to know $H^* D_{n,q} X$ as a module over the Steenrod algebra A. The solution to this dualization problem for $n = 2$ and X a sphere is basic to the work of the next section, and we shall also say what little is known when $n > 2$. Before turning to this, however, I should mention the related work of Wellington [51]. He has solved the analogous dualization problem for the algebra structure, giving a precise global description of $H^* \Omega^n \Sigma^n X$ for all connected X (or all X if $p = 2$; corrections of [51] are needed when $p > 2$.) He has also studied the problem of determining the A-annihilated primitive elements in $H_*\Omega^n\Sigma^nX$. More is said about this in [39, §4] (but the description there should have been restricted to connected X).

§2. The spaces $D_q(R^n, S^r)$ and the Brown-Gitler spectra

The work discussed in this section began with and was inspired by Mahowald's brilliant paper [33]. I shall reverse historical order by first discussing the work of Brown and Peterson [7,8] and Ralph Cohen [21] on the structure of the spaces $D_q(R^n,S^r) \simeq D_{n,q}S^r$ and then briefly explaining the use of this analysis for the detection of elements of the stable stems. I am very grateful to Cohen for lucid explanations of some of this material. (In case anyone has not yet noticed, it is to be emphasized that there are two different Cohens at work in this area.)

Let $\zeta_{n,q}$ be the q-plane bundle

$$F(R^n,q) \times_{\Sigma_q} (R^1)^q \to B(R^n, q).$$

With Thom spaces of vector bundles defined by one-point compactification of fibres followed by identification of all points at infinity, it is obvious that the Thom space of $\zeta_{n,q}$ is precisely $D_q(R^n,S^1)$. Replacing R^1 by R^r in this construction, the resulting bundle is the r-fold Whitney sum of $\zeta_{n,q}$ with itself and the resulting Thom space is $D_q(R^n,S^r)$. Here $D_1(R^n,S^r) \simeq S^r$ and $D_q(R^n,S^r)$ is $(rq-1)$-connected. Let $j_{n,q}$ denote the order of $\zeta_{n,q}$ (or better, of its associated S^q-fibration). We have the following evident periodicity (see e.g. [37, III §1]).

<u>Lemma 2.1.</u> $D_q(R^n, S^{r+j_{n,q}})$ is equivalent to $\Sigma^{qj_{n,q}}D_q(R^n, S^r)$.

Thus the first problem in analyzing the spaces $D_q(R^n, S^r)$ is to determine the numbers $j_{n,q}$. The following lemma summarizes what is presently known. Let $\nu_p(j)$ denote the p-order of j (the exponent of p in j).

<u>Theorem 2.2.</u> (i) $j_{2,q} = 2$ for all $q \geq 2$.

(ii) $j_{n,2} = 2^{\phi(n-1)}$, where $\phi(n-1)$ is the vector fields number (namely the number of $i \equiv 0, 1, 2, 4 \mod 8$ with $0 < i < n$).

(iii) $j_{n,q}$ divides $j_{n,q+1}$ (and, trivially, $j_{n,q}$ divides $j_{n+1,q}$).

(iv) For an odd prime p, $\nu_p(j_{n,q}) = 0$ for $q < p$ and $\nu_p(j_{n,p}) = [n-1/2]$.

(v) For any prime p, $\nu_p(j_{n,q}) = \nu_p(j_{n,p^i})$ if $p^i \leq q < p^{i+1}$.

(vi) $j_{4,4} = 12$.

None of these is very hard. Part (i) was first proven by Cohen, Mahowald, and Milgram by use of various results of mine in infinite loop space theory, but Brown later found the trivial trivialization of $2\zeta_{2,q} = \zeta_{2,2q}$ displayed in [15]. For (ii), $B(R^n, 2) \simeq RP^{n-1}$ and $\zeta_{n,2}$ is the canonical bundle $\eta \oplus \varepsilon$. Parts (iii)-(v) are in Yang [52] and were also proven by Kuhn. Part (vi) is an unpublished result of F. Cohen. It remains to determine the numbers $\nu_p(j_{n,p^i})$ for $i \geq 2$ and $n \geq 3$, and this is an interesting and apparently difficult problem.

In connection with this, the only general work on the K-theory of spaces $\Omega^n\Sigma^n X$ for $1 < n < \infty$ that I am aware of is the computation by Saitoti [43] and Snaith [48] of $K_*(\Omega^2\Sigma^3 X; Z_2)$ for X a finite torsion-free CW-complex. However, there is work in progress by Kuhn.

As would be expected from the lemma, much more is known about the spaces $D_q(R^n, S^r)$ when $n = 2$ than when $n > 2$. Before restricting to $n = 2$, however, we summarize the results of Brown and Peterson [8] in the general case. Their main result gives the following splitting of certain of the $D_q(R^n, S^r)$. It is proven by using (ii) of Theorem 2.2, the Thom construction, and various structure maps from [36] to write down explicit splitting maps and then using F. Cohen's calculations in [14, III and IV] to check that they do indeed produce a splitting. We adopt the convention that $D_0(R^n, X) = S^0$.

Theorem 2.3. Let $t \geq 1$ (except that $t > 1$ if $n = 2, 4,$ or 8). Then $D_q(R^n, S^{2^{\phi(n-1)t} - n})$ is homotopy equivalent to

$$\bigvee_{i=0}^{[q/2]} D_{q-2i}(R^{n-1}, S^{2^{\phi(n-1)}t-n}) \wedge D_i(R^n, S^{2^{\phi(n-1)+1}t-n-1}).$$

As usual, [m] denotes the greatest integer $\leq m$. Brown and Peterson [8] also observe that the Nishida relations imply the following 2-primary cohomological periodicity.

Lemma 2.4. Let $\psi(n-1)$ be defined by $2^{\psi(n-1)-1} < n \leq 2^{\psi(n-1)}$. Then

$$\tilde{H}^* D_q(R^n, S^{2^{\psi(n-1)}+r}) \cong \Sigma^{q2^{\psi(n-1)}} \tilde{H}^* D_q(R^n, S^r).$$

Since one can add a multiple $2^{\psi(n-1)}s$ to any $2^{\psi(n-1)}t-n$ so as to reach a number $2^{\phi(n-1)}u-n$, the previous lemma and theorem have the following consequence.

Corollary 2.5. As a module over the mod 2 Steenrod algebra, $\tilde{H}^*D_q(R^n, S^{2^{\psi(n-1)}t-n})$ is isomorphic to

$$\sum_{i=0}^{[q/2]} \tilde{H}^*D_{q-2i}(R^{n-1}, S^{2^{\psi(n-1)}t-n}) \otimes \tilde{H}^*D_i(R^n, S^{2^{\psi(n-1)}t-n-1}) .$$

Brown and Peterson [8] note one further cohomological splitting.

Proposition 2.6. As a module over the mod 2 Steenrod algebra,

$$\tilde{H}^*D_q(R^n, S^{2^{\psi(n-1)}t}) \cong \sum_{i=0}^{[q/2]} \Sigma^{(q-2i)2^{\psi(n-1)}t} K_i^*$$

where $K_0^* = Z_2$ and K_i^*, $i > 0$, is dual to the sub A-module of $\tilde{H}_*D_{2i}(R^n, S^{2^{\psi(n-1)}t})$ spanned by all monomials in the $Q^I(\iota)$ not divisible by ι (where ι is the fundamental class of $S^{2^{\psi(n-1)}t}$ and the suspensions are realized by multiplication with ι^{q-2i}).

These last two cohomological splittings may or may not be realizable geometrically. Brown and Peterson conjecture that they exhaust the 2-primary possibilities in the sense that $\tilde{H}^*D_q(R^n, S^r)$ has no non-trivial direct summands as an A-module unless $r \equiv 0$ or $r \equiv -n \mod 2^{\psi(n-1)}$. They point out explicitly at the end of [8] that, at least when $n = 3$, there can be finer splittings than those displayed when the specified congruences are satisfied. The analysis is not yet complete and there remains much work to be done. In particular, virtually nothing is known about the explicit global A-module structure of the indecomposable summands when $n \geq 3$.

The splittings of Cohen, Taylor, and myself in [17] (see Theorem 3.7 below) together with Lemma 2.1 and an easy homological inspection (compare [17, 3.3]) imply the following analog of Theorem 2.3.

<u>Theorem 2.7.</u> $D_q(R^n, S^{j_{n,q}t})$ is stably equivalent to

$$\Sigma^{j_{n,q}tq}(S^0 \vee \bigvee_{i=2}^{[q/2]} B(R^n, 2i)/B(R^n, 2i-1)) .$$

Specializing now to the case $n = 2$, note that Lemma 2.1 and Theorem 2.2(i) imply that

(1) $D_q(R^2, S^{2r+1}) \simeq \Sigma^{2qr} D_q(R^2, S^1)$ and $D_q(R^2, S^{2r}) \simeq \Sigma^{2qr} D_q(R^2, S^0)$.

