ON TRAIN TRACK SPLITTING SEQUENCES

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ABSTRACT. We present a structure theorem for the subsurface projections of train track splitting sequences. For the proof we introduce *induced tracks*, *efficient position*, and *wide curves*. As a consequence of the structure theorem we prove that train track sliding and splitting sequences give quasi-geodesics in the train track graph; this generalizes a result of Hamenstädt [Invent. Math.].

1. INTRODUCTION

Thurston, in his revolution of geometric topology, introduced train tracks to the study of surface diffeomorphisms and of hyperbolic three-manifolds. The geometry and combinatorics of individual tracks have been carefully studied [25, 20, 19]. Equally important is the dynamical idea of splitting a train track. This leads to a deep connections between splitting sequences of tracks and the curves they carry, on the one hand, and Teichmüller geodesics and measured foliations on the other.

Another notion that has recently proved useful, for example in the resolution of the ending lamination conjecture [18, 3], is subsurface projection. A combinatorial or geometric object τ in a surface S has a projection $\pi_X(\tau)$ contained in the given essential subsurface $X \subset S$. Subsurface projection arises naturally in the study of the metric properties of entities such as the marking graph and the pants graph [15] and also when obtaining coarse control over Teichmüller geodesics [21, 22].

In this light it becomes important to understand how a splitting sequence of tracks $\{\tau_i\}$ interacts with the subsurface projection map π_X. We give a structure theorem (Theorem 5.3) that explains this interaction in great detail. As a major consequence we obtain:

Theorem 5.5. *For any surface S with $\xi(S) \geq 1$ there is a constant $Q = Q(S)$ with the following property: For any sliding and splitting sequence $\{\tau_i\}_{i=0}^N$ of birecurrent train tracks in S and for any essential*

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subsurface $X \subset S$ if $\pi_X(\tau_N) \neq \emptyset$ then the sequence $\{\pi_X(\tau_i)\}_{i=0}^N$ is a Q–unparameterized quasi-geodesic in the curve complex $\mathcal{C}(X)$.

There are a number of further consequences of the work in this paper. For example we generalize, via a very different proof, a result of Hamenstädt [10, Corollary 3]:

Theorem 6.2. *For any surface S with $\xi(S) \geq 1$ there is a constant $Q = Q(S)$ with the following property: If $\{\tau_i\}_{i=0}^N$ is a sliding and splitting sequence in the train track graph $\mathcal{T}(S)$, injective on slide subsequences, then $\{\tau_i\}$ is a Q–quasi-geodesic.*

Our techniques have further applications: Theorem 5.5 is used in Masur and Schleimer's proof [17] that the disk complex is Gromov hyperbolic. Maher and Schleimer [13] use our structure theorem and the local finiteness of the train track graph to prove the *stability* of disk sets in the curve complex. As a consequence they prove that the *graph of handlebodies* has infinite diameter and they also produce, in every genus, a pseudo-Anosov map so that no non-trivial power extends over any non-trivial compression body. Additionally, our notion of efficient position is used by Gadre [7] to show that harmonic measures on $\mathcal{PMF}(S)$ for distributions with finite support on $\mathcal{MCG}(S)$ are singular with respect to Lebesgue measure on $\mathcal{PMF}(S)$.

For the proof of Theorem 5.3 we introduce *induced tracks*, *efficient position*, and *wide curves*. For any essential subsurface $X \subset S$ and track $\tau \subset S$ there is an induced track $\tau|X$. Induced tracks generalize the notion of subsurface projection of curves. Efficient position of a curve with respect to a track τ is a simultaneous generalization of curves carried by τ and curves dual to τ (called *hitting τ efficiently* in [20]). Efficient position of ∂X allows us to pin down the location of the induced track $\tau|X$. Wide curves are our combinatorial analogue of curves of definite modulus in a Riemann surface. The structure theorem (5.3) then implies Theorem 5.5: this, together with subsurface projection, controls the motion of a splitting sequence through the complex of curves $\mathcal{C}(X)$.

The quasi-geodesic behaviour of splitting sequences in the train track graph (Theorem 6.2) is a direct consequence of Theorem 6.1. Theorem 6.1 requires a delicate induction, conceptually similar to the hierarchy machine developed in [15]. We do not deduce Theorem 6.1 directly from the results of [15]; in particular it is not known if splitting sequences fellow-travel resolutions of hierarchies.

After this work was submitted we discovered a paper of Takarajima [23], introducing *quasi-transverse* curves. Quasi-transverse and efficient position (see Definition 2.3) are equivalent concepts for simple curves. Takarajima goes on to give an intricate proof that quasi-transverse position exists, relying on a lexicographic ordering of various combinatorial geodesic curvatures. That implies the existence statement of our Theorem 4.1. We here give a completely independent and somewhat simpler proof. Note, however, that Takarajima's existence proof is in principle constructive, while ours is not.

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2. Background

We provide the definitions needed for Theorem 5.3 and its corollaries.

2.1. **Coarse geometry.** Suppose that $Q \geq 1$ is a real number. For real numbers r, s we write $r \leq_Q s$ if $r \leq Qs + Q$ and say that r is *quasi-bounded* by s. We write $r =_Q s$ if $r \leq_Q s$ and $s \leq_Q r$; this is called a *quasi-equality*.

For a metric space $(\mathcal{X}, d_\mathcal{X})$ and finite diameter subsets $A, B \subset \mathcal{X}$ define $d_\mathcal{X}(A, B) = \operatorname{diam}_\mathcal{X}(A \cup B)$. Following Gromov [8], a relation $f\colon \mathcal{X} \to \mathcal{Y}$ of metric spaces is a Q–*quasi-isometric embedding* if for all $x, y \in \mathcal{X}$ we have $d_\mathcal{X}(x, y) =_Q d_\mathcal{Y}(f(x), f(y))$. (Here $f(x) \subset \mathcal{Y}$ is the set of points related to x.) If, additionally, the Q–neighborhood of $f(\mathcal{X})$ equals $\mathcal{Y}$ then f is a Q–*quasi-isometry* and $\mathcal{X}$ and $\mathcal{Y}$ are *quasi-isometric*.

If $[m, n]$ is an interval in $\mathbb{Z}$ and $f\colon [m, n] \to \mathcal{Y}$ is a quasi-isometric embedding then f is a Q–*quasi-geodesic*. Now suppose that $Q > 1$ is a real number, $[m, n]$ and $[p, q]$ are intervals in $\mathbb{Z}$, and $f\colon [m, n] \to \mathcal{Y}$ is a relation. Then f is a Q–*unparameterized quasi-geodesic* if there is a strictly increasing function $\rho\colon [p, q] \to [m, n]$ so that $f \circ \rho$ is a Q–quasi-geodesic and for all $i \in [p, q-1]$ the diameter of $f\big([\rho(i), \rho(i+1)]\big)$ is at most Q.

2.2. **Surfaces, arcs, and curves.** Let $S = S_{g,n}$ be a compact, connected, orientable surface of genus g with n boundary components. The *complexity* of S is $\xi(S) = 3g - 3 + n$. A *curve* in S is an embedding of the circle into S. An *arc* in S is a proper embedding of the interval $[0, 1]$ into S. A curve or arc $\alpha \subset S$ is *trivial* if α separates S and one component of $S \smallsetminus \alpha$ is a disk; otherwise α is *essential*. A curve α is *peripheral* if α separates S and one component of $S \smallsetminus \alpha$ is an annulus; otherwise α is *non-peripheral*. A connected subsurface $X \subset S$ is *essential* if every component of ∂X is essential in S and X is neither a pair of pants ($S_{0,3}$) nor a peripheral annulus (the core curve is peripheral). Note that X inherits an orientation from S. This, in turn, induces an orientation on ∂X so that X is to the left of ∂X.

Define $\mathcal{C}(S)$ to be the set of isotopy classes of essential, non-peripheral curves in S. Define $\mathcal{A}(S)$ to be the set of proper isotopy classes of essential arcs in S. Let $\mathcal{AC}(S) = \mathcal{C}(S) \cup \mathcal{A}(S)$. If $\alpha, \beta \in \mathcal{AC}(S)$ then the *geometric intersection number* [6] of α and β is

$$i(\alpha, \beta) = \min\left\{|a \cap b| : a \in \alpha, b \in \beta\right\}.$$

A finite subset $\Delta \subset \mathcal{AC}(S)$ is a *multicurve* if $i(\alpha, \beta) = 0$ for all $\alpha, \beta \in \Delta$.

If $T \subset S$ is a subsurface with ∂T a union of smooth arcs, meeting perpendicularly at their endpoints, then define

$$\operatorname{index}(T) = \chi(T) - \frac{c^+(T)}{4} + \frac{c^-(T)}{4}$$

where $c^\pm(T)$ is the number of outward (inward) corners of ∂T. Note that index is additive: $\operatorname{index}(T \cup T') = \operatorname{index}(T) + \operatorname{index}(T')$ as long as the interiors of T, T' are disjoint.

2.3. **Train tracks.** For detailed discussion of train tracks see [25, 20, 19]. A *pretrack* $\tau \subset S$ is a properly embedded graph in S with additional structure. The vertices of τ are called *switches*; every switch x is equipped with a tangent $v_x \in T^1_x S$. We require every switch to have valence three; higher valence is dealt with in [20]. The edges of τ are called *branches*. All branches are smoothly embedded in S. All branches incident to a fixed switch x have derivative $\pm v_x$ at x.

An immersion $\rho\colon \mathbb{R} \to S$ is a *train-route* (or simply a *route*) if

- $\rho(\mathbb{R}) \subset \tau$ and
- $\rho(n)$ is a switch if and only if $n \in \mathbb{Z}$.

The restriction $\rho|[0, \infty)$ is a *half-route*. If ρ factors through $\mathbb{R}/m\mathbb{Z}$ then ρ is a *train-loop*. We require, for every branch b, a train-route travelling along b.

For each branch b and point $p \in b$, a component b' of $b \smallsetminus \{p\}$ is a *half-branch*. Two half-branches $b', b'' \subset b$ are equivalent if $b' \cap b''$ is again a half-branch. Every switch divides the three incident half-branches into a pair of *small* half-branches on one side and a single *large* half-branch on the other. A branch b is *large* (*small*) if both of its half-branches are large (small); if b has one large and one small half-branch then b is called *mixed*.

Let $\mathcal{B} = \mathcal{B}(\tau)$ be the set of branches of τ. A function $w\colon \mathcal{B} \to \mathbb{R}_{\geq 0}$ is a *transverse measure* on τ if w satisfies the *switch conditions*: for every switch $x \in \tau$ we have $w(a) + w(b) = w(c)$, where a', b' are the small half-branches and c' is the large half-branch meeting x. Let $P(\tau)$ be the projectivization of the cone of transverse measures; define $V(\tau)$ to be the vertices of the polyhedron $P(\tau)$.

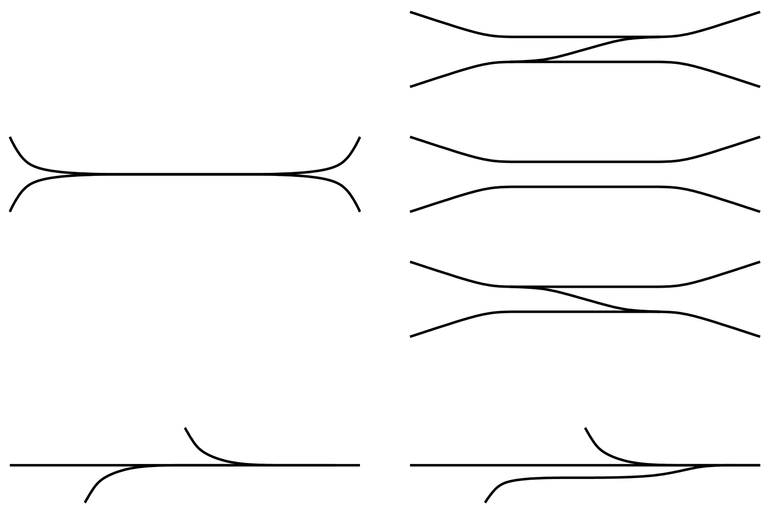

FIGURE 1. Top: A large branch admits a left, central, or right splitting. Bottom: a mixed branch admits a slide.

We may *split* a pretrack along a large branch or *slide* it along a mixed branch; see Figure 1. (Slides are called *shifts* in [20].) The inverse of a split or slide is called a *fold*. Note that the inverse of a slide may be obtained via a slide followed by an isotopy.

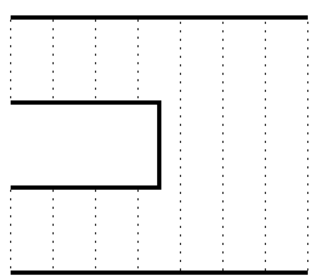

FIGURE 2. The local model for $N(\tau)$ near a switch, with horizontal and vertical boundary in the correct orientation. The dotted lines are ties.

Suppose that $\tau \subset S$ is a pretrack. Let $N = N(\tau) \subset S$ be a *tie neighborhood* of τ: so N is a union of rectangles $\{R_b \mid b \in \mathcal{B}\}$ foliated by vertical intervals (the *ties*). At a switch, the upper and lower thirds of the vertical side of the large rectangle are identified with the vertical side of the small rectangles, as shown in Figure 2. Since N is a union of rectangles it follows that $\operatorname{index}(N) = 0$. The *horizontal boundary*

$\partial_h N$ is the union of $\partial_h R_b$, for $b \in \mathcal{B}$, while the *vertical boundary* is $\partial_v N = \overline{\partial N \smallsetminus \partial_h N}$.

Let $N = N(\tau)$ be a tie neighborhood. Let T be a *complementary region* of τ: a component of the closure of $S \smallsetminus N$. Define the horizontal and vertical boundary of T to be $\partial_h T = \partial T \cap \partial_h N$ and $\partial_v T = \partial T \cap \partial_v N$. Note that all corners of T are outward, so $\operatorname{index}(T) = \chi(T) - \frac{1}{4}|\partial \partial_h T|$.

Suppose $\tau \subset S$ is a pretrack. The subsurface *filled* by τ is the union of N with all complementary regions T of τ that are disks or peripheral annuli.

Definition 2.1. Suppose that $\tau \subset S$ is a pretrack and $N = N(\tau)$. We say that τ is a *train track* if τ is compact, every component of ∂N has at least one corner and every complementary region T of τ has negative index.

Definition 2.2. In a *sliding and splitting sequence* $\{\tau_i\}$ of train tracks each τ_{i+1} is obtained from τ_i by a slide or a split.

2.4. **Carrying, duality, and efficient position.** Suppose that $\tau \subset S$ is a train track. If σ is also a track, contained in $N = N(\tau)$ and transverse to the ties, then we write $\sigma \prec \tau$ and say that σ is *carried* by τ. For example, if τ is a fold of σ then σ is carried by τ.

A properly embedded arc or curve $\beta \subset N$ is *carried* by τ if β is transverse to the ties and $\partial \beta \cap \partial_h N = \emptyset$. Thus if β is carried then $\partial \beta \subset \partial_v N$. Again we write $\beta \prec \tau$ for carried arcs and curves.

Definition 2.3. Suppose that $\alpha \subset S$ is a properly embedded arc or curve. Then α is in *efficient position* with respect to τ, denoted $\alpha \dashv \tau$, if

- every component of $\alpha \cap N$ is a tie or is carried by τ and
- every region $T \subset S \smallsetminus (N \cup \alpha)$ has negative index or is a rectangle.

Suppose that $\alpha \dashv \tau$. If $\alpha \subset N$ then α is carried, $\alpha \prec \tau$. If no component of $\alpha \cap N$ is carried then α is *dual* to τ and we write $\alpha \pitchfork \tau$. If $\Delta \subset \mathcal{AC}(S)$ is a multicurve then we write $\Delta \prec \tau$, $\Delta \pitchfork \tau$, or $\Delta \dashv \tau$ if all elements of Δ are disjointly and simultaneously carried, dual, or in efficient position.

Remark 2.4. Our notion of duality is called *hitting efficiently* by Penner and Harer [20, page 19]. Note that $\alpha \pitchfork \tau$ if and only if α is carried by some extension of the *dual track* τ^*, also defined in [20]. Likewise, if $\alpha \dashv \tau$ and $\alpha \cap N$ consists of carried arcs then α is carried by some extension of τ.

An index argument proves:

Lemma 2.5. *If α is a properly embedded curve or arc in efficient position with respect to a train track $\tau \subset S$ then α is essential and non-peripheral in S.* □

One of the goals of this paper is to prove the converse of Lemma 2.5; this is done in Theorem 4.1. Following Lemma 2.5 we may define $\mathcal{C}(\tau) = \{\alpha \mid \alpha \prec \tau\}$ and $\mathcal{C}^*(\tau) = \{\alpha \mid \alpha \pitchfork \tau\}$. Notice that if $\sigma \prec \tau$ is a track then $\mathcal{C}(\sigma) \subset \mathcal{C}(\tau)$ and $\mathcal{C}^*(\tau) \subset \mathcal{C}^*(\sigma)$.

A branch $b \in \mathcal{B}(\tau)$ is *recurrent* if there is some $\alpha \prec \tau$ that meets R_b. The track τ is *recurrent* if every branch is recurrent. *Transverse recurrence* is defined by replacing carrying by duality [20, page 20]. The track τ is *birecurrent* if τ is recurrent and transversely recurrent [20, Section 1.3]. In a slight departure from Penner and Harer's terminology [20, page 27] we will call a birecurrent track τ *complete* if all complementary regions have index $-1/2$. (When $S = S_{1,1}$ there is, instead, a single complementary region with index -1.)

Lemma 2.6. *Suppose that $\sigma \subset S$ is a birecurrent track. Then $\mathcal{C}^*(\sigma)$ has infinite diameter inside of $\mathcal{C}(S)$.*

Proof. Let τ be a complete track extending σ [20, Corollary 1.4.2]. Section 3.4 of [20] and a dimension count gives a lamination $\lambda \pitchfork \tau$ so that $i(\lambda, \alpha) \neq 0$ for all $\alpha \in \mathcal{C}(S)$. Now an argument of Kobayashi [12], refined by Luo [14, page 124], implies that $\mathcal{C}^*(\tau) \subset \mathcal{C}^*(\sigma)$ has infinite diameter. □

2.5. **Vertex cycles.** When $\alpha \prec \tau$ is a curve there is a transverse measure w_α defined by taking $w_\alpha(b) = |\alpha \cap t|$ where t is any tie of the rectangle R_b. Conversely, for any integral transverse measure w there is a multicurve α_w — take $w(b)$–many horizontal arcs in R_b and glue endpoints as dictated by the switch conditions.

Note that if $v \in V(\tau)$ then there is a minimal integral measure w projecting to v. Since v is an extreme point of $P(\tau)$ deduce that α_w is an embedded curve. We call α_w a *vertex cycle* of τ and henceforth use $V(\tau)$ to denote the set of vertex cycles.

2.6. **Wide curves.** Let $N = N(\tau)$ be a tie neighborhood.

Definition 2.7. A multicurve $\Delta \dashv \tau$ is *wide* if there is an orientation of the components of Δ so that

- for every $b \in \mathcal{B}(\tau)$, all arcs of $\Delta \cap R_b$ are to the right of each other (see the top of Figure 3) and
- for every complementary region T of τ, all arcs of $\Delta \cap T$ are to the right of each other (see the bottom of Figure 3).

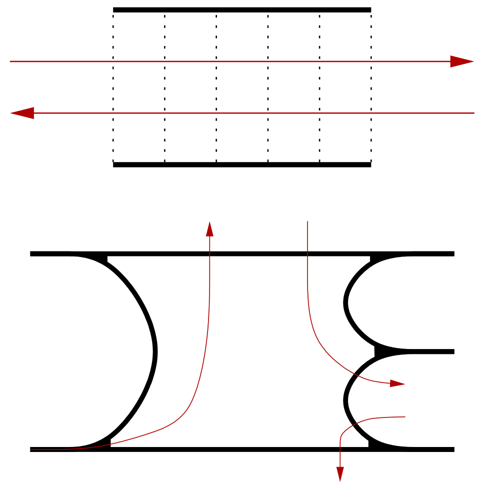

FIGURE 3. Above: Arcs of $\Delta \cap R_b$ to the right of each other. The vertical dotted line are ties; the heavy horizontal lines are arcs of $\partial_h N$. Below: Arcs meeting the complementary region T, all to the right of each other.

It follows from the definition that if $\Delta \dashv \tau$ is wide then for any branch $b \in \mathcal{B}(\tau)$ the intersection $\Delta \cap R_b$ has at most two components.

Lemma 2.8. *Every vertex cycle $\alpha \in V(\tau)$ is wide.*

For an even more precise characterization of vertex cycles see Lemma 3.11.3 of [19]. The proof below recalls a surgery technique used in the sequel.

Proof of Lemma 2.8. We prove the contrapositive. Suppose that α is not wide. Orient α. There are three cases.

Suppose there is a branch $b \subset \tau$ and an oriented tie $t \subset R_b$ where x and y are consecutive (along t) points of $\alpha \cap t$ so that the signs of intersection at x and y are equal. Let $[x, y]$ be the subarc of t bounded by x and y. Surger α along $[x, y]$ to form curves $\beta, \gamma \prec \tau$. See Figure 4. Thus $w_\alpha = w_\beta + w_\gamma$ and α is not a vertex cycle.

Suppose instead that x, y, z are consecutive (along t) points of $\alpha \cap t$ with alternating sign. In this case there is again a surgery along $[x, z]$ producing curves β and γ. See Figure 5 for one of the possible arrangements of α, β and γ. Again $w_\alpha = w_\beta + w_\gamma$ is a non-trivial sum and α is not a vertex cycle.

In the remaining case $w_\alpha(a) \leq 2$ for all $a \in \mathcal{B}$ and there are branches $b, c \in \mathcal{B}$ where the arcs of $\alpha \cap R_b$ are to the right of each other while

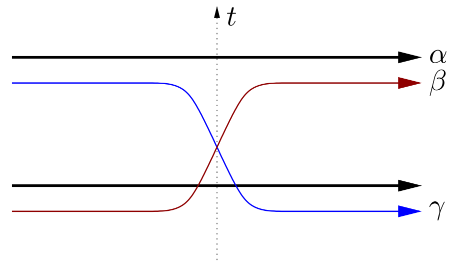

FIGURE 4. Surgery when adjacent intersections have the same sign.