Here $D_q(R^2, S^0) = B(R^2, q)^+$. (The disjoint basepoint was omitted in [15, p. 226].) Since $\phi(1) = 1$ and $D_q(R^1, S^t) \simeq S^{tq}$, Theorem 2.3 implies that

$$D_q(R^2, S^{2t}) \simeq \bigvee_{i=0}^{[q/2]} \Sigma^{2t(q-2i)} D_i(R^2, S^{4t+1}) , \quad t \geq 1 .$$

Setting $t = 1$ and combining with (1), we find the splitting

(2) $$\Sigma^{2q} D_q(R^2, S^0) \simeq S^{2q} \vee (\bigvee_{i=1}^{[q/2]} \Sigma^{2q} D_i (R^2, S^1)) .$$

This splitting is also immediate from Theorem 2.7 and [17, 3.3]. Its original proof is in Brown and Peterson [7].

Clearly, then, analysis of the stable homotopy type of $\Omega^2 S^{r+2} \overset{s}{\simeq} \bigvee_{q \geq 1} D_q(R^2, S^r)$ reduces to analysis of the stable homotopy type of the spaces $D_q(R^2, S^1)$. We therefore abbreviate

$$X_q = D_q(R^2, S^1)$$

in what follows. We fix a prime p and localize all spaces and spectra at p. The results to follow are due to Mahowald [33], Brown and Gitler [6], and Brown and Peterson [7] at $p = 2$ and to R. Cohen [21] at $p > 2$.

The starting point of the analysis of the X_q is the determination of their mod p cohomologies. Let χ be the conjugation in the mod p Steenrod algebra A. Define

$$M(q) = A/A\{\chi(\beta^{\varepsilon} P^i) \mid i > q \text{ and } \varepsilon = 0 \text{ or } 1\} .$$

If $p = 2$, we let $P^i = Sq^i$ and suppress the Bockstein. Davis' result [23] that

$$\chi P^{p(i+1)} = (-1)^{i+1} P^{p^i} P^{p^{i-1}} \cdots P^p P^1 , \quad p(i+1) = 1 + p + \cdots + p^i ,$$

greases some of the computations. The following result is not too hard to prove by direct inductive calculation from F. Cohen's results on $H_* X_q$ [14, III].

Mahowald's original argument when $p = 2$ [33] is somewhat different. Abbreviate $M([q/p]) = M[q/p]$.

Theorem 2.8. $\tilde{H}^*(X_q; Z_2) \cong \Sigma^q M[q/2]$ and there is a stable 2-primary cofibration sequence

$$\Sigma X_{2q-1} \xrightarrow{f} X_{2q} \xrightarrow{g} \Sigma^{2q} X_q$$

which on mod 2 cohomology realizes the $2q^{\text{th}}$ suspension of the short exact sequence

$$0 \longleftarrow M(q-1) \xleftarrow{\beta} M(q) \xleftarrow{\alpha} \Sigma^q M[q/2] \longleftarrow 0,$$

where α is the A-map specified by $\alpha(\Sigma^q 1) = \chi(Sq^q)$ and β is the natural A-map.

Here ΣX_{2q-1} may be replaced by its 2-local equivalent $\Sigma^2 X_{2q-2}$. The key to the cofibration is the construction of g which, as Milgram pointed out, is an easy exercise in the use of the classical James maps. The rest follows by use of the geometric and homological properties of the spaces $C_2 S^r$ in [36] and [14,III]; see [15, Thm. 2] or [33, 5.5]. The analog at odd primes is more complicated but proceeds along similar lines [21]. Let M^r denote the Moore space $S^r \cup_p e^{r+1}$.

Theorem 2.9. Localize all spaces at $p > 2$. Then the following conclusions hold.

(i) X_q is contractible unless $q \equiv 0$ or $q \equiv 1 \mod p$.

(ii) $X_1 \simeq S^1$ and $X_{pq+1} \simeq \Sigma X_{pq}$ if $q > 0$.

(iii) $X_{p^2 q} \wedge M^{2r(p-1)} \simeq X_{p^2 q + pr}$ if $q > 0$ and $1 \leq r \leq p-1$.

(iv) $\tilde{H}^*(X_{pq}; Z_p) \cong \Sigma^{2q(p-1)} A/A\{\chi(\beta^{\varepsilon} P^i) \mid pi+\varepsilon > 0\}$; in particular, $\tilde{H}^*(X_{pq}; Z_p) \cong \Sigma^{2q(p-1)} M[q/p]$ if $q \not\equiv 0 \mod p$.

(v) There is a stable map

$$g: X_{p^{i+2}+p} \to \Sigma^{2p^{i+1}(p-1)} X_{p^{i+1}+p}$$

which on mod p cohomology realizes $\Sigma^{2(p^{i+1}+1)(p-1)}\alpha$, where $\alpha: \Sigma^{2p^i(p-1)} M(p^{i-1}) \to M(p^i)$ is the A-map specified by $\alpha(\Sigma^{2p^i(p-1)} 1) = \chi(P^{p^i})$.

In order to obtain homotopical information from these cohomological calculations, one wants to determine k-invariants. Now Brown and Gitler [6] have displayed certain spectra with the same cohomology as the X_q when $p = 2$, and R. Cohen [21] has generalized their constructions to odd primes. Combining results, one obtains the following theorem.

Theorem 2.10. There exist finite p-local CW-spectra $B(q)$ with the following properties.

(i) $H^*(B(q); Z_p)$ is isomorphic to the A-module $M(q)$.

(ii) If $i: B(q) \to K(Z_p, 0)$ represents the generator, then the induced map $i_*: B(q)_r(X) \to H_r(X; Z_p)$ is an epimorphism for all spaces X if either

$$p = 2 \quad \text{and} \quad r \leq 2q+1 \quad \text{or} \quad p > 2 \quad \text{and} \quad r \leq 2p(q+1) - 1.$$

(iii) If M is a compact smooth n-manifold embedded in R^{n+j} with normal bundle ν and Thom space $T\nu$, then $i_*: B(q)^r(T\nu) \to H^r(T\nu; Z_p)$ is an epimorphism if either

$$p = 2 \quad \text{and} \quad n+j-r \leq 2q+1 \quad \text{or} \quad p > 2 \quad \text{and} \quad n+j-r \leq 2p(q+1) - 1.$$

(iv) $\pi_k B(q)$ is a known Z_p-vector space (a certain quotient of the Λ-algebra in degree k) if either

$$p = 2 \quad \text{and} \quad k < 2q \quad \text{or} \quad p > 2 \quad \text{and} \quad k < 2p(q+1) - 2 .$$

Here (3) follows from (2) by Alexander duality, since M is an n+j S-dual of $T\nu$. Particularly for the case $p > 2$, a little generalization of this situation is appropriate. Let T be an m S-dual of some finite CW-complex and let $\nu \in H^s(T; Z_p)$. Say that (T, ν) is adapted to $M(q)$ if

(a) $p = 2$ and $m-s \leq 2q+1$ or $p > 2$ and $m-s \leq 2p(q+1) - 1$; and

(b) the kernel of $\bar{\nu}: A \to H^*(T; Z_p)$, $\bar{\nu}(a) = a\nu$, is $A\{\chi(\beta^\varepsilon P^i) \mid i > q\}$.

Brown and Peterson [7] at $p = 2$ and R. Cohen [21] at $p > 2$ proved the following characterization of the spectra $B(q)$.

Theorem 2.11. Let E be a spectrum which is trivial at all primes other than p and satisfies $H^*(E; Z_p) \cong \Sigma^s M(q)$. Suppose that for some (T, ν) adapted to $M(q)$ there exists a map $\tilde{\nu}: \Sigma^\infty T \to E$ such that $\tilde{\nu}^*$ carries the generator of $\Sigma^s M(q)$ to ν. Then E is equivalent to $\Sigma^s B(q)$.

They used this characterization to prove the following basic theorem on the structure of the X_q. Recall that Σ^∞ denotes the stabilization functor from spaces to spectra.

Theorem 2.12. At $p = 2$, $\Sigma^\infty X_q$ is equivalent to $\Sigma^q B[q/2]$. At $p > 2$, $\Sigma^\infty X_{pq}$ is equivalent to $\Sigma^{2q(p-1)} B[q/p]$ if $q \not\equiv 0 \mod p$.

Because of Theorem 9 (iii), it suffices to consider $q \equiv 2 \mod p$ when $p > 2$. In both cases, one inductively constructs (T_q, ν_q) adapted to $M[q/p]$, with ν_q of degree q or $2q(p-1)$, together with maps $\tilde{\nu}_q$ from $\Sigma^\infty T_q$ to $\Sigma^\infty X_q$ or $\Sigma^\infty X_{pq}$ which realize ν_q. Modulo certain constructions and calculations based on the geometric and homological properties of the $C_2 S^r$ in [36] and [14, III], the construction of the (T_q, ν_q) and $\tilde{\nu}_q$ reduces to the verification of the following result.

Theorem 2.13. For each $i \geq 0$, there is a compact smooth closed manifold of dimension 2^i if $p = 2$ or $p^{i+2} + p - 2$ if $p > 2$ with stable normal bundle ν_i and a bundle map

$$f_i : \nu_i \to \zeta_{2,2^i} \quad \text{if } p = 2 \qquad \text{or} \qquad f_i : \nu_i \to \zeta_{2,p^{i+2}+p} \quad \text{if } p > 2$$

such that $w_{2^i - 1}(\nu_i) \neq 0$ if $p = 2$ or $(Tf_i)^* P^{p(i+1)} u_i) \neq 0$ if $p > 2$, where $p(i+1) = 1 + p + \dots + p^i$ and u_i generate $H^*(T\zeta_{2,p^{i+2}+p})$ as an A-module.