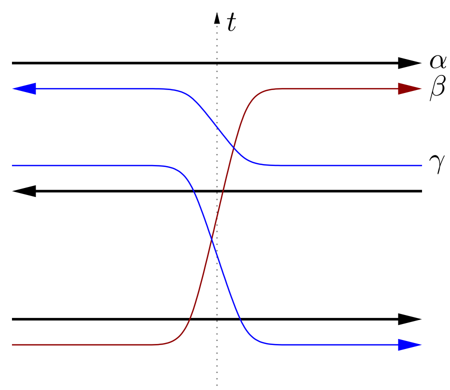

FIGURE 5. Surgery when the three intersections have alternating sign.

the arcs of $\alpha \cap R_c$ are to the left of each other. See Figure 6 for the two ways α may be carried by τ.

If α is carried as in the top line of Figure 6 then surger α in both rectangles R_b and R_c as was done in Figure 4. This shows that w_α is a non-trivial sum and so α is not a vertex cycle. Now suppose that α is carried as in the bottom line of Figure 6. Note that the closure of $\alpha \smallsetminus (R_b \cup R_c)$ is a union of four arcs. Two of these, β' and γ', meet both R_b and R_c. Since S is orientable no tie-preserving isotopy of N throws β' onto γ'. Let $\alpha' \cup \alpha'' = \alpha \smallsetminus (\beta' \cup \gamma')$. Create an embedded curve β by taking two parallel copies of β' and joining them to α' and α''. Similarly create γ by joining two parallel copies of γ' to the arcs α' and α''. It follows that $w_\beta \neq w_\gamma$. Since $2w_\alpha = w_\beta + w_\gamma$ again α is not a vertex cycle. $\square$

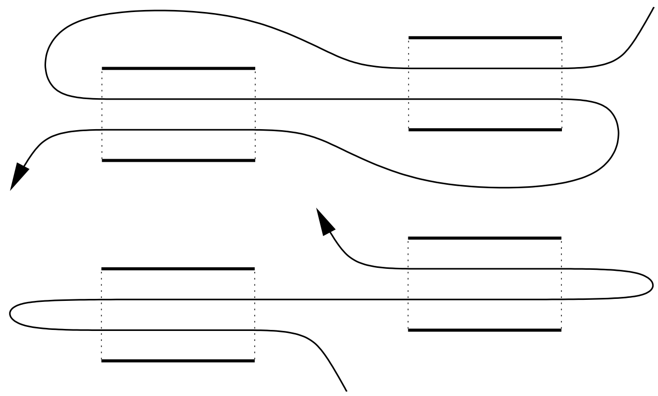

FIGURE 6. The curve α meets R_b and R_c twice.

2.7. **Combing.** Suppose that $\alpha \prec \tau$ has $w_\alpha(b) \leq 1$ for every branch $b \subset \tau$. Orient and transversely orient α to agree with the orientation of the surface S. We think of the orientation as pointing in the x–direction and the transverse orientation pointing in the y–direction. A half-branch $b \subset \tau \smallsetminus \alpha$, sharing a switch with α, *twists to the right* if any train-route through b locally has positive slope. Otherwise b twists to the left. If all branches on one side of α twist to the right then that side of α has a *right combing*, and similarly for a left combing. See Figure 8 for an example where both sides are combed to the left.

2.8. **Curve complexes and subsurface projection.** For more information on the curve complex see [14, 15]. Impose a simplicial structure on $\mathcal{AC}(S)$ where $\Delta \subset \mathcal{AC}(S)$ is a simplex if and only if Δ is a multicurve. The complex of curves $\mathcal{C}(S)$ and the arc complex $\mathcal{A}(S)$ are the subcomplexes spanned by curves and arcs, respectively. Note that if the complexity $\xi(S)$ is at least two then $\mathcal{C}(S)$ is connected [11, Proposition 2]. For surfaces of lower complexity we alter the simplicial structure on $\mathcal{C}(S)$.

Define the Farey tessellation $\mathcal{F}$ to have vertex set $\mathbb{Q} \cup \{\infty\}$. A collection of *slopes* $\Delta \subset \mathcal{F}$ spans a simplex if $ps - rq = \pm 1$ for all $p/q, r/s \in \Delta$. If $S = S_{1,1}$ or $S_{0,4}$ we take $\mathcal{C}(S) = \mathcal{F}$; that is, there is an edge between curves that intersect in exactly one point (for $S_{1,1}$) or two points (for $S_{0,4}$). Note that for surfaces with $\xi(S) \geq 1$ the inclusion of $\mathcal{C}(S)$ into $\mathcal{AC}(S)$ is a quasi-isometry.

Suppose now that $X \cong S_{0,2}$ is an annulus. Define $\mathcal{A}(X)$ to be the set of all essential arcs in X, up to isotopy fixing ∂X pointwise. For

$\alpha, \beta \in \mathcal{A}(X)$ define

$$i(\alpha, \beta) = \min \big\{ |(a \cap b) \smallsetminus \partial X| : a \in \alpha, b \in \beta \big\}.$$

As usual, multicurves give simplices for $\mathcal{A}(X)$.

If α, β are vertices of $\mathcal{C}(S)$, $\mathcal{A}(S)$, or $\mathcal{AC}(S)$ then define $d_S(\alpha, \beta)$ to be the minimal number of edges in a path, in the one-skeleton, connecting α to β; the containing complex will be clear from context. Note that if α β are distinct arcs of $\mathcal{A}(X)$, when X is an annulus, then $d_X(\alpha, \beta) = 1 + i(\alpha, \beta)$ [15, Equation 2.3].

As usual, suppose that $\xi(S) \geq 1$. Fix an essential subsurface $X \subset S$ with $\xi(X) < \xi(S)$. We suppose that X is either a non-peripheral annulus or a surface of complexity at least one. (The case of an essential annulus inside of S_1 is not relevant here.) Following [15], we will define the *subsurface projection* relation $\pi_X \colon \mathcal{AC}(S) \to \mathcal{AC}(X)$. Let S^X be the cover of S corresponding to the inclusion $\pi_1(X) < \pi_1(S)$. The surface S^X is not compact; however, there is a canonical (up to isotopy) homeomorphism between X and the Gromov compactification of S^X. This identifies the arc and curve complexes of X and S^X. Fix $\alpha \in \mathcal{AC}(S)$. Let α^X be the preimage of α in S^X.

If α^X contains a non-peripheral curve in S^X then $\pi_X(\alpha) = \{\alpha\}$. Otherwise, place every essential arc of α^X into the set $\pi_X(\alpha)$. If neither obtains then $\pi_X(\alpha) = \emptyset$. If $\pi_X(\alpha) = \emptyset$ then we say that α *misses* X. If $\pi_X(\alpha) \neq \emptyset$ then α *cuts* X.

Suppose that $\alpha, \beta \in \mathcal{C}(S)$. If $\pi_X(\alpha)$ and $\pi_X(\beta)$ are nonempty define

$$d_X(\alpha, \beta) = \operatorname{diam}_X \big(\pi_X(\alpha) \cup \pi_X(\beta)\big).$$

Likewise define the distance $d_X(A, B)$ between finite sets $A, B \subset \mathcal{C}(S)$. When τ is a track, we use the shorthand $\pi_X(\tau)$ for the set $\pi_X(V(\tau))$. If σ is also a track, we write $d_X(\tau, \sigma)$ for the distance $d_X(\pi_X(\tau), \pi_X(\sigma))$.

We end with a lemma connecting the subsurface projection of carried (or dual) curves to the behavior of wide curves.

Lemma 2.9. *Suppose that $X \subset S$ is an essential surface and τ is a track. If some $\alpha \prec \tau$ ($\alpha \pitchfork \tau$) cuts X then there is a vertex cycle $\beta \prec \tau$ (wide dual $\beta \pitchfork \tau$) cutting X.*

Proof. Some multiple of $\alpha \prec \tau$ is a sum of vertices: $m \cdot w_\alpha = \sum n_i w_i$ where w_i is the integral transverse measure for the vertex $\beta_i \in V(\tau)$. Via a sequence of tie-preserving isotopies of N we may arrange for all of the β_i to realize their geometric intersection with each other. Note that there is an isotopy representative of α contained inside of a small neighborhood of the union $B = \cup \beta_i$.

To prove the contrapositive, suppose that none of the β_i cut X. It follows that X may be isotoped in S to be disjoint from B. Thus α misses X, as desired. A similar discussion applies when $\alpha \pitchfork \tau$. □

3. Induced tracks

Suppose that $\tau \subset S$ is a train track. Suppose that $X \subset S$ is an essential subsurface with $\xi(X) < \xi(S)$. Let S^X be the corresponding cover of S. Let τ^X be the preimage of τ in S^X; note that the pretrack τ^X satisfies all of the axioms of a train track except compactness.

Define $\mathcal{AC}(\tau^X)$ to be the set of essential arcs and essential, non-peripheral curves properly embedded in the Gromov compactification of S^X with interior a train-route or train-loop carried by τ^X. A bit of caution is required here — inessential arcs and peripheral curves may be carried by τ^X but these are not admitted into $\mathcal{AC}(\tau^X)$. Define $\mathcal{A}(\tau^X), \mathcal{C}(\tau^X) \subset \mathcal{AC}(\tau^X)$ to be the subsets of arcs and curves respectively. Define $\mathcal{AC}^*(\tau^X)$ to be the set of dual essential arcs and dual essential, non-peripheral curves, up to isotopies fixing τ^X setwise.

3.1. **Induced tracks for non-annuli.**

If X is not an annulus define $\tau|X$, the *induced track*, to be the union of the branches of τ^X crossed by an element of $\mathcal{C}(\tau^X)$.

Lemma 3.1. *If X is not an annulus then the induced track $\tau|X$ is compact.*

Proof. Note train-routes in τ^X that are mapped properly to S^X are uniform quasi-geodesics in S^X [19, Proposition 3.3.3]. Thus there is a compact core $X' \subset S^X$, homeomorphic to X, so that any route meeting $S^X \setminus X'$ has one endpoint on the Gromov boundary of S^X. It follows that $\tau|X \subset X'$. □

Note that $\tau|X$ may not be a train track: $N = N(\tau|X)$ may have smooth boundary components and complementary regions with non-negative index. However, since all complementary regions of τ^X have negative index it follows that if a complementary region T of $\tau|X$ has non-negative index then T is a peripheral annulus meeting a smooth component of ∂N.

The definition of $\tau|X$ implies that $\tau|X$ is recurrent. Carrying, duality, efficient position and wideness with respect to an induced track are defined as in Section 2.4. Define $\mathcal{C}(\tau|X) \subset \mathcal{C}(X)$, the subset of curves carried by $\tau|X$. Note that $\mathcal{C}(\tau|X) = \mathcal{C}(\tau^X)$. Define $\mathcal{AC}^*(\tau|X) \subset \mathcal{AC}(X)$ to be the subset of arcs and curves dual to $\tau|X$. Note that $\mathcal{AC}^*(\tau|X) \supset \mathcal{AC}^*(\tau^X)$.

Now, $\tau|X$ fails to be transversely recurrent exactly when it carries a peripheral curve. We say that a branch $b \subset \tau|X$ is *transversely recurrent with respect to arcs and curves* if there is $\alpha \in \mathcal{AC}^*(\tau|X)$ meeting b. Then $\tau|X$ is *transversely recurrent with respect to arcs and curves* if every branch b is.

Lemma 3.2. *Suppose that τ is transversely recurrent in S. Then $\tau|X$ is transversely recurrent with respect to arcs and curves in X. Furthermore: suppose that $\tau|X$ is transversely recurrent with respect to arcs and curves in X. If $\sigma \subset \tau|X$ is a train track then σ is transversely recurrent in X.*

Proof. The first claim follows from the definitions. An index argument proves the second claim. □

Here is our second surgery argument.

Lemma 3.3. *Suppose that τ is a track and $X \subset S$ is an essential subsurface, yet not an annulus. For every $\alpha \in \mathcal{A}(\tau^X)$ at least one of the following holds:*

- *There is an arc $\beta \in \mathcal{A}(\tau^X)$ so that β is wide and $i(\alpha, \beta) = 0$.*
- *There is a curve $\gamma \in \mathcal{C}(\tau|X)$ so that $i(\alpha, \gamma) \leq 2$.*

The statement also holds replacing $\mathcal{A}, \mathcal{C}$ by $\mathcal{A}^*, \mathcal{C}^*$.

Proof. The proof is modelled on that of Lemma 2.8. If $\alpha \prec \tau^X$ is wide we are done. If not, as α is a quasi-geodesic [19, Proposition 3.3.3], orient α so that α is wide outside of a compact core for S^X. Now we induct on the total number of arcs of intersection between α and rectangles $R_b \subset N(\tau^X)$ meeting the compact core.

Let t be a tie of R_b. Orient t. Suppose that x, y are consecutive (along t) points of $\alpha \cap t$. Suppose that the sign of intersection at x equals the sign at y. Let $[x, y]$ be the subarc of t bounded by x and y. As in Lemma 2.8 surger α along $[x, y]$ to form an arc β' and a curve γ. See Figure 4 with β' substituted for β.

Note that γ is essential in S^X, by an index argument. If γ is non-peripheral then the second conclusion holds. So suppose that γ is peripheral. Then α is obtained by Dehn twisting β' about γ. So β' is properly isotopic to α and has smaller intersection with R_b; thus we are done by induction.

Suppose instead that x, y, z are consecutive (along t) points of $\alpha \cap t$, with alternating sign. Surger α along $[x, z]$ to form an arc β' and a curve γ. See Figure 5, with β' substituted for β, for one of the possible arrangements of α, β', and γ. Again γ is essential. If γ is non-peripheral then the second conclusion holds and we are done. If γ is peripheral then, as α and β' differ by a half-twist about γ, we find that β' is properly isotopic to α. Since β' has smaller intersection with R_b we are done by induction.

All that remains is the case that α meets every rectangle R_b in at most a pair of arcs of opposite orientation. For every branch b where α meets R_b twice, choose a subarc t_b of a tie in R_b so that $\alpha \cap t_b = \partial t_b$. We call t_b a *chord* for α. For every t_b there is a subarc $\alpha_b \subset \alpha$ so that $\partial t_b = \partial \alpha_b$. A chord t_b is *innermost* if there is no chord t_c with α_c strictly contained in α_b. Let t_b be the first innermost chord. Let α' be the component of $\alpha \smallsetminus \alpha_b$ before α_b. Build a route β by taking two copies of α' and joining them to α_b. Since t_b is the first innermost chord the intersection $\beta \cap R_c$ is a single arc or a pair of arcs depending on whether it is α_b or α' that meets R_c. Thus β is wide. Also, β is essential: otherwise $t_b \cup \alpha_b$ bounds a disk with index one-half, a contradiction. By construction $i(\alpha, \beta) = 0$ and Lemma 3.3 is proved. □

3.2. **Induced tracks for annuli.** Suppose that $X \subset S$ is an annulus. Define $\tau|X$ to be the union of branches $b \subset \tau^X$ so that some element of $\mathcal{A}(\tau^X)$ travels along b. (Note that $\tau|X$, if nonempty, is not compact.) Define $\mathcal{A}(\tau|X) = \mathcal{A}(\tau^X)$ and also the duals $\mathcal{A}^*(\tau|X) \supset \mathcal{A}^*(\tau^X)$.

Define $V(\tau|X)$ in $\mathcal{A}(\tau|X)$ to be the set of wide carried arcs. Define $V^*(\tau|X)$ dually.

Lemma 3.4. *Suppose that $X \subset S$ is an essential annulus. If $\mathcal{A}^{(*)}(\tau|X)$ is nonempty then $V^{(*)}(\tau|X)$ is nonempty. Let $N = N(\tau|X)$. If $\gamma \dashv \tau|X$ is a wide essential arc then γ meets each rectangle of N and each region of $S^X \smallsetminus N$ in at most a single arc.*

For example, if $\gamma \prec \tau|X$ is a wide essential arc then γ embeds into $\tau|X$.

Proof of Lemma 3.4. We prove the second conclusion; the first is similar. Suppose that R is either a rectangle or region so that $\gamma \cap R$ is a pair of arcs to the right of each other. Let δ be an arc properly embedded in $R \smallsetminus \gamma$ so that $\delta \cap \gamma = \partial \delta$. Let γ' be the component of $\gamma \smallsetminus \partial \delta$ so that $\partial \gamma' = \partial \delta$. If $\gamma' \cup \delta$ bounds a disk in S^X then this disk has index one-half and we contradict efficient position. If $\gamma' \cup \delta$ bounds an annulus then γ was not essential, another contradiction. □

Suppose that α is the core curve of the annulus X.

Lemma 3.5. *If α is not carried by $\tau|X$ then $V(\tau|X) = \mathcal{A}(\tau|X)$. If α is not dual to $\tau|X$ then $V^*(\tau|X) = \mathcal{A}^*(\tau|X)$.* □

Lemma 3.6. *Suppose that $\alpha \prec \tau|X$. One side of α is combed if and only if both sides are combed in the same direction if and only if some isotopy representative of α is dual to $\tau|X$.* □

4. Finding efficient position

After discussing the various sources of non-uniqueness we prove in Theorem 4.1 that efficient position exists.

Let $N = N(\tau)$; suppose that $\alpha \dashv \tau$. A rectangle $T \subset S \setminus (N \cup \alpha)$ is *vertical* if ∂T has a pair of opposite sides meeting α and $\partial_v N$ respectively. Define *horizontal* rectangles similarly. Figure 7 depicts the two kinds of *rectangle swap*.

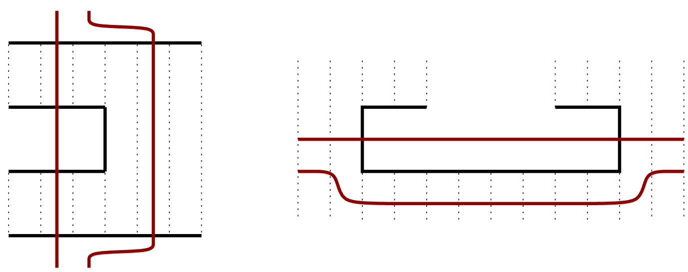

Figure 7. Pieces of N are shown, with vertical and horizontal boundary in the correct orientation; the dotted lines are ties. The left and right pictures show a vertical and horizontal rectangle swap, respectively.

Now suppose that $\alpha \prec \tau$, every rectangle $R_b \subset N$ meets α in at most a single arc, and one side of α is combed. Let A be a small regular neighborhood of α. Then an *annulus swap* interchanges α and the component of ∂A on the combed side. See Figure 8.

Theorem 4.1. *Suppose that $\xi(S) \geq 1$ and $\tau \subset S$ is a birecurrent train track. Suppose that $\Delta \subset \mathcal{AC}(S)$ is a multicurve. Then efficient position for Δ with respect to τ exists and is unique up to rectangle swaps, annulus swaps, and isotopies of S preserving the foliation of $N(\tau)$ by ties.*

Remark 4.2. When $S = S_1$ is a torus, Lemma 14 of [9] proves the existence of efficient position for curves with respect to Reebless bigon tracks. Uniqueness of efficient position follows from a slight generalization of Section 4.1 using *bigon swaps*.

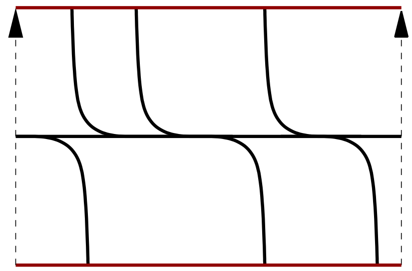

FIGURE 8. Both sides of α are combed (to the left). Thus both boundary components of the annulus shown are dual to τ and both differ from the carried core curve by an annulus swap.

4.1. **Uniqueness of efficient position.** Suppose that α and β are isotopic curves and in efficient position with respect to τ. We induct on $i(\alpha, \beta)$. For the base case suppose that $|\alpha \cap \beta| = 0$. Then α and β cobound an annulus $A \subset S$ so that ∂A has no corners [4, Lemma 2.4]. Since $N = N(\tau)$ is a union of rectangles the intersection $A \cap N$ is also a union of rectangles. Thus $\operatorname{index}(N \cap A) = 0$. By the hypothesis of efficient position any region $T \subset \overline{A \smallsetminus N}$ has non-positive index and has all corners outwards. By the additivity of index it follows that $\operatorname{index}(T) = 0$. It follows that each region T is either an annulus without corners or a rectangle.

Suppose that some region T is an annulus without corners. Then we must have $T = A$. For if ∂T meets ∂N then ∂N has a component without corners, contrary to assumption. Since $T = A$ it follows that α and β are isotopic in the complement of N and we are done.

So we may assume that all regions of $A \smallsetminus N$ are rectangles. (In particular, $A \cap N \neq \emptyset$.) Note that if a region R is a horizontal rectangle then there is no obstruction to doing a rectangle swap across R. After doing all such swaps we may assume that $A \smallsetminus N$ contains no horizontal rectangles.

We now abuse terminology slightly by assuming that the position of N determines that of τ. So if A contains vertical rectangles then there are switches of τ contained in A. This implies that A contains half-branches of τ. Let b' be a half-branch in A, meeting ∂A. If b' is large then there is a vertical rectangle swap removing three half-branches from A. After doing all such swaps we may assume that any such b' is small. If R is a vertical rectangle meeting ∂A then R has two horizontal sides. If neither of these meets a switch on its interior then again there is a swap removing three half-branches from A.

After doing all such swaps if there are still vertical rectangles in A then we proceed as follows: every vertical rectangle must have a horizontal side that properly contains the horizontal side of another vertical rectangle. (For example, in Figure 8 number the rectangles above the core curve R_0, R_1, R_2 from left to right. Note that the left horizontal side of R_i strictly contains the right horizontal side of R_{i-1}.) It follows that the union of these vertical rectangles gives an annulus swap which we perform. Thus, we are reduced to the situation where A contains no horizontal or vertical rectangles.

If $A \subset N$ then α and β are both carried. For any tie $t \subset N$, any component $t' \subset t \cap A$ is an essential arc in A. (To see this, suppose that t' is inessential. Let $B \subset A$ be the bigon cobounded by t' and $\alpha' \subset \alpha$, say. Since α is carried, α' is transverse to the ties. We define a continuous involution on α'; for every tie s and for every component $s' \subset s \cap B$ transpose the endpoints of s'. As this involution is fixed point free, we have reached a contradiction.) It follows that A is foliated by subarcs of ties and we are done.