When $p > 2$, the cohomological condition looks like but isn't a statement about Wu classes (since the relevant bundles $\zeta_{n,q}$ are not orientable).

Unfortunately, the proof of this last theorem is not very pleasant, being based on a detailed study of the Adams spectral sequance for the suspension spectrum of $T(\zeta_{2,q}) \wedge K(Z_p, 1)$ for the relevant q. This analysis is based on ideas of Mahowald [33]; the maps g displayed in Theorems 2.8 and 2.9 provide the key geometric input.

Several people have considered the problem of directly constructing explicit manifolds as claimed in the theorem above. However, as far as I know, not even the requisite four-dimensional manifold has been concretely identified. The maps f_i are called Mahowald orientations, or braid orientations. On the classifying space level, one is asking for a lift of $\nu_i : M_i \to BO$ to the appropriate braid space $B(R^2, q) = K(B_q, 1)$.

F. Cohen [12] has used results of [36 and 37] to prove that there is a commutative diagram

$$\begin{array}{ccc} K(B_\infty, 1) & \xrightarrow{i} & K(\Sigma_\infty, 1) \\ \bar{\alpha}\big\downarrow & & \big\downarrow j \\ \Omega^2 S^3 & \xrightarrow{\bar{\eta}} & BO \end{array},$$

where i is induced by the natural homomorphism $B_\infty \to \Sigma_\infty$, j is induced by the regular representation $\Sigma^\infty \to O$, $\bar{\alpha}$ is a homology isomorphism, and $\bar{\eta}$ is the second loop map induced from the non-trivial map $S^1 \to BO$. (This implies, and was the original proof of, (i) of Theorem 2.2.) Mahowald [33] proved that the Thom spectrum $M\bar{\eta}$ of $\bar{\eta}$ is $K(Z_2, 0)$, and it follows that the Thom spectrum of ji is also $K(Z_2, 0)$. (See Lewis [31] for a thorough study of the Thom spectra of maps.) In turn, this implies that any element of $H_*(X; Z_2)$ for any space X is represented by a braid orientable manifold. This fact says that such manifolds must abound, but they are still very hard to find directly. F. Cohen [12] shows that solv-manifolds and certain Bieberbach manifolds admit braid orientations.

In view of all this, it is very natural to ask precisely what it means geometrically for an n-manifold M to admit a braid orientation. Sanderson [44] answers this question by associating to a braid orientation of M a canonical immersion $N \times I \to M$, $\dim N = n-1$, factored through an embedding of $N \times I$ in $M \times R^2$ and showing how the normal bundle of M can be recovered from the immersion. I shall say no more about this here since Sanderson's paper appears in these proceedings.

Incidentally, if $\bar{\eta}_n$ is the restriction of $\bar{\eta}$ to $F_n C_2 S^1 \subset C_2 S^1 \simeq \Omega^2 S^3$, then Mahowald [33] also proved that $H^*(M\bar{\eta}_n; Z_2) \cong M(n)$. R. Cohen [22] has just recently proven that $M\bar{\eta}_n$ is in fact equivalent to $B(n)$.

Returning to our main line of development, we sketch how the results above lead to infinite families of elements in the ring π_*^s of stable homotopy groups of spheres. The essential point is the connection of the spaces X_q to $\Omega^2 S^{r+2}$ on the one hand and the relative ease of analyzing the homotopy groups $\pi_* B(q)$ on the other. Mahowald's work actually preceded that summarized above. In particular, he conjectured Theorem 2.12 (its implicit assumption being an error caught by Adams in a preliminary version of [33]). However, given the results above, Mahowald's arguments go as follows.

Let $b: S^8 \to BO$ generate $\pi_8 BO = Z$ and let $\bar{b}: \Omega S^9 \to BO$ be the unique loop map which restricts to b on S^8. Let $\tilde{b}: \Sigma\Omega^2 S^9 \to BO$ be the adjoint of $\Omega\bar{b}$, namely the composite of $\bar{b}$ with the evaluation map $\Sigma\Omega^2 S^9 \to \Omega S^9$. It is easy to check that $\tilde{b}^*(w_{2i}) \neq 0$ for $i \geq 3$ [33, p. 250]. It is well-known (see [14, II. 5.15] and [31, 2.3.4]) that the Thom spectrum $M\tilde{b}$ is the cofibre of the map $\phi: \Sigma^\infty \Omega^2 S^9 \to \Sigma^\infty S^0 = S$ of spectra adjoint to the composite map of spaces

$$\Omega S^9 \xrightarrow{\Omega \bar{b}} \Omega_0 BO \simeq SO \xrightarrow{j} SF = Q_1 S^0 \xrightarrow{*[-1]} Q_0 S^0 \subset QS^0$$

(where the subscripts indicate the relevant components). Of course, $Sq^{2^i} \mu \neq 0$ for $i \geq 3$, where μ is the Thom class of $M\tilde{b}$. Now

$$\Sigma^\infty \Omega^2 S^9 \simeq \bigvee_{q \geq 1} \Sigma^\infty D_q(R^2, S^7) \simeq \bigvee_{q \geq 1} \Sigma^\infty(\Sigma^{6q} X_q),$$

hence ϕ is a sum of maps $\phi_q: \Sigma^\infty(\Sigma^{6q} X_q) \to S$. Let

$$Y_i = \Sigma^{2^{i-2}+2^{i-1}} X_{2^{i-3}} \quad \text{and} \quad f_i = \phi_{2^{i-3}}: \Sigma^\infty Y_i \to S, \ i > 3.$$

Mahowald's main result [33, Thm. 2] reads as follows. Let Cf denote the cofibre of a map f (of spaces or spectra).

Theorem 2.14. The spaces Y_i and stable maps f_i have the following properties.

(i) Y_i has dimension $2^i - 1$, $H^{2^i-1}(Y_i; Z_2) = Z_2$, and Y_i is $(2^i - 2^{i-3} - 1)$-connected.

(ii) $Sq^{2^i} \mu \neq 0$, where μ generates $H^0(Cf_i; Z_2)$.

(iii) There is a (stable) map $g_i: S^{2^i} \to Y_i$ whose composite with the projection $Y_i \to Y_i/Y_i^{2^i-2} \simeq S^{2^i-1}$ is essential.

Here (i) is immediate from the filtration and homology of $C_2 S^1$ and (ii) is immediate from the paragraph above. Part (iii) requires a little more work since $\pi_{2^i} Y_i = \pi_{2^i-3} B(2^{i-4})$, and this is the first group beyond the range of Theorem 2.10 (iv) and the first in which 4-torsion can occur. (The preprint version of [7] gave a range one higher, this being an error caught by Mahowald.) Nevertheless, the explicit construction of the B(q) makes detection of the required g_i via the Adams spectral sequence quite easy.

Standard arguments show that $f_i \circ \Sigma^\infty g_i: \Sigma^\infty S^{2^i} \to S$ projects to $h_1 h_i$ in $E_2^{2, 2^i+2}$ of the mod 2 Adams spectral sequence. Thus the $h_1 h_i$ are permanent cycles and represent non-zero elements η_i in π_*^s. This was the starting point of Mahowald's Aarhus talk, in which he noted that one could start the argument above with $b: S^4 \to BO$ instead of $b: S^8 \to BO$ and so obtain a different family of

η_i's. He asserted that $2\eta_{i+1} = \eta_i^2$ for both families, this being zero if $i = 4$ and non-zero if $i \geq 5$ (for both families) but the reader is referred to Mahowald's contribution [35] to these proceedings for further information.

R. Cohen [21] gives the following analogous development at odd primes. Let $b: S^{2p-2} \to BSO$ generate $\pi_{2p-2}BSO = Z$. The composite $Bj \circ b: S^{2p-2} \to BSF$ determines a loop map $\bar{b}: \Omega S^{2p-1} \to BSF$ and $\Omega\bar{b}$ has adjoint $\tilde{b}: \Sigma\Omega^2 S^{2p-1} \to BSF$. Here $\tilde{b}^*(w_{p^i}) \neq 0$ for $i \geq 1$, where w_k is the k^{th} mod p Wu class. The Thom spectrum $M\tilde{b}$ is the cofibre of the adjoint $\phi: \Sigma^\infty \Omega^2 S^{2p-1} \to S$ of the composite

$$\Omega^2 S^{2p-1} \xrightarrow{\Omega\bar{b}_p} \Omega BSF \simeq SF = Q_1 S^0 \xrightarrow{*[-1]} Q_0 S^0 \subset QS^0.$$

Of course, $P^{p^i}\mu \neq 0$ for $i \geq 1$, where μ is the Thom class of $M\tilde{b}$. Now

$$\Sigma^\infty \Omega^2 S^{2p-1} \simeq \bigvee_{q \geq 1} \Sigma^\infty D_q(R^2, S^{2p-3}) \simeq \bigvee_{q \geq 1} \Sigma^\infty(\Sigma^{(2p-4)q} X_q),$$

hence ϕ is a sum of maps $\phi_q: \Sigma^\infty(\Sigma^{(2p-4)q} X_q) \to S$. Let

$$Y_i = \Sigma^{(2p-4)p^i} X_{p^i} \quad \text{and} \quad f_i = \phi_{p^i}: \Sigma^\infty Y_i \to S, \; i > 1.$$

<u>Theorem 2.15.</u> The spaces Y_i and stable maps f_i have the following properties (where cohomology is taken mod p).