There is one remaining possibility in the base case of our induction: $A \cap N \neq \emptyset$, $A \not\subset N$, and A contains no switches of τ. Thus every region of $A \cap N$ and of $A \setminus N$ is a rectangle meeting both α and β. Any region R of $A \cap N$ is foliated by (subarcs of) ties and, as above, all ties meet R essentially. Thus R gives a parallelism between (carried arcs) ties of α and β. It follows that A gives an isotopy between α and β, sending ties to ties. This completes the proof of uniqueness when $|\alpha \cap \beta| = 0$.

For the induction step assume $|\alpha \cap \beta| > 0$. Since α is isotopic to β the Bigon Criterion [4, Lemma 2.5], [5, Proposition 1.3] implies that there is a disk $B \subset S$ with exactly two outward corners x and y so that $B \cap (\alpha \cup \beta) = \partial B$. Suppose that x is a *dual intersection*: an intersection of a tie of α and a carried arc of β. See Figure 9.

Let $\alpha' = \alpha \cap B$ and $\beta' = \beta \cap B$. Orient β' away from x. Let $z \in \alpha'$ be immediately adjacent to x. Without loss of generality we may assume that z is to the left of β', near x. Let ρ be the half-route starting at z, initially agreeing with β, and turning left at every switch. If $\rho \subset B$ then eventually ρ repeats a branch b in the same direction; it follows that there is a curve $\gamma \prec \tau$ contained in B contradicting Lemma 2.5. However, if ρ exits B through α' (β') then we contradict efficient position of α (β).

It follows that the corner x either lies in $S \setminus N$ or is the intersection of carried arcs of α and β. The same holds for y. In either case we remove from B a small neighborhood of x and of y: when the corner

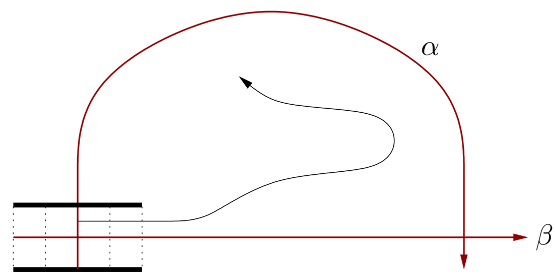

FIGURE 9. The left corner is a dual intersection, between a tie of α and a carried arc of β. If the half-route ρ exits through α or β then a bigon or non-trivial trigon is created.

lies in N we use a subarc of a tie to do the cutting. The result B' is a rectangle with the components of $\partial_h B$ contained in α and β respectively. As $\text{index}(B') = 0$ the argument given in the case of an annulus gives a sequence of rectangle swaps moving α across B. This reduces $|\alpha \cap \beta|$ by two and so completes the induction step.

The proof when α and β are arcs follows the above but omitting any mention of annulus swaps.

Finally, suppose that Δ, Γ are isotopic multicurves, both in efficient position. We may isotope Γ to Δ, as above, being careful to always use innermost bigons. This completes the proof that efficient position is unique.

We end this subsection with a useful corollary:

Corollary 4.3. *Suppose that $\Gamma \subset \mathcal{AC}(S)$ is a finite collection of arcs and curves in efficient position. Then we may perform a sequence of rectangle swaps to realize the pairwise geometric intersection numbers.*

Proof. Let $\Gamma = \{\gamma_i\}_{i=1}^k$. By induction, the curves of $\Gamma' = \Gamma \setminus \{\gamma_k\}$ realize their pairwise geometric intersection numbers. If γ_k meets some $\gamma_i \in \Gamma'$ non-minimally then by the Bigon Criterion [6, page 46] there is an innermost bigon between γ_k and some $\gamma_j \in \Gamma'$. We now may reduce the intersection number following the proof of uniqueness of efficient position. □

4.2. **Existence of efficient position.** Our hypotheses are weaker, and thus our discussion is more detailed, but the heart of the matter is inspired by [14, pages 122-123].

We may assume that τ fills S; for if not we replace S by the subsurface filled by τ. Since τ is transversely recurrent for any $\epsilon, L > 0$ there is a finite area hyperbolic metric on the interior of S and an isotopy of τ so that: every branch of τ has length at least L and every train-route $\rho \prec \tau$ has geodesic curvature less than ϵ at every point [20, Theorem 1.4.3].

Let $\tau^{\mathbb{H}}$ be the lift of τ to $\mathbb{H} = \mathbb{H}^2$, the universal cover of S. Every train-route $\rho \prec \tau^{\mathbb{H}}$ cuts $\mathbb{H}$ into a pair $H^{\pm}(\rho)$ of open *half-planes*. Fix a route $\rho \prec \tau^{\mathbb{H}}$ and a half-branch $b' \subset \tau^{\mathbb{H}}$ so that there is some $n \in \mathbb{Z}$ with $b' \cap \rho = \rho(n)$. We say the branch b is *rising* or *falling* with respect to ρ as the large half-branch at the switch $\rho(n)$ is contained in $\rho|[n, \infty)$ or contained in $\rho|(-\infty, n]$.

Claim 4.4. For any route $\rho \prec \tau^{\mathbb{H}}$ one side of ρ has infinitely many rising branches while the other side has infinitely many falling branches.

Proof. Note that there are infinitely many half-branches on both sides of ρ: if not then $\partial_h N(\tau)$ would have a component without corners, contrary to assumption. Suppose that there are only finitely many rising branches along ρ. Then there is a curve $\gamma \prec \tau$ so that $w_\gamma(b) \leq 1$ for every branch b and so that the two sides of γ are combed in opposite directions. Thus τ is not recurrent, a contradiction. The same contradiction is obtained if there are only finitely many falling branches along ρ. $\square$

Claim 4.5. For any route $\rho \prec \tau^{\mathbb{H}}$ and for any family of half-routes $\{\beta_n\}$ if $\beta_n \cap \rho = \rho(n)$ then $\lim_{n\to\infty} \beta_n(\infty) = \rho(\infty)$.

Proof. Let $x = \rho(\infty) \in \partial_\infty\mathbb{H}$. Consider the subsequence $\{\beta_n\}$ where the first branch of each β_n is falling. Let $P_n = \rho|(-\infty, n] \cup \beta_n$, oriented away from $\rho(-\infty)$. Note that $P_n(\infty) = \beta_n(\infty)$. Recall that ρ and P_n are both uniformly close to geodesics [20, pages 61–62]. Thus $P_n(\infty) \to x$ as $n \to \infty$.

Now consider the subsequence $\{\beta_n\}$ where the first branch of each β_n is rising. Let $P_n = \beta_n \cup \rho|[n, \infty)$ oriented towards x; so $P_n(\infty) = x$ for all n. Note that $P(-\infty) = \beta_n(\infty)$. Since all complementary regions of $\tau^{\mathbb{H}}$ have negative index none of the P_n may cross each other. It follows that either the P_n exit compact subsets of $\mathbb{H}$, and we are done, or the P_n converge [20, Theorem 1.5.4] to P, a train-route with $P(\infty) = x$. Since P does not cross any P_n deduce that P and ρ are disjoint. But this contradicts [19, Corollary 3.3.4]: train-routes that share an endpoint must share a half-route. $\square$

Given distinct points $x, y, z \in S^1 = \partial_\infty\mathbb{H}$, arranged counterclockwise, let (y, z) be the component of $S^1 \setminus \{y, z\}$ that does not contain x. Let $[y, z]$ be the closure of (y, z). Thus $x \in (z, y)$, $(y, z) \cap [z, y] = \emptyset$, and $(y, z) \cup [z, y] = S^1$.

Claim 4.6. For any distinct $z, y \in S^1$ there is a train-route ρ so that one of the intervals $\partial_\infty H^\pm(\rho)$ is contained in (z, y).

Proof. The endpoints of train-routes are dense in $S^1 = \partial_\infty \mathbb{H}$. Fix $x \in (z, y)$ so that x is the endpoint of a train-route γ. Since there are infinitely many rising branches along γ (Claim 4.4) the claim follows from the rising case of Claim 4.5. □

Let $H_{x,y} \subset \mathbb{H}$ be the convex hull of $(x, y) \subset \partial_\infty \mathbb{H}$. Let $\mathcal{H}_{x,y}$ be the union of all open half-planes $H(\rho)$ so that $\partial_\infty H(\rho) \subset (x, y)$. Since train-routes have geodesic curvature less than ϵ at every point:

Claim 4.7. The union $\mathcal{H}_{x,y}$ is contained in an δ–neighborhood of $H_{x,y}$, where δ may be taken as small as desired by choosing appropriate ϵ, L. □

A set $X \subset \mathbb{H}$ is ϵ'*–convex* if every pair of points in X can be connected by a path in X which has geodesic curvature less than ϵ' at every point.

Claim 4.8. $\mathbb{H} \setminus \mathcal{H}_{x,y}$ is closed and ϵ'–convex, where ϵ' may be taken as small as desired by choosing appropriate ϵ, L.

Proof. This is proved in detail on pages 122-123 of [14]. □

Claim 4.9. The point x is an accumulation point of $\partial(\mathbb{H} \setminus \mathcal{H}_{x,y})$.

Proof. Pick a sequence of subintervals $(x_n, y_n) \subset (x, y)$ so that $x_n, y_n \to x$ as $n \to \infty$. By Claim 4.6 for every n there is a route ρ_n and a half-plane $H_n = H(\rho_n)$ so that $\partial_\infty H_n \subset (x_n, y_n)$. It follows that $H_n \subset \mathcal{H}_{x,y}$. Let r_n be any bi-infinite geodesic perpendicular to $\partial H_{x,y}$ and meeting H_n. Thus $r_n \to x$ as $n \to \infty$. By Claim 4.7 the intersection $r_n \cap \partial(\mathbb{H} \setminus \mathcal{H}_{x,y})$ is nonempty, and we are done. □

The next lemma is not needed for the proof of Theorem 4.1: we state it and give the proof in order to introduce necessary techniques and terminology.

Lemma 4.10. *For any non-parabolic point $x \subset S^1$ there is a sequence of train-routes $\{\rho_n\}$ with associated half-planes $\{H(\rho_n)\}$ forming a neighborhood basis for x.*

Proof. Let y, z be arbitrary points of S^1 so that x, y, z are ordered counterclockwise. It suffices to construct a train-route separating x from (y, z).

First assume that x is the endpoint of a route ρ. Claim 4.4 implies that there are infinitely many rising branches $\{a_m\}$ on one side of ρ and infinitely many falling branches $\{c_n\}$ on the other side. Run half-routes α_m and γ_n through a_m and c_n; so each half-route meets ρ in a single switch. By Claim 4.5 the endpoints converge: $\alpha_m(\infty), \gamma_n(\infty) \to x$. Thus sufficiently large m, n give a train-route

$$\alpha_m \cup \rho|_{[m,n]} \cup \gamma_n$$

that separates x from (y, z), as desired.