(i) Y_i has dimension $2(p-1)p^i - 1$, $H^{2(p-1)p^i-1}(Y_i)$ and $H^{2(p-1)p^i-2}(Y_i)$ are both Z_p with respective generators y and x such that $\beta(x) = y$, and Y_i is $(2(p-1)p^i - 2p^{i-1} - 1)$-connected.

(ii) $P^{p^i}\mu = \hat{x}$ and $\Gamma_{i-1}\mu = \hat{y}$ (up to non-zero constants), where μ generates $H^0(Cf_i)$, $\hat{x}$ and $\hat{y}$ are the images in $H^*(Cf_i)$ of Σx and Σy in $H^*(\Sigma Y_i)$, and Γ_i is the secondary cohomology operation associated to the Adem relation for $P^{(p-1)p^i}P^{p^i}$; the second equation holds modulo zero indeterminacy.

(iii) There is a (stable) map $g_i: S^{2(p^i+1)(p-1)-3} \to Y_i$ such that $P^1\bar{x} \neq 0$ in $H^*(Cg_i)$, where $\bar{x}$ pulls back to $x \in H^*(Y_i)$.

Here (i) is again immediate from the filtration and homology of C_2S^1, the first part of (ii) is immediate from the paragraph above, and the second part of (ii) is a direct consequence in view of Liulevicius' factorization of P^{p^i} via secondary cohomology operations [32]. For (iii), Theorems 2.9(iii) and 2.12 give that

$$X_{p^i} \wedge M^{2(p-1)} \simeq X_{p^i+p} \overset{s}{\simeq} \Sigma^{2(p-1)(p^{i-1}+1)} B(p^{i-2}).$$

This time the desired element is just within the range of Theorem 2.10(iv), which gives a homotopy class

$$\mu_{p(i-1)} \in \pi_q B(p^{i-2}), \quad q = 2p(p^{i-2}+1) - 4 .$$

The required map g_i is obtained by suspension of the composite of $\Sigma^{2(p-1)(p^{i-1}+1)}\mu_{p(i-1)}$ with the projection $X_{p^i+p} \to \Sigma^{2p-1}X_{p^i}$ induced by the projection $M^{2(p-1)} \to S^{2p-1}$.

Standard arguments now show that $f_i \circ \Sigma^{\infty} g_i : \Sigma^{\infty} S^{2(p-1)(p^i+1)-3} \to S$ projects to $h_0\lambda_{i-1}$ in $E_2^{3,2(p-1)(p^i+1)}$ of the mod p Adams spectral sequence. Here $\lambda_i \in E_2^{2,2(p-1)p^{i+1}}$ corresponds to Γ_i and is also denoted b_1^i (or $b_{1,i}$ or b_i); up to sign, it is the p-fold symmetric Massey product $\langle h_i, \ldots, h_i \rangle$. Thus the $h_0\lambda_i$ are permanent cycles which represent non-zero elements ζ_i of order p in π_*^s. Of course, ζ_i extends to $\overline{\zeta}_i : \Sigma^{2(p-1)(p^{i+1}+1)-2} M^{-1} \to S$, where M^{-1} is the mod p Moore spectrum with bottom cell in dimension -1. If $\pi: M^{-1} \to S$ is the projection onto the top cell, then Cohen shows further that $\overline{\zeta}_i$ is represented by $\pi^*(h_0 h_{i+1})$ in E_2 of the mod p Adams spectral sequence converging to the stable cohomotopy of M^{-1}, but the method fails to detect the $h_0 h_{i+1}$ themselves.

§ 3. Splitting theorems; James maps and Segal maps

The material to be discussed here is simpler, but in an earlier stage of development, than that discussed in the previous section. It is potentially at least as rich, and should lead to a later generation of concrete homotopical applications. The calculations of the previous section presupposed from iterated loop space theory only the geometric properties of the approximation $C(R^n, X) \simeq \Omega^n\Sigma^n X$, the existence of the stable splitting $C(R^n,X) \overset{s}{\simeq} \bigvee_{q\geq 1} D_q(R^n,X)$, and understanding of the homologies of $C(R^n,X)$ and the pieces $D_q(R^n,X)$. The analogous information for first loop spaces would be the approximation $MX \simeq \Omega\Sigma X$, the splitting $\Sigma MX \simeq \bigvee_{q\geq 1} \Sigma X^{[q]}$, and the homologies of MX and the $X^{[q]}$. The latter information is utterly trivial, and the James approximation acquires much of its force from homological calculation of the James maps $j_q : MX \to MX^{[q]} \simeq \Omega\Sigma X^{[q]}$ whose adjoint James-Hopf maps $h_q : \Sigma MX \to \Sigma X^{[q]}$ yield the splitting. For example, it was just such homological information which led to the homological understanding of the key maps g of Theorems 2.8 and 2.9.

The deepest part of the theory to follow (and the part in most rudimentary form) will in principle lead to complete information on the homological behavior of the James maps $j_q : C(R^n, X) \to QD_q(R^n, X)$ whose adjoint stable James-Hopf maps $h_q^s : \Sigma^\infty C(R^n, X) \to \Sigma^\infty D_q(R^n, X)$ yield the stable splitting. However, while the geometry leading to such computations is more or less understood, we have not yet begun the actual calculations. Thus the present state of the theory is analogous to the status of the original approximation theorem after the work of [36] but before that of [14].

Before proceeding further, I should say that virtually everything discussed in this section is joint work of Cohen, Taylor, and myself [16-19] and also Caruso [10], the only exception being the closely related work of Koschorke and Sanderson [30].

I shall first explain the various splitting theorems of [16 and 17] and then discuss the multiplicative properties of the James maps and certain analogous maps, the definition of which is based on ideas of Segal [46]. We shall see that an unstable version of the Kahn-Priddy theorem follows directly from these properties, and we shall obtain a result on the 2-primary exponent of the homotopy group groups of spheres as an obvious corollary. Another fairly immediate application is a simple proof of Mahowald's theorem [34, 6.2.8] on how to represent $K(Z,0)$ as a Thom spectrum. Nevertheless, I am sure that the most interesting applications belong to the future.

Return to the general context established in section one. A collection of Σ_j-spaces $\mathcal{C}_j$ with degeneracy operators is denoted $\mathcal{C}$ and called a coefficient system. A collection $\underline{X} = \{X_q\}$ of based spaces with all the formal properties that would be present if X_q were the q^{th} power of a based space X is called a Π-space. Given $\mathcal{C}$ and $\underline{X}$, there results a filtered based space $C\underline{X}$. See [16, §§1,2] for details of this generalization of the construction CX of section one. X_q might be $P \wedge X^q$ for based spaces P and X, and this example leads to useful "parametrized" splitting theorems. However, the reader may prefer to think of X_q as X^q.

The splitting theorems of [16] all fit into a single general framework which we now sketch. Let $\mathcal{C}$ and $\mathcal{C}'$ be coefficient systems and let q be given. Let

$$D_q(\mathcal{C}, \underline{X}) = F_q C\underline{X} / F_{q-1} C\underline{X} = \mathcal{C}_q^+ \wedge_{\Sigma_q} X_{[q]} ,$$

where $X_{[q]}$ is the quotient of X_q by the generalized fat wedge present in X_q for a

Π-space $\underline{X}$. These spaces for $\mathcal{C}'$ will be irrelevant, and we abbreviate $D_q(\mathcal{C},\underline{X}) = D_q\underline{X}$.

A James system $\mathcal{C} \to \mathcal{C}'$ is a collection of maps $\mathcal{C}_r \to \mathcal{C}'_{(q,r-q)}$ such that certain simple diagrams commute [16, 4.1]. A James system induces a James map

$$j_q : C\underline{X} \to C'D_q\underline{X}$$

for any Π-space $\underline{X}$ [16, 4.2]. In practice, $C'X$ is an H-space for spaces X (but not for general Π-spaces). If we are given James systems $\mathcal{C} \to \mathcal{C}'$ for $1 \leq q \leq r$, then we can define $k_r : C\underline{X} \to C'(\bigvee_{q=1}^{r} D_q\underline{X})$ to be the sum over q of the composites of the j_q with the evident inclusions $C'D_q\underline{X} \to C'(\bigvee_{q=1}^{r} D_q\underline{X})$. We can also restrict k_r to the finite filtrations of $C\underline{X}$. Any chosen basepoint of $\mathcal{C}'_1$ gives rise to a natural map $\eta : X \to C'X$ for spaces X. We can thus write down the following diagram:

(*)
$$\begin{array}{ccccc}
F_{r-1}C\underline{X} & \xrightarrow{\iota} & F_rC\underline{X} & \xrightarrow{\pi} & D_r\underline{X} \\
\downarrow k_{r-1} & & \downarrow k_r & & \downarrow \eta \\
C'(\bigvee_{q=1}^{r-1} D_q\underline{X}) & \xrightarrow{C'\iota} & C'(\bigvee_{q=1}^{r} D_q\underline{X}) & \xrightarrow{C'\pi} & C'D_r\underline{X}
\end{array}$$

Here ι and π are used generically for evident cofibrations and quotient maps. In practice, the left square always commutes on the nose (which allows passage to limits over r when we have James systems for all q) and the right square at least commutes up to homotopy.

We are interested in homotopical splittings, but we digress momentarily to discuss homological splittings. The infinite symmetric product NX comes from the coefficient system $\mathcal{N}$ with each $\mathcal{N}_j$ a point. The unique maps $\mathcal{C}_r \to \mathcal{N}_{(q,r-q)}$ give a James system for each q. Here (*) commutes. Applying N to (*) and using the natural transformation $NN \to N$, we obtain the commutative diagram

$$\begin{array}{ccccc}
NF_{r-1}C\underline{X} & \xrightarrow{N\iota} & NF_rC\underline{X} & \xrightarrow{N\pi} & ND_r\underline{X} \\
\downarrow & & \downarrow & & \| \\
N(\bigvee_{q=1}^{r-1} D_q\underline{X}) & \xrightarrow{N\iota} & N(\bigvee_{q=1}^{r} D_q\underline{X}) & \xrightarrow{N\pi} & ND_r\underline{X}
\end{array}$$

Using the fact that N converts cofibrations to quasifibrations and the relationship between N and integral homology [24], and playing games with parameter spaces P, we derive the following general homological splitting theorem [16, 4.10].

Theorem 3.1. For all coefficient systems $\mathcal{C}$, Π-spaces $\underline{X}$, Abelian groups G, and $r \geq 1$ (including $r = \infty$), $\tilde{H}_*(F_r C\underline{X}; G)$ is isomorphic to $\sum_{q=1}^{r} \tilde{H}_*(D_q\underline{X}; G)$. These isomorphisms are natural in $\mathcal{C}, \underline{X}$, and G and commute with Bockstein operations.

The case $\mathcal{C} = \mathcal{N}$ recovers Steenrod's homological splitting [49] of the reduced symmetric powers $F_r NX$, and this example shows that we could not hope for a stable homotopical splitting without some restriction on $\mathcal{C}$.

For the homotopical splittings, $\mathcal{C}$ will always be Σ-free in the sense that Σ_j acts freely on $\mathcal{C}_j$ for each j. Returning to the general context, assume given a natural map $\beta_t : C'X \to \Omega^t\Sigma^t X$ for spaces X and some $t \geq 1$. The key example is $C'X = C(R^t, X)$ and $\beta_t = \alpha_t g_t^{-1}$ as in section one. Composing the diagram (*) with β_t and taking adjoints, we obtain a homotopy commutative diagram

$$\begin{array}{ccccccc}
\Sigma^{t-1}D_r\underline{X} & \xrightarrow{\delta} & \Sigma^t F_{r-1}C\underline{X} & \xrightarrow{\iota} & \Sigma^t F_r C\underline{X} & \xrightarrow{\pi} & \Sigma^t D_r\underline{X} \\
 & & \Big\downarrow \tilde{k}_{r-1} & & \Big\downarrow \tilde{k}_r & & \Big\| \\
 & & \bigvee_{q=1}^{r-1} \Sigma^t D_q\underline{X} & \overset{\iota}{\rightleftarrows} & \bigvee_{q=1}^{r} \Sigma^t D_q\underline{X} & \xrightarrow{\pi} & \Sigma^t D_r\underline{X}
\end{array}$$

Assuming inductively that $\tilde{k}_{r-1}$ is an equivalence, a trivial diagram chase implies that $\delta \simeq 0$ in the top cofibration sequence. This implies that $\tilde{k}_r$ is an equivalence. The same sort of argument works when $t = \infty$.

With $\mathcal{C} = \mathcal{C}' = \mathcal{M}$, where $\mathcal{M}_j = \Sigma_j$, this recovers and generalizes Milnor's splitting [42] of ΣMX [16, 3.7].

Theorem 3.2. For all Π-spaces $\underline{X}$ and $r \geq 1$ (including $r = \infty$), $\Sigma F_r M\underline{X}$ is naturally equivalent to $\bigvee_{q=1}^{r} \Sigma X_{[q]}$. The equivalence is given by sums over q of restrictions of James-Hopf maps $h_q : \Sigma MX \to \Sigma X_{[q]}$.

Note that no connectivity hypothesis is needed; we use a map $\beta_1 : MX \to \Omega\Sigma X$ but do not require it to be an equivalence. It is an immediate consequence that there is a natural weak equivalence $\Sigma\Omega\Sigma X \simeq \bigvee_{q \geq 1} \Sigma X^{[q]}$ for all connected based spaces X.

In [16, § 5], we introduce "separated" coefficient systems. For such $\mathcal{C}$, if $\mathcal{B}_q = \mathcal{C}_q/\Sigma_q$ and $\mathcal{C}^{(q)}$ is the coefficient system given by the configuration spaces $F(\mathcal{B}_q, j)$, there are tautological James systems $\mathcal{C} \to \mathcal{C}^{(q)}$ for each $q \geq 1$. When $\mathcal{C} = \mathcal{C}(Y)$ is itself the configuration space coefficient system of a space Y, we shall write down the resulting James maps explicitly below. If $\mathcal{B}_q$ embeds in R^t for $q \leq r$, then $\mathcal{C}^{(q)}$ maps to $\mathcal{C}(R^t)$ and we can apply the theory above with $\mathcal{C}' = \mathcal{C}(R^t)$. The product of any coefficient system with a separated system is separated, and by use of tricks involving both pairs of projections

$$\mathcal{C} \longleftarrow \mathcal{C} \times \mathcal{C}(R^\infty) \longrightarrow \mathcal{C}(R^\infty) \quad \text{and} \quad \mathcal{C}^{(q)} \longleftarrow \mathcal{C}^{(q)} \times \mathcal{C}(R^\infty) \longrightarrow \mathcal{C}(R^\infty)$$

we prove the following general splitting theorem [16, 8.2]. The second pair of projections plays a critical role in the naturality, leads to a uniqueness assertion for the stable James-Hopf maps, and allows one to avoid any choices of embeddings. This precision is crucial to the deeper theory discussed at the end of the section. On the other hand, as discussed in [16, § 5], use of embeddings gives a good hold on the destabilization properties of the James-Hopf maps.

Theorem 3.3. For all Σ-free coefficient systems $\mathcal{C}$, Π-spaces $\underline{X}$, and $r \geq 1$ (including $r = \infty$), $\Sigma^\infty F_r C\underline{X}$ is equivalent to $\bigvee_{q=1}^{r} \Sigma^\infty D_q \underline{X}$, naturally in $\mathcal{C}$ and $\underline{X}$. The equivalence is given by restrictions of stable James-Hopf maps $h_q^s : \Sigma^\infty C\underline{X} \to \Sigma^\infty D_q \underline{X}$.

Specializing either to $\mathcal{C} = \mathcal{C}(R^n)$ or $\mathcal{C} = \mathcal{C}_n$, compatibly in view of $g: \mathcal{C}_n \to \mathcal{C}(R^n)$, we obtain the following immediate consequence [16, 8.4].

Corollary 3.4. For all connected based spaces X, there is a natural equivalence in the stable category between $\Sigma^\infty \Omega^n \Sigma^n X$ and $\bigvee_{q \geq 1} \Sigma^\infty D_q(R^n, X)$, $n \geq 1$ or $n = \infty$, and these equivalences are compatible as n varies.

Such equivalences were first obtained by Snaith [48], but our proof has a number of advantages (discussed in [16, § 7]). In particular, it is not clear that Snaith's splitting maps $\Sigma^t F_r C_n X \to \Sigma^t D_{n,q} X$ can be extended over all of $\Sigma^t C_n X$; that is, they are not given by globally defined James-Hopf maps.

We shall come back to these splittings shortly, but I want first to explain the further splittings obtained in [17], which partially remove the restriction to connected spaces in the corollary above.

In [17, § 1], we introduce the notion of a "directed" coefficient system. The details are rather delicate and the range of examples is peculiar; $\mathcal{M}$ and $\mathcal{N}$ are directed but the $\mathcal{C}_n$ are not; $\mathcal{C}(Y)$ is directed if Y is an open manifold but is not directed if Y is a compact ANR. When $\mathcal{C}$ is directed, there are inclusions

$$\zeta_r : \mathcal{C}_r \times_{\Sigma_r} X_r \to \mathcal{C}_{r+1} \times_{\Sigma_{r+1}} X_{r+1}$$

for a Π-space $\underline{X}$, and we define $\overline{C}\underline{X}$ to be the resulting colimit. If $\mathcal{C} \to \mathcal{C}'$ is a James system, there result James maps

$$\overline{j}_q : \overline{C}\underline{X} \to C'\overline{D}_q\underline{X},$$

where $\overline{D}_q\underline{X}$ is a certain space equivalent to the cofibre of ζ_{q-1}. Just as before, we define $\overline{k}_r : \overline{C}\underline{X} \to C'(\bigvee_{q=1}^{r} \overline{D}_q\underline{X})$ by summing the $\overline{j}_q$ for $q \leq r$ and write down the diagram

$$\begin{array}{ccccc}
\mathcal{C}_{r-1} \times_{\Sigma_{r-1}} X_{r-1} & \xrightarrow{\iota} & \mathcal{C}_r \times_{\Sigma_r} X_r & \xrightarrow{\pi} & \overline{D}_r\underline{X} \\
\downarrow{\scriptstyle \overline{k}_{r-1}} & & \downarrow{\scriptstyle \overline{k}_r} & & \downarrow{\scriptstyle \eta} \\
C'(\bigvee_{q=1}^{r-1} \overline{D}_q\underline{X}) & \xrightarrow{C'\iota} & C'(\bigvee_{q=1}^{r} \overline{D}_q\underline{X}) & \xrightarrow{C'\pi} & C'\overline{D}_r\underline{X}
\end{array}$$

From here, the derivation of splitting theorems is precisely the same as in the discussion above, and we obtain the following theorems [17, §2].