For the general case consider

$$\mathcal{K} = \mathbb{H} \smallsetminus (\mathcal{H}_{z,x} \cup \mathcal{H}_{x,y}).$$

Note that x is an accumulation point of $\mathcal{K}$ (by Claim 4.9 and because $\mathcal{H}_{z,x}$ cannot contain points of $\partial(\mathbb{H} \smallsetminus \mathcal{H}_{x,y})$). Fix any basepoint $w \in \mathcal{K}$. By Claim 4.8 $\mathcal{K}$ is ϵ'–convex. Thus there is a path $r \subset \mathcal{K}$ from w to x which has geodesic curvature less than ϵ' at every point. Since x is not a parabolic point the projection of r to S recurs to the thick part of S; thus r meets infinitely many branches $\{b_n\}$ of $\tau^{\mathbb{H}}$.

Suppose that b is a branch of $\tau^{\mathbb{H}}$ lying in $\mathcal{K}$. If the two sides of b meet $\mathcal{H}_{z,x}$ and $\mathcal{H}_{x,y}$ then b is a *bridge* of $\mathcal{K}$. If the sides of b meet neither $\mathcal{H}_{z,x}$ nor $\mathcal{H}_{x,y}$ then b is an *interior branch* of $\mathcal{K}$. If exactly one side of b lies in $\mathcal{K}$ then b is a *boundary branch*. If both sides lie in $\mathcal{H}_{z,x}$ (or both sides lie in $\mathcal{H}_{x,y}$) then b is an *exterior branch*. See Figure 10 for an illustration of the case $z = y$.

By convexity the path r is disjoint from the exterior branches. After a small isotopy the path r is also disjoint from the boundary branches, meets the interior branches transversely, and still has geodesic curvature less than ϵ' at every point.

Now, if r travels along a bridge then there are routes $\rho^{\pm}$ cutting off half-planes $H^{\pm}$ lying in $\mathcal{H}_{z,x}$ and $\mathcal{H}_{x,y}$ respectively. Then either x is the endpoint of a train-route or a cut and paste of $\rho^{\pm}$ gives the desired route Γ separating x from (y, z). In either case we are done.

So suppose that r only meets interior branches $\{b_n\}$ of $\mathcal{K}$. Let γ_n be any train-route travelling along b_n. If any of the γ_n land at x we are done, as above. Supposing not: Fixing orientations and passing to a subsequence we may assume that $\gamma_n(\infty) \to x$ as $n \to \infty$. There are now two cases: suppose that for infinitely many n we find that $\gamma_n(-\infty) \in [y, z]$. Then passing to a further subsequence we have that $\gamma_n \to \Gamma$ where $\Gamma(\infty) = x$ [20, Theorem 1.5.4]; thus x is the endpoint of a train-route and we are done as above. The other possibility is that for some sufficiently large n both endpoints $\gamma_n(\pm\infty)$ lie in (z, y). Since b_n is an interior branch γ_n separates x from (y, z) and Lemma 4.10 is proved. □

4.3. **Finding invariant efficient position.** Fix $\alpha \in \mathcal{C}(S)$. (The case where $\alpha \in \mathcal{A}(S)$ is dealt with at the end.) Let α' be a component of the lift of α to the universal cover $\mathbb{H}$. Let $\pi_1(\alpha)$ be the cyclic subgroup (of the deck group) preserving α'. Let $\{x, y\} = \partial_\infty \alpha' \subset S^1$. We take

$$\mathcal{K} = \mathbb{H} \smallsetminus (\mathcal{H}_{y,x} \cup \mathcal{H}_{x,y}).$$

By construction $\mathcal{K}$ is $\pi_1(\alpha)$–invariant. By Claims 4.8 and 4.9 the set $\mathcal{K}$ is closed, ϵ'–convex, and has $\{x, y\} \subset \partial_\infty \mathcal{K}$. By Lemma 4.10 the only non-parabolic points of $\partial_\infty \mathcal{K}$ are x and y. As in the proof of Lemma 4.10 we find a bi-infinite path $r \subset \mathcal{K}$ connecting y to x, with geodesic curvature less than ϵ' at every point. See Figure 10.

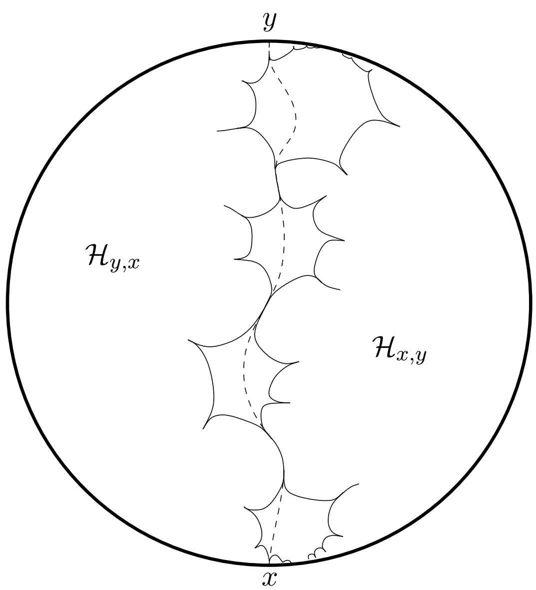

FIGURE 10. The path r runs from y to x. To simplify the figure, no interior branches are shown.

Let $H(r)$ be the open half-plane to the right of r. If we remove the union

$$\bigcup_{g \in \pi_1(\alpha)} g \cdot H(r)$$

from $\mathbb{H}$ then, as with Claim 4.8, what remains is closed and ϵ''–convex for some small ϵ''. It follows that we may homotope the path r to become a $\pi_1(\alpha)$–invariant smooth path, contained in $\mathcal{K}$ and transverse to the interior branches, and avoiding the exterior branches of $\mathcal{K}$. A further equivariant isotopy ensures that r also avoids the boundary branches of $\mathcal{K}$. Orient r from y to x.

Remark 4.11. Suppose that $\gamma \prec \tau^{\mathbb{H}}$ is a train-route that separates x from y. Note that if there exists a non-identity element $g \in \pi_1(\alpha)$ so that γ and $g \cdot \gamma$ meet then r is carried by $\tau^{\mathbb{H}}$, thus $\alpha \prec \tau$, and we are done. We will henceforth assume that train-routes separating x from y are disjoint from their non-trivial translates.

Let b be any interior branch of $\mathcal{K}$ and let γ be any train-route travelling along b. Since b is interior, γ must separate x from y. Orient γ from $\mathcal{H}_{y,x}$ to $\mathcal{H}_{x,y}$. (Thus if γ and r meet once then the tangent vectors to r and γ, in that order, form a positive frame.) The orientation of γ gives an orientation to b. Moreover, as b is an interior branch a cut and paste argument shows that the orientation on b is independent of our choice of γ. Orient all interior branches in this fashion and note that these orientations agree at interior switches.

We say that $p \in r \cap b$ has *positive* or *negative sign* as the tangent vectors to r and b (in that order) form a positive or negative frame. Suppose that there are $\mathsf{N} \in \mathbb{N}$ orbits of points of negative sign, under the action of $\pi_1(\alpha)$. We now induct on N.

Suppose that N is zero. Any bigon between r and a train-route is contained in $\mathcal{K}$ and so contributes one point of positive and one point of negative sign. So if there are no points of negative sign then there are no bigons and r is in efficient position with respect to $\tau^{\mathbb{H}}$. Recall that r is $\pi_1(\alpha)$–invariant. So $\beta \subset S$, the image of r under the universal covering map, is an immersed curve in S homotopic to α. If β is embedded then we are done. If not then the Bigon Criterion for immersed curves [24] implies that β must have either a monogon or a bigon of self-intersection. If β has a monogon B of self-intersection then, since index is additive, τ must be disjoint from B. Thus we can homotope β to remove B while fixing τ pointwise. If β has a bigon B of self-intersection then, as in the proof of uniqueness in Section 4.1, we may remove B via a sequence of rectangle swaps. After removing all monogons and bigons of self-intersection the curve β is embedded and in efficient position.

Suppose that N is positive. Let b be a branch with a point $p \in r \cap b$ of negative sign. Let γ_R (γ_L) be the half-route starting at p, travelling in the direction of b, and thereafter turning only right (left). Each of γ_R and γ_L must have at least one bigon with r, as their points at infinity lie in (x, y). There are now two (essentially identical) cases:

- There is a bigon B between r and γ_R, to the right of r, so that the corners of B appear in the same order along r and γ_R.
- There is a bigon B between r and γ_L, to the right of r, so that the corners of B appear in opposite order along r and γ_L.

See Figure 11. If neither case holds then any half-route ending at p must originate in (x, y), contradicting the fact that p has negative sign. Note that, by Remark 4.11, γ_R (γ_L) is disjoint from its non-trivial $\pi_1(\alpha)$ translates. It follows that the bigon B is also disjoint from its non-trivial

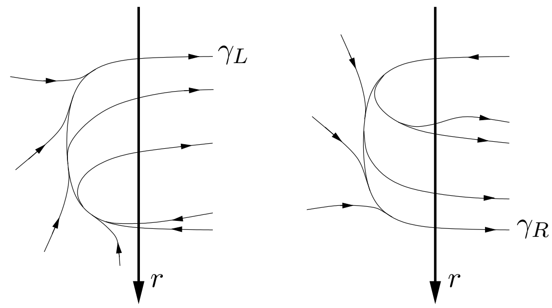

FIGURE 11. Left: The lowest point shown has negative sign. The paths r and γ_L form a bigon. Right: The corresponding figure for γ_R.

translates. Finally, we may equivariantly isotope r across $\pi_1(\alpha) \cdot B$. Since the arc of $\gamma_R \cap \partial B$ (respectively $\gamma_L \cap \partial B$) is combed outside of B this isotopy reduces N by at least one. (Again, see Figure 11.) This completes the proof of Theorem 4.1 when α is a curve.

4.4. **Efficient position for arcs and multicurves.** Now suppose that $\alpha \subset S$ is an essential arc. Let α' be a lift of α to $\mathbb{H}$, the universal cover of S. Note that $\{x, y\} = \partial_\infty \alpha'$ is a pair of parabolic points. Construct $\mathcal{K} = \mathbb{H} \smallsetminus (\mathcal{H}_{y,x} \cup \mathcal{H}_{x,y})$ as before. The proof now proceeds as above, omitting any mention of $\pi_1(\alpha)$, equivariance, or annulus swaps.

Finally suppose that Δ is a multicurve. As shown in Section 4.2 we may isotope, individually, every $\alpha \in \Delta$ into efficient position. By Corollary 4.3 all $\alpha \in \Delta$ may be realized disjointly in efficient position. This completes the proof of Theorem 4.1. □

5. THE STRUCTURE THEOREM

5.1. **Bounding diameter.** We now bound the diameters of the sets of wide arcs and curves carried by the induced track.

Lemma 5.1. *Suppose that τ is a birecurrent track and $X \subset S$ is an essential annulus. If $\tau|X \neq \emptyset$ then the diameter of $V(\tau|X) \cup V^*(\tau|X)$ inside of $\mathcal{A}(X)$ is at most eight.*

Proof. In the proof we use V, V^* to represent $V(\tau|X)$ and $V^*(\tau|X)$. Since $\tau|X \neq \emptyset$ it follows that $\mathcal{A}(\tau|X)$ is nonempty. The first conclusion of Lemma 3.4 now implies that V is nonempty.

Claim. $V^* \neq \emptyset$.