Theorem 3.5. For all directed coefficient systems $\mathcal{C}$, Π-spaces $\underline{X}$, Abelian groups G, and $r \geq 1$, there are isomorphisms

$$\tilde{H}_*(\mathcal{C}_r \times_{\Sigma_r} X_r; G) \cong \sum_{q=1}^{r} \tilde{H}_*(D_q\underline{X}; G) \quad \text{and} \quad \tilde{H}_*(\overline{C}X; G) \cong \sum_{q \geq 1} \tilde{H}_*(D_q\underline{X}; G)$$

which are natural in $\mathcal{C}$, $\underline{X}$, and G and commute with Bocksteins.

With $\mathcal{C} = \mathcal{N}$, this recovers Steenrod's isomorphisms [49]

$$\tilde{H}_*(X^r/\Sigma_r; G) \cong \sum_{q=1}^{r} \tilde{H}_*(X^q/\Sigma_q, X^{q-1}/\Sigma_{q-1}; G)$$

for the unreduced symmetric powers of a space.

Theorem 3.6. For all Π-spaces $\underline{X}$ and $r \geq 1$, there are natural equivalences

$$\Sigma X_r \simeq \bigvee_{q=1}^{r} \Sigma(X_q/\mathrm{Im}\, X_{q-1}) \quad \text{and} \quad \Sigma \overline{M}\underline{X} \simeq \bigvee_{q \geq 1} \Sigma(X_q/\mathrm{Im}\, X_{q-1}).$$

For spaces X, $\overline{M}X$ is the weak infinite product of countably many copies of X and the successive quotients are X^q/X^{q-1}, X^{q-1} being embedded as points with last coordinate the basepoint.

Theorem 3.7. For all Σ-free directed coefficient systems $\mathcal{C}$ and all Π-spaces $\underline{X}$, there are natural equivalences

$$\Sigma^{\infty}(\mathcal{C}_r \times_{\Sigma_r} X_r) \simeq \bigvee_{q=1}^{r} \Sigma^{\infty}\overline{D}_q\underline{X} \quad \text{and} \quad \Sigma^{\infty}\overline{C}\underline{X} \simeq \bigvee_{q\geq 1} \Sigma^{\infty}\overline{D}_q\underline{X}.$$

For example, with $\mathcal{C} = \mathcal{C}(R^{\infty})$, the case of spaces BG for a topological monoid G gives that $B(\Sigma_{\infty}\int G)$ is stably equivalent to the wedge of the cofibres of the natural maps $B(\Sigma_{q-1}\int G) \to B(\Sigma_q\int G)$.

For the promised analog of Corollary 3.4, we use an approximation of the form

$$\overline{C}(R^n, X) \xleftarrow{\ \overline{g}\ } \overline{C}_n(X^+) \xrightarrow{\ \overline{\alpha}_n\ } \Omega_0^n\, \Sigma^n(X^+)\ .$$

Here X is a connected space and X^+ is the union of X and a disjoint basepoint. The space $\overline{C}_n(X^+)$ is the telescope of a sequence of "right translations"

$$\mathcal{C}_{n,r} \times_{\Sigma_r} X^r \to \mathcal{C}_{n,r+1} \times_{\Sigma_{r+1}} X^{r+1}\ ,$$

and the map $\overline{\alpha}_n$ is a homology isomorphism constructed at the end of [14, I§ 5]. While $\overline{\alpha}_n$ is defined there for general X, it is only a homology isomorphism for connected X; in general, the two-variable Browder operations mix components non-trivially in $H_*\Omega_0^n\, \Sigma^n(X^+)$ but not in $H_*\overline{C}_n(X^+)$. The map $\overline{g}$ is an equivalence analogous to g [17, 3.1].

Corollary 3.8. For all connected based spaces X, there is a natural equivalence in the stable category between $\Sigma^{\infty}\Omega_0^n\Sigma^n(X^+)$ and $\bigvee_{q\geq 1}\Sigma^{\infty}\overline{D}_q(R^n, X)$, $n \geq 2$ or $n = \infty$, and these equivalences are compatible as n varies.

These results by no means exhaust the possibilities of the basic line of argument, and there are various other such splittings known to Cohen, Taylor, and myself but not written down. For example, Joe Neisendorfer reminded us of [36,6.6], in which I introduced a relative construction $E_n(X, A)$ for a based pair (X, A). When $A \to X$ is a cofibration and A is connected, there is a quasifibering

$$C_nA \to E_n(X, A) \to C_{n-1}(X/A)\ , \qquad n \geq 1\ ,$$

where C_0 is the identity functor [36, 7.3]. There are filtration preserving inclusions $C_nA \subset E_n(X, A) \subset C_nX$ and it is perfectly straightforward to trace through the proof of the stable splitting of C_nX and see that it restricts to give a stable splitting of $E_n(X, A)$.

Theorem 3.9. Let $A \to X$ be a cofibration. For all $r \geq 1$ (including $r = \infty$), and all $n \geq 1$ (including $n = \infty$), there is a natural equivalence

$$\Sigma^{\infty} F_r E_n(X, A) \simeq \bigvee_{q=1}^{r} \Sigma^{\infty}(F_q E_n(X, A)/F_{q-1} E_n(X, A)).$$

These equivalences are compatible as r and n vary and are also compatible with the stable splittings of $C_n A$ and $C_n X$.

The relationship between the splittings of $E_n(X, A)$ and of $C_{n-1}(X/A)$ is unclear and deserves study.

Again, it is a simple matter to give equivariant versions of our splitting theorems, putting actions of a finite group G on all spaces in sight (see [39, § 5]), and this in turn is surely a special case of a general categorical version of the argument.

We return to the original splitting theorem and specialize to configuration space coefficient systems $\mathcal{C}(Y)$, the case $Y = R^n$ being of most interest. Actually, we are wholly uninterested in splitting theorems in the rest of the paper, being concerned instead with the analysis of the James maps as a topic of independent interest.

As in section one, think of points of $C(Y, X)$ as pairs (L, λ), where L is a finite subset of Y and $\lambda: L \to X$ is a function. Recall that $B(Y, q) = F(Y, q)/\Sigma_q$. As mentioned above, there are canonical James systems which give rise to James maps

$$j_q: C(Y, X) \to C(B(Y, q), D_q(Y, X)).$$

Explicitly, $j_q(L, \lambda) = (M, \mu)$, where M is the set of all subsets of L with q elements (such a set of q elements of Y being a typical point of $B(Y, q)$) and $\mu: M \to D_q(Y, X)$ sends a point $m \in M$ to the image in $D_q(Y, X)$ of $(m, \lambda | m) \in F_q C(Y, X)$. Of course, it is not immediately apparent that j_q is well-defined. To check this, the more combinatorial description in [16, § 5] is perhaps more appropriate. To proceed further, one can assume that $B(Y, q)$ embeds in R^t, say via e_q, and then compose with $C(e_q, 1)$ to obtain a James map

$$j_q: C(Y, X) \to C(R^t, D_q(Y, X)).$$

When $Y = R^n$, we may take $t = 2qn$ (or $(2q-1)n$, by [16, 5.7]).

This functional description of these James maps is due to Koschorke and Sanderson [30], who discovered them independently of Cohen and Taylor. (To see the comparison, their $C^r_m(X)$ is our $C(B(R^m, k), X)$.) Their emphasis is not on the maps and their homotopical implications but rather on their geometrical interpretation. Let V be a smooth manifold without boundary with one-point compactification V_c. Also, let ξ be a vector bundle over some space B, with Thom complex $T\xi$, and let $\xi_{m,k}$ be the evident derived bundle over $B_k = F(R^m, k) \times_{\Sigma_k} B^k$. Consider immersion data consisting of a smooth closed manifold M, an immersion $g_1: M \to V$ with normal bundle ν, and a bundle map $\overline{g}: \nu \to \xi_{m,k}$ such that $(g_1, g_2): M \to V \times B(R^m, k)$ is an embedding, where $g_2: M \to B(R^m, k)$ is the composite of the base space map of $\overline{g}$ and the projection $B_k \to B(R^m, k)$. Let $\mathcal{I}^k_m(V, \xi)$ be the set of bordism classes of such immersions. Koschorke and Sanderson first prove that $C^k_m(T\xi)$ classifies this set,

$$\mathcal{I}^k_m(V, \xi) \cong [V_c, C^k_m(T\xi)],$$

and then explain how to interpret the maps j_q above (for X a Thom space) in terms of certain operations $\mathcal{I}^1_m(V,\xi) \to \mathcal{I}^k_m(V,\xi)$ specified by associating to an immersion $g_1: M \to V$ with normal bundle ξ an immersion $g^k_1: M(k) \to V$ with normal bundle mapping appropriately to $\xi_{m,k}$, where $M(k) \subset B(M, k)$ is the manifold of k-tuple self-intersection points of g_1. In this context, they obtain geometric proofs and interpretations of some of the multiplicative properties of James maps we are about to discuss.