Proof. By Lemma 2.6 there is a dual curve $\beta \in \mathcal{C}^*(\tau)$ so that $i(\alpha, \beta) > 0$. Thus there is a lift $\beta' \subset S^X$ with closure an essential arc. Since $\tau|X \subset \tau^X$ it follows that $\beta' \in \mathcal{A}^*(\tau|X)$. The first conclusion of Lemma 3.4 now implies that V^* is nonempty. □

Claim. If $\beta \in V$ and $\gamma \in V^*$ then $i(\beta, \gamma) \leq 3$.

Proof. Suppose that $i(\beta, \gamma) = n \geq 4$. Let $\{\gamma_i\}_{i=1}^{n-1}$ be the components of $\gamma \setminus \beta$ with both endpoints on β. Let R_i be the components of $S^X \setminus (\beta \cup \gamma)$ with compact closure. We arrange matters so that opposite sides of R_i are on γ_i and γ_{i+1}. Let R be the union of the R_i. Since $\mathrm{index}(R) = 0$ every region T of the closure of $R \setminus N(\tau|X)$ also has index zero and so is a rectangle. If T meets both γ_i and γ_{i+1} then γ was not wide, a contradiction. As $n - 1 \geq 3$ any region T meeting γ_2 is a compact rectangle component of the closure of $S^X \setminus N(\tau|X)$. An index argument implies that τ^X and thus $\tau \subset S$ has a complementary region with non-negative index, a contradiction. □

Since V, V^* are nonempty it follows that $\mathrm{diam}(V \cup V^*) \leq 8$. □

Now suppose that X is not an annulus. Prompted by Lemma 2.8 we define

$$W(\tau^X) = \left\{\alpha \in \mathcal{AC}\big(\tau^X\big) \mid \alpha \text{ is wide}\right\}.$$

Define $W^*(\tau^X)$ similarly, replacing $\mathcal{AC}(\tau^X)$ by $\mathcal{AC}^*(\tau^X)$.

Lemma 5.2. *There is a constant $\mathsf{K}_1 = \mathsf{K}_1(S)$ with the following property: Suppose that τ is a track and $X \subset S$ is an essential subsurface (not an annulus) with $\pi_X(\tau) \neq \emptyset$. Then the diameter of $W(\tau^X) \cup W^*(\tau^X)$ inside of $\mathcal{AC}(X)$ is at most K_1. Furthermore if ∂X, after isotopy into efficient position and with the induced orientation, is not wide then either $\mathcal{C}(\tau|X)$ or $\mathcal{C}^*(\tau|X)$ has diameter at most two in $\mathcal{C}(X)$.*

Proof. In the proof we use $W, W^*, \mathcal{AC}, \mathcal{AC}^*$ to represent $W(\tau^X)$ and so on. Since $\pi_X(\tau) \neq \emptyset$ there is some vertex cycle $\alpha \in V(\tau)$ so that α cuts X. Since α is wide (Lemma 2.8) there is a lift $\alpha' \prec \tau^X$ which is also wide; deduce that W is nonempty.

Claim. $W^* \neq \emptyset$.

Proof. By Lemma 2.6 there is a dual curve $\alpha \in \mathcal{C}^*(\tau)$ cutting X. By Lemma 2.9 there is a wide dual β that also cuts X. Thus there is a lift $\beta' \subset S^X$ with closure an essential wide arc or wide essential, non-peripheral curve. So $\beta' \in W^*$ as desired. □

Now isotope ∂X into efficient position. Let X' be the compact component of the preimage of X under the covering map $S^X \to S$.

Note that $\partial X'$ is in efficient position with respect to τ^X. Note that the covering map $S^X \to S$ induces a homeomorphism between X' and X. Let $N^X = N(\tau^X) \subset S^X$ be the preimage of $N = N(\tau)$. Let $N' = X' \cap N^X$. Again, the covering map induces a homeomorphism between N' and $N \cap X$.

Suppose that ∂X, with its induced orientation, is not wide. If ∂X fails to be wide in $S \smallsetminus N(\tau)$ then there is a properly embedded, essential arc $\gamma \subset X$ disjoint from $N(\tau)$. Lift γ to $\gamma' \subset X'$. Adjoin to γ' geodesic rays in $S^X \smallsetminus X'$ to obtain an essential, properly embedded arc $\gamma'' \subset S^X$. Note that $i(\gamma'', \alpha) = 0$ for every $\alpha \in \mathcal{AC}$; only intersections in X' contribute to geometric intersection number as computed in S^X. This implies that $\operatorname{diam}_X(\mathcal{C}(\tau|X)) \leq 2$. Furthermore, $i(\gamma'', \beta) \leq 2$ for every $\beta \in W^*$. This gives the desired diameter bound for $W \cup W^*$.

If, instead, ∂X fails to be wide in $N(\tau)$ then there is a properly embedded, essential arc $\gamma \subset X$ that is a subarc of a tie. Again, lift and extend to an essential arc $\gamma'' \subset S^X$ so that $i(\gamma'', \beta) = 0$ for any $\beta \in \mathcal{AC}^*$. This implies that $\operatorname{diam}_X(\mathcal{C}^*(\tau|X)) \leq 2$. We also have $i(\gamma'', \alpha) \leq 2$ for any $\alpha \in W$. Again the diameter is bounded.

Now suppose that ∂X is wide. Thus, for every $b \in \mathcal{B}(\tau)$, the rectangle R_b meets ∂X in at most a pair of arcs. It follows that $N \cap X$, and thus N', is a union of at most $2|\mathcal{B}(\tau)|$ subrectangles of the form $R', R'' \subset R_b$. Suppose that $\alpha \in W$ and $\beta \in W^*$. Then α and β each meet a subrectangle R' in at most two arcs. Thus α and β intersect in at most four points inside of R'. Thus $i(\alpha, \beta) \leq 8|\mathcal{B}(\tau)|$. Since $|\mathcal{B}(\tau)| \leq 6 \cdot \xi(S) = 18g - 18 + 6n$, Lemma 5.2 is proved. □

5.2. **Accessible intervals.** Suppose that $\{\tau_i\}_{i=0}^N$ is a sliding and splitting sequence of birecurrent train tracks. Suppose $X \subset S$ is an essential subsurface, yet not an annulus, with $\xi(X) < \xi(S)$. Define

$$m_X = \min\left\{i \in [0, N] \mid \operatorname{diam}_X(\mathcal{C}^*(\tau_i|X)) \geq 3\right\}$$

and

$$n_X = \max\left\{i \in [0, N] \mid \operatorname{diam}_X(\mathcal{C}(\tau_i|X)) \geq 3\right\}.$$

If either m_X or n_X is undefined or if $n_X < m_X$ then I_X, the *accessible interval* is empty. Otherwise, $I_X = [m_X, n_X]$.

If $X \subset S$ is an annulus, then I_X is defined by replacing $\mathcal{C}$ by $\mathcal{A}$ and increasing the lower bound on diameter from 3 to 9. We may now state the structure theorem:

Theorem 5.3. *For any surface S with $\xi(S) \geq 1$ there is a constant $\mathsf{K}_0 = \mathsf{K}_0(S)$ with the following property: Suppose that $\{\tau_i\}_{i=0}^N$ is a sliding and splitting sequence of birecurrent train tracks in S and suppose that $X \subset S$ is an essential subsurface.*

- *For every $[a,b] \subset [0,N]$ if $[a,b] \cap I_X = \emptyset$ and $\pi_X(\tau_b) \neq \emptyset$ then $d_X(\tau_a, \tau_b) \leq \mathsf{K}_0$.*

Suppose $i \in I_X$. If X is an annulus:

- *The core curve α is carried by and wide in τ_i.*
- *Both sides of α are combed in the induced track $\tau_i|X$.*
- *If $i+1 \in I_X$ then $\tau_{i+1}|X$ is obtained by taking subtracks, slides, or at most a pair of splittings of $\tau_i|X$.*

If X is not an annulus:

- *When in efficient position ∂X is wide with respect to τ_i.*
- *The track $\tau_i|X$ is birecurrent and fills X.*
- *If $i+1 \in I_X$ then $\tau_{i+1}|X$ is either a subtrack, a slide, or a split of $\tau_i|X$.*

Proof. Fix an interval $[a,b] \subset [0,N]$. Note that $\tau_b \prec \tau_a$ and so $\tau_b^X \prec \tau_a^X$. Thus $\mathcal{AC}(\tau_b^X) \subset \mathcal{AC}(\tau_a^X)$ while $\mathcal{AC}^*(\tau_a^X) \subset \mathcal{AC}^*(\tau_b^X)$.

Claim. If $[a,b] \cap I_X = \emptyset$ and $\pi_X(\tau_b) \neq \emptyset$ then $d_X(\tau_a, \tau_b) \leq \mathsf{K}_0$.

Proof. Fix, for the duration of the claim, a vertex cycle $\beta \in V(\tau_b)$ so that β cuts X. Since β is also carried by τ_a there is, by Lemma 2.9, a vertex cycle $\alpha \in V(\tau_a)$ cutting X. Pick $\alpha' \in \pi_X(\alpha)$ and $\beta' \in \pi_X(\beta)$. Note Lemma 2.8 implies that α' is wide in τ_a^X while β' is wide in τ_b^X. The proof divides into cases depending on the relative positions of a, b, m_X and n_X.

Case I. Suppose $n_X < a$ or n_X is undefined.

Note that $\beta' \prec \tau_a^X$. If X is an annulus then since $a \notin I_X$ the diameter of $\mathcal{A}(\tau_a|X)$ is at most eight; thus $d_X(\alpha, \beta) \leq 8$ and we are done.

Suppose that X is not an annulus. If β' is an arc then Lemma 3.3 gives two cases: we may replace β' by γ which is either a wide arc in τ_a^X or is an essential non-peripheral curve in $\tau_a|X$. (If β' is a curve then let $\gamma = \beta'$.) In either case Lemma 3.3 ensures that $i(\gamma, \beta') \leq 2$ and so $d_X(\gamma, \beta') \leq 4$. If γ is an arc then both α' and γ are wide so Lemma 5.2 gives $d_X(\alpha, \beta) \leq \mathsf{K}_1 + 4$. If γ is a curve pick any $\delta \in V(\tau_a|X)$. Then Lemma 5.2 implies that $d_X(\alpha', \delta) \leq \mathsf{K}_1$. Also, $a \notin I_X$ implies that $d_X(\delta, \gamma) \leq 2$. Thus $d_X(\alpha, \beta) \leq \mathsf{K}_1 + 6$.

Case II. Suppose $b < m_X$ or m_X is undefined.

If X is an annulus, then Lemma 5.1 gives wide duals $\alpha^* \in V^*(\tau_a|X)$ and $\beta^* \in V^*(\tau_b|X)$ so that $d_X(\alpha', \alpha^*), d_X(\beta', \beta^*) \leq 8$. It follows that the arc α^* also lies in $\mathcal{A}^*(\tau_b|X)$. Since $b \notin I_X$ we have $d_X(\alpha^*, \beta^*) \leq 8$. Thus $d_X(\alpha, \beta) \leq 24$, as desired.