In [10], we shall discuss multiplicative properties of James maps in full axiomatic generality. Given suitably related James systems $\mathcal{C} \to \mathcal{C}^{(q)}$ and suitable structure on $\mathcal{C}$ and the $\mathcal{C}^{(q)}$, there is a ring space structure on the infinite product $\times_{q \geq 0} C^{(q)} D_q X$ and the map

$$(j_q): CX \to \times_{q \geq 0} C^{(q)} D_q X$$

is an exponential H-map. Here $D_0 X = S^0$ and j_0 carries CX to $1 \in S^0 \subset C^{(0)} S^0$. For any coefficient system $\mathcal{C}$ with appropriate sums $\mathcal{C}_p \times \mathcal{C}_q \to \mathcal{C}_{p+q}$, the trivial James systems $\mathcal{C} \to \mathcal{N}$ used to prove Theorem 3.1 satisfy the relevant axioms. For any separated $\mathcal{C}$ with sums, the canonical James systems $\mathcal{C} \to \mathcal{C}(\mathcal{B}_q)$ satisfy the axioms. If $\mathcal{C} = \mathcal{C}(Y)$, where Y admits an injection $Y \amalg Y \to Y$ each component of which is homotopic through injections to the identity map, then $\mathcal{C}$ admits sums of the sort required. In particular, this applies to $Y = R^n$. Here the following are all H-maps with respect to the appropriate multiplication on the infinite products:

$$\begin{array}{ccc}
\underset{q\geq 0}{\times}\; C(B(R^n, q), D_q(R^n,X)) & \xrightarrow{\underset{q\geq 0}{\times}\; C(e_q, 1)} & \underset{q\geq 0}{\times}\; C(R^{2qn}, D_q(R^n,X)) \\
\Big\uparrow (j_q) & & \Big\downarrow \underset{q\geq 0}{\times}\; \alpha_{2qn}\, g^{-1} \\
C(R^n, X) & \overset{(j_q)}{\dashrightarrow} & \underset{q\geq 0}{\times}\; \Omega^{2qn}\Sigma^{2qn} D_q(R^n,X)
\end{array}$$

We continue to write j_q for the composite $\alpha_{2qn} g^{-1} C(e_q, 1) j_q$. We could also have stabilized, replacing $2qn$ by ∞ on the right. The product on the loop space level is induced in an evident way from smash products $\Omega^i Y \times \Omega^j Z \to \Omega^{i+j}(Y \wedge Z)$ and the pairings

$$D_s(R^n,X) \wedge D_t(R^n,X) \;\to\; D_{s+t}(R^n,X)$$

induced by the additive H-space structure on $C(R^n,X)$.

We digress to mention an application to Thom spectra in [18]. There we give a simple proof, based solely on use of Steenrod operations, of the following theorem. Let $S^3\langle 3\rangle$ denote the 3-connective cover of S^3.

Theorem 3.10. (i) The Thom spectrum associated to any H-map $\Omega^2 S^3 \to BF$ with non-zero first Stiefel-Whitney class is $K(Z_2, 0)$.
(ii) The Thom spectrum associated to any H-map $\Omega^2 S^3\langle 3\rangle \to BSF$ with non-zero second Stiefel-Whitney class and non-zero first Wu class at each odd prime is $K(Z, 0)$.

Part (i) gives a new proof of Mahowald's result that $M\eta = K(Z_2, 0)$. At $p > 2$, $D_i(R^2, S^{2q-1}) \simeq 0$ for $1 < i < p$ and $D_p(R^2, S^{2q-1}) \simeq M^{2pq-2}$. It follows from the discussion above that

$$j_p : \Omega^2 S^{2q+1} \simeq C_2 S^{2q-1} \to QM^{2pq-1}$$

is a p-local H-map. As explained in [18], with $q = 1$ this easily leads to an H-map as prescribed in part (ii) and so gives Mahowald's result that $K(Z, 0)$ is a Thom spectrum.

Returning to the work in [10], we now head towards the Kahn-Priddy theorem. We follow the ideas of Segal [46], but we work unstably with general spaces X and thus introduce a great deal of new structure into iterated loop space theory. We want first to extend the James maps over $\Omega^n\Sigma^n X$. There is no problem when X is connected, but it is the case $X = S^0$ in which we are most interested. As Segal points out [46], the following obstruction theoretical observation allows use of the exponential H-map property above to extend the j_q simultaneously for all q. Henceforward, all H-spaces are to be homotopy associative and commutative.

Lemma 3.11. Let $g: X \to Y$ be a group completion of H-spaces, where $\pi_0 X$ has a countable cofinal sequence. Then for any grouplike H-space Z and weak H-map $f: X \to Z$, there is a unique weak H-map $\tilde{f}: Y \to Z$ such that $\tilde{f}g$ is weakly homotopic to f.

The "weak" aspect is that we are ignoring $\lim^1$ terms. The interpretation is that, on finite-dimensional CW-complexes A, $g: [A,X] \to [A,Y]$ is universal with respect to natural transformations of monoid-valued functors from $[A,X]$ to group-valued represented functors $[A,Z]$. We take $[\ ,\ ]$ in the sense of based homotopy classes.

By a power series argument, $(1, \times_{q \geq 1} \Omega^{2qn}\Sigma^{2qn} D_q(R^n, X))$ is grouplike; that is, its monoid of components is a group. This gives the following generalization of results of Segal [46]. We assume that $\pi_0 X$ is countable and write $\eta(r,s)$ for the natural inclusion $\Omega^r \Sigma^r X \to \Omega^s \Sigma^s X$, $s \geq r$; $\eta(r,s)$ induces $(s-r)$-fold suspension on homotopy groups.

Theorem 3.12. For $n \geq 2$ and all X, there exist maps

$$j_q : \Omega^n \Sigma^n X \to \Omega^{2nq} \Sigma^{2nq} D_q(R^n, X)$$

such that j_0 is constant at $1 \in S^0$, j_1 is $\eta(n, 2n)$, and

$$j_r(\alpha + \beta) = \sum_{p+q=r} j_p(\alpha) j_q(\beta)$$

for $\alpha, \beta \in [A, \Omega^n \Sigma^n X]$.

Here the sums are loop addition and the products are those specified above. Segal [46] also introduced very special cases of the general maps

$$s_q : C(Z, D_q(Y,X)) \to C(Z \times Y^q, X^{[q]})$$

specified by $s_q(M, \mu) = (N, \nu)$ where if $\mu(m)$ is the image in $D_q(Y,X)$ of an element $(L_m, \lambda_m) \in F_q C(Y,X)$ such that $L_m \subset Y$ has q elements, then

$$N = \bigcup_{m \in M, \ell_i \in L_m, \sigma \in \Sigma_q} (m, \ell_{\sigma(1)}, \ldots, \ell_{\sigma(q)}) \subset Z \times Y^q$$

and

$$\nu(m, \ell_{\sigma(1)}, \ldots, \ell_{\sigma(q)}) = \lambda_m(\ell_{\sigma(1)}) \wedge \cdots \wedge \ell_m(\ell_{\sigma(q)}) .$$

It is easy to analyze the additive and multiplicative properties of the s_q combinatorially, and we arrive at the following complement to the previous result. Let $X^{[0]} = S^0$.

Theorem 3.13. For $m \geq 2$, $n \geq 1$, and all X, there exist weak H-maps

$$s_q : \Omega^m \Sigma^m D_q(R^n, X) \rightarrow \Omega^{m+nq} \Sigma^{m+nq} X^{[q]}$$

such that s_0 is the identity map of $\Omega^m S^m$, s_1 is $\eta(n, m+n)$, and

$$s_r(\beta\gamma) = (p,q) s_p(\beta) s_q(\gamma), \quad r = p+q,$$

for $\beta \in [A, \Omega^{tp} \Sigma^{tp} D_p(R^n, X)]$ and $\gamma \in [A, \Omega^{tq} \Sigma^{tq} D_q(R^n, X)]$, $t \geq 2$. Moreover, for $\alpha \in [A, \Omega^m \Sigma^m X]$,

$$(s_q \circ \Omega^m \Sigma^m \bar{\Delta})(\alpha) = q!(\eta(m, m+nq) \circ \Omega^m \Sigma^m \Delta)(\alpha),$$

where $\Delta : X \rightarrow Y^{[q]}$ is the diagonal and $\bar{\Delta} : X \rightarrow D_q(R^n, X)$ is induced from Δ (via any chosen basepoint in $F(R^n, q)$).

The product $\beta\gamma$ is that above, while that on the right is just smash product of maps. Here s_q is obtained by application of Lemma 3.11 to the additive H-space structures, and the uniqueness clause of that lemma implies the last formula. The passage from the combinatorial level product formula to the loop space level is more subtle and requires use of the following result (the need for which was overlooked in [46]).

Lemma 3.14. Let $g: X \rightarrow Y$ and $g': X' \rightarrow Y'$ be group completions, where $\pi_0 X$ and $\pi_0 X'$ have countable cofinal sequences. Then for any grouplike H-space Z and weakly homotopy bilinear map $f: X \wedge X' \rightarrow Z$ there exists a unique weakly homotopy bilinear map $\tilde{f}: Y \wedge Y' \rightarrow Z$ such that $\tilde{f}(g \wedge g')$ is weakly homotopic to f.

Setting $m = 2nq$, we can compose s_q with j_q. To analyze this composite, we need the map

$$k_q : C(Y, X) \rightarrow C(Y^q, X^{[q]})$$

specified by $k_q(L, \lambda) = (M, \mu)$ where M is the set of all ordered q-tuples of elements of L and $\mu(\ell_1, \ldots, \ell_q) = \lambda(\ell_1) \wedge \ldots \wedge \lambda(\ell_q)$. Again, easy combinatorics, a power series argument, and use of Lemma 3.11 give the following result.

Theorem 3.15. For $n \geq 2$ and all X, there exist maps

$$k_q : \Omega^n \Sigma^n X \rightarrow \Omega^{nq} \Sigma^{nq} X^{[q]}$$

such that k_0 is constant at $1 \in S^0$, k_1 is the identity map, and

$$k_r(\alpha + \beta) = \sum_{p+q=r} (p,q) k_p(\alpha) k_q(\beta)$$

for $\alpha, \beta \in [A, \Omega^n \Sigma^n X]$. Moreover,

$$(\eta(nq, 3nq) \circ k_q)(\alpha) = (s_q \circ j_q)(\alpha) .$$

While all this general structure is bound to prove useful, the combinatorics for the last step towards the Kahn-Priddy theorem require $\Delta : X \to X^{[q]}$ to be the identity map, that is to say $X = S^0$. Let c_{iq} be the number of ways of dividing a set of q elements into i unordered subsets. The "therefore" in the following result comes from a purely algebraic argument.

Theorem 3.16. For $n \geq 2$ and $\alpha \in [A, \Omega^n S^n]$,

$$\alpha^q = \sum_{i=1}^{q} c_{iq}(\eta(ni, nq) \circ k_i)(\alpha).$$

Therefore $k_q(\alpha) = \alpha(\alpha-1)\cdots(\alpha-q+1)$ if $A = B^+$, where $r \in [A, \Omega^n S^n]$ is r times the map which sends B to 1. If, further, B is a suspension and α maps B to $\Omega_0^n S^n$, then

$$k_q(\alpha) = (-1)^{q-1}(q-1)!\,\eta(n, nq)(\alpha) .$$

The last assertion holds since $\alpha\alpha = 0$ by the standard argument that cup products are trivial for a suspension.

Note that $D_q(R^n, S^0) = B(R^n, q)^+$ and let $\delta : D_q(R^n, S^0) \to S^0$ map 0 to 0 and $B(R^n, q)$ to 1. Let $F_q(n, t)$ be the fibre of $\Omega_0^t \Sigma^t \delta : \Omega_0^t \Sigma^t D_q(R^n, S^0) \to \Omega_0^t S^t$ and note that $F_q(n, \infty) \simeq QB(R^n, q)$. Any choice of basepoint in $B(R^n, q)$ yields $S^0 \to D_q(R^n, S^0)$, and there results a composite equivalence

$$F_q(n,t) \times \Omega^t S^t \to \Omega^t \Sigma^t D_q(R^n, S^0) \times \Omega^t \Sigma^t D_q(R^n, S^0) \to \Omega^t \Sigma^t D_q(R^n, S^0) .$$

Let $j_q : \Omega^n S^n \to \Omega^{nq} \Sigma^{nq} D_q(R^n, S^0)$ have components j'_q and j''_q in $F_q(n,t)$ and $\Omega^t S^t$. Theorem 3.15 gives a homotopy commutative diagram

$$\begin{array}{ccc}
\Omega^n S^n & \xrightarrow{(j'_q, j''_q)} & F_q(n, 2nq) \times \Omega^{2nq} S^{2nq} \simeq \Omega^{2nq} \Sigma^{2nq} D_q(R^n, S^0) \\
\downarrow{\scriptstyle k_q} & & \downarrow{\scriptstyle s_q} \\
\Omega^{nq} S^{nq} & \xrightarrow{\eta(nq, 3nq)} & \Omega^{3nq} S^{3nq}
\end{array}$$

On $\pi_r \Omega_0^n S^n$, $r > 0$, Theorems 3.13 and 3.16 yield the formula

$$(s_q j'_q)(\alpha) = (-1)^{q-1}(q-1)!\ \Sigma^{3nq-n} \alpha - q!\ \Sigma^{nq} j''_q(\alpha) .$$

This is our unstable version of the Kahn-Priddy theorem. Taking q to be a prime p, we conclude that, up to a constant, $s_q j'_q$ is congruent mod p to the iterated suspension homomorphism. All maps in sight are compatible as n varies. Since $B(R^{\infty}, p) \simeq B\Sigma_p$, we obtain Segal's version [46] of the usual Kahn-Priddy theorem on passage to limits.

Theorem 3.17. The composite $Q_0S^0 \xrightarrow{j'_p} QB\Sigma_p \xrightarrow{s_p} Q_0S^0$ is a p-local homotopy equivalence.

It is not clear to us that s_p is an infinite loop map. According to Adams [1], this is a necessary and sufficient condition that s_p agree with the map used by Kahn and Priddy [28].

By construction, we have the commutative diagram

$$\begin{array}{ccccc} \Omega_0^n S^n & \xrightarrow{j'_2} & QB(R^n, 2) \simeq Q(RP^{n-1}) & & \\ \eta \downarrow & & \downarrow & & \\ Q_0S^0 & \xrightarrow{j'_2} & QB(R^{\infty},2) \simeq Q(RP^{\infty}) & \xrightarrow{s_2} & Q_0S^0 . \end{array}$$

Thus stabilization factors through $Q(RP^{n-1})$. This has the following consequence.

Theorem 3.18. If $\alpha \in \pi_r^s$ is a 2-torsion element in the image under stabilization of $\pi_{2n+1+r}S^{2n+1}$, then $2^{n+\varepsilon}\alpha = 0$, where $\varepsilon = 0$ if $n \equiv 0$ or 3 mod 4 and $\varepsilon = 1$ if $n \equiv 1$ or 2 mod 4.

Indeed, Toda [50] proves that the identity of $\Sigma^{2n}RP^{2n}$ has this order.

All of this is quite easy. We close with some remarks onthe deeper theory, to appear in [19], which explains what structure the James maps $j_q: C(R^n,X) \to QD_q(R^n,X)$ really carry. As mentioned before, $C(R^n,X)$ is not just an H-space but a $\mathcal{C}_n$-space. Since $H_*C(R^n,X)$ is functorially determined by H_*X via homology operations derived from this structure, one wants to know how this structure behaves with respect to the James maps. Consider the infinite product $\times_{q\geq 0} QD_q(R^n,X)$. We have said that this is a ring space. In fact, it is an E_n ring space (more precisely, it has an equivalent subspace so structured). This means that there is an operad pair $(\mathcal{C}, \mathcal{J})$ in the sense of [37, VI. 1.6] such that $\mathcal{C}$ is an E_{∞} operad and $\mathcal{J}$ is an E_n operad (that is, $\mathcal{J}$ is equivalent to $\mathcal{C}_n$) and there is an action in the sense of [37,VI. 1.10] of $(\mathcal{C}, \mathcal{J})$ on $\times_{q\geq 0} QD_q(R^n,X)$. (For the aficionados, $\mathcal{C}$ is the little convex bodies operad $\mathcal{K}_{\infty}$ and $\mathcal{J} = \mathcal{C}_n \times \mathcal{L}$, where $\mathcal{L}$ is the linear isometries operad.) The additive action, by $\mathcal{C}$, is the evident product action. The multiplicative action, by $\mathcal{J}$, is a parametrization of the multiplicative

H-space structure described earlier. $\mathcal{J}$ also acts on $C(R^n, X)$ (via the projection $\mathcal{J} \to \mathcal{C}_n$), and the crucial fact is that

$$(j_q): C(R^n, X) \to \underset{q \geq 0}{\times} QD_q(R^n, X)$$

is a map of $\mathcal{J}$-spaces. Upon restriction of its target to the unit space (zeroth coordinate 1), the recognition principle of [36] implies that the extension

$$(j_q): \Omega^n \Sigma^n X \to (1, \underset{q \geq 1}{\times} QD_q(R^n, X))$$

is actually an n-fold loop map for a suitable n-fold delooping of the target (not, of course, the obvious additive one).

To compute all the j_q on homology, it suffices to determine the multiplicative homology operations on the target. In principle, these are completely determined by the known additive operations and general mixed Cartan and mixed Adem relations for E_n ring spaces like those developed for E_∞ ring spaces in [14, II]. I have no doubt that such calculations will eventually become a powerful tool for the working homotopy theorist, just as the earlier calculations of [14], which once seemed impossibly complicated, are now being assimilated and exploited by many workers in the field.

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