If X is not an annulus, then by Lemma 5.2 there is a wide dual $\alpha^* \in W^*(\tau_a^X)$ so that $d_X(\alpha', \alpha^*) \leq \mathsf{K}_1$. Again, α^* is also an element of $\mathcal{AC}^*(\tau_b^X)$ but may not be wide there. If α^* is an arc then Lemma 3.3 gives two cases: we may replace α^* by γ^* which is either a wide dual arc to τ_b^X or is an essential non-peripheral dual curve to τ_b^X. (If α^* is a curve then let $\gamma^* = \alpha^*$.) So $i(\alpha^*, \gamma^*) \leq 2$ and thus $d_X(\alpha^*, \gamma^*) \leq 4$. If γ^* is a wide dual arc then Lemma 5.2 implies that $d_X(\gamma^*, \beta') \leq \mathsf{K}_1$ and so $d_X(\alpha, \beta) \leq 2\mathsf{K}_1 + 4$. If γ^* is a dual curve then, as $b \notin I_X$, any dual wide curve $\delta^* \in V^*(\tau_b|X)$ has $d_X(\gamma^*, \delta^*) \leq 2$. Again, Lemma 5.2 implies that $d_X(\delta^*, \beta') \leq \mathsf{K}_1$ and so $d_X(\alpha, \beta) \leq 2\mathsf{K}_1 + 6$.

Case III. Suppose $a \leq n_X < c < m_X \leq b$.

The first two cases bound $d_X(\tau_c, \tau_b)$ and $d_X(\tau_a, \tau_c)$; thus we are done by the triangle inequality.

Case IV. Suppose $a \leq n_X < m_X \leq b$ and $m_X = n_X + 1$.

Let $c = n_X$ and $d = m_X$. The first two cases bound $d_X(\tau_d, \tau_b)$ and $d_X(\tau_a, \tau_c)$. Since $V(\tau_c)$ and $V(\tau_d)$ have bounded intersection $d_X(\tau_c, \tau_d)$ is also bounded and the claim is proved. □

Now fix $i \in I_X$.

Claim. If X is an annulus:

- The core curve α is carried by and is wide in τ_i.
- Both sides of α are combed in the induced track $\tau_i|X$.
- If $i+1 \in I_X$ then $\tau_{i+1}|X$ is obtained by taking subtracks, slides, or at most a pair of splittings of $\tau_i|X$.

Proof. Since $i \in I_X$, both $\mathcal{A}(\tau_i|X)$ and $\mathcal{A}^*(\tau_i|X)$ have diameter at least nine. From Lemma 5.1 deduce that the inclusions $V \subset \mathcal{A}$ and $V^* \subset \mathcal{A}^*$ are strict. Thus by Lemma 3.5 the core curve α is both carried by and dual to $\tau_i|X$. The second statement now follows from Lemma 3.6. Thus at least one side of α is combed in τ_i^X. Projecting from S^X to S we find that $\alpha \prec \tau_i$. If α is not wide in τ_i then we deduce that neither side of α is combed in τ^X, a contradiction.

Suppose that τ_i slides to τ_{i+1}. Then, up to isotopy, τ_{i+1} slides to τ_i. Since slides do not kill essential arcs it follows that $\tau_{i+1}|X$ is obtained from $\tau_i|X$ by an at most countable collection of slides.

Now suppose that τ_{i+1} is obtained by splitting τ_i along a large branch b. Thus τ_{i+1}^X is obtained from τ_i^X by splitting all of the countably many lifts of b. Every essential arc carried by $\tau_{i+1}|X$ is also carried by τ_i^X. Let $\tau' \subset \tau_i^X$ be the union of these essential routes. It follows that $\tau_{i+1}|X$ is obtained from τ' by splitting along lifts of b that are also large branches of τ'. Since both sides of α are combed in $\tau_i|X$ the same is true in τ' and so any component of $\tau' \setminus \alpha$ is a tree without large branches. The track τ' therefore has only finitely many large branches, all contained in α. Since α is wide in τ_i there are at most two preimages of the large branch b contained in $\alpha \subset S^X$. Thus $\tau_{i+1}|X$ is obtained from τ' by at most two splittings. This proves the claim. (See Figure 12 for pictures of how α may be carried by τ_i and how splitting b effects $\tau_i|X$.) $\square$

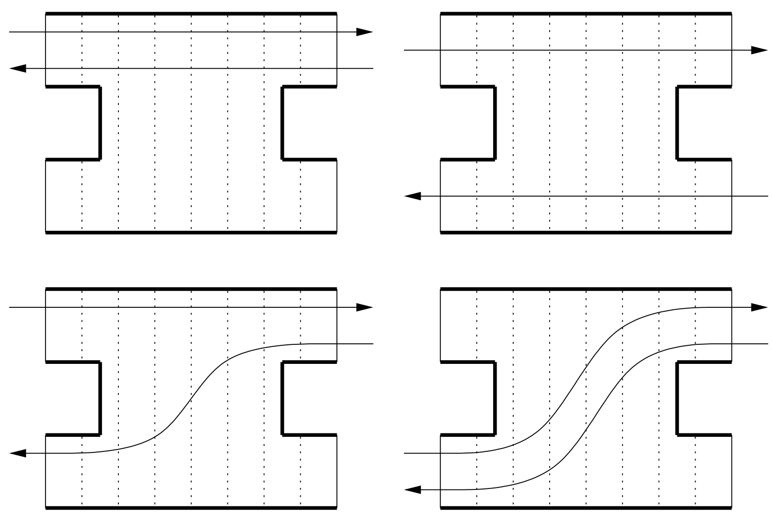

FIGURE 12. Four of the possible ways for an oriented, carried, wide curve α to meet a large rectangle R_b of $N(\tau)$. Note that when X is an annulus and α is the core curve the upper left picture implies that neither side of α is combed in $\tau|X$. Splitting the upper right deletes zero or two components of $\tau_i|X \setminus \alpha$. In the bottom row, only the left splitting is possible when $i, i+1 \in I_X$. On the bottom left one component of $\tau_i|X \setminus \alpha$ is deleted and $\tau_i|X$ is split once. On the bottom right $\tau_i|X$ is split twice.

Claim. Suppose X is not an annulus.

- When in efficient position ∂X is wide with respect to τ_i.
- The track $\tau_i|X$ is birecurrent and fills X.
- If $i+1 \in I_X$ then $\tau_{i+1}|X$ is either a subtrack, a slide, or a split of $\tau_i|X$.

Proof. Since $i \in I_X$ the second conclusion of Lemma 5.2 implies that ∂X is wide. The induced track $\tau_i|X$ carries a pair of curves at distance at least three, so fills X. Also, $\tau_i|X$ is recurrent by definition. For any branch $b' \in \tau_i|X \subset S^X$, let $b \subset \tau$ be the image in S. Since τ is transversely recurrent there is a dual curve β meeting b. Lifting β to a curve or arc $\beta' \subset S^X$ gives a dual to $\tau|X$ meeting b'. Thus $\tau|X$ is transversely recurrent with respect to arcs and curves, as defined in Section 3.1.

Now, if τ_i slides to τ_{i+1} then, as in the annulus case, $\tau_i|X$ slides to $\tau_{i+1}|X$. Suppose instead that τ_i splits to τ_{i+1} along the branch $b \in \mathcal{B}(\tau_i)$. Thus $\tau_i|X$ splits (or isotopes) to a track τ' so that $\tau_{i+1}|X$ is a subtrack. Let $R_b \subset N(\tau_i)$ be the rectangle corresponding to the branch b. Isotope ∂X into efficient position with respect to $N(\tau_i)$ and recall that X is to the left of ∂X. Note that, by an isotopy, we may arrange for all curves in $\mathcal{C}(\tau_i|X)$ to be disjoint from ∂X. Let $\beta \subset R_b$ be a *central splitting arc*: a carried arc completely contained in R_b.

If $\beta \cap X$ is empty then $\tau_i|X$ is identical to $\tau_{i+1}|X$. If $\beta \subset X$ then $\tau_{i+1}|X$ is either a subtrack or a splitting of $\tau_i|X$, depending how the carried curves of $\tau_i|X$ meet the lift of R_b. In all other cases $\tau_{i+1}|X$ is a subtrack of $\tau_i|X$. See Figure 12 for some of the ways carried subarcs of ∂X may meet R_b. If $\partial X \cap R_b$ contains a tie then $\tau_i|X$ is identical to $\tau_{i+1}|X$. This completes the proof of the claim. □

Thus Theorem 5.3 is proved. □

We now rephrase a result of Masur and Minsky using the *refinement* procedure of Penner and Harer [20, page 122].

Theorem 5.4. [16, Theorem 1.3] *For any surface S with $\xi(S) \geq 1$ there is a constant $Q = Q(S)$ with the following property: For any sliding and splitting sequence $\{\tau_i\}_{i=0}^N$ of birecurrent train tracks in S the sequence $\{V(\tau_i)\}_{i=0}^N$ forms a Q–unparameterized quasi-geodesic in $\mathcal{C}(S)$.*

Theorem 5.3 implies that the same result holds after subsurface projection.

Theorem 5.5. *For any surface S with $\xi(S) \geq 1$ there is a constant $Q = Q(S)$ with the following property: For any sliding and splitting sequence $\{\tau_i\}_{i=0}^N$ of birecurrent train tracks in S and for any essential subsurface $X \subset S$ if $\pi_X(\tau_N) \neq \emptyset$ then the sequence $\{\pi_X(\tau_i)\}_{i=0}^N$ is a Q–unparameterized quasi-geodesic in $\mathcal{C}(X)$.*

Proof. By the first conclusion of Theorem 5.3 we may restrict attention to the subinterval $[p, q] = I_X \subset [0, N]$.

Fix any vertex $\alpha \in V(\tau_q|X)$. Note α is carried by $\tau_i|X$ for all $i \leq q$. So define $\sigma_i \subset \tau_i|X$ to be the minimal pretrack carrying α. Since σ_i

does not carry any peripheral curves σ_i is a train track. Note that σ_i is recurrent by definition and transversely recurrent by Lemma 3.2. Applying Theorem 5.3, for all $i \in [p, q-1]$ the track σ_{i+1} is a slide, a split, or identical to the track σ_i.

Theorem 5.4 implies the sequence $\{V(\sigma_i)\}$ is a Q–unparameterized quasi-geodesic in $\mathcal{C}(X)$. Note that $d_X(\sigma_i, \tau_i|X)$ is uniformly bounded because σ_i is a subtrack.

Since $\alpha \prec \tau_i|X$ the curve α is also carried by τ_i. By Lemma 2.9 there is a vertex cycle $\beta_i \prec \tau_i$ that cuts X. Since β_i is wide (Lemma 2.8) any element $\beta_i' \in \pi_X(\beta)$ is carried by and wide in τ_i^X. It follows that β_i' and the vertex cycles of $\tau_i|X$ have bounded intersection. Thus $d_X(\tau_i, \tau_i|X)$ is uniformly bounded and we are done. □

6. Further applications of the structure theorem

We now turn to Theorems 6.1 and 6.2; both are slight generalizations of a result of Hamenstädt [10, Corollary 3]. Our proofs, however, rely on Theorem 5.3 and are quite different from the proof found in [10].

6.1. **The marking and train track graphs.** Suppose that S is not an annulus. A finite subset $\mu \subset \mathcal{AC}(S)$ *fills* S if for all $\beta \in \mathcal{C}(S)$ there is a $\gamma \in \mu$ so that $i(\beta, \gamma) \neq 0$. If $\mu, \nu \subset \mathcal{AC}(S)$ then we define

$$i(\mu, \nu) = \sum_{\alpha \in \mu, \beta \in \nu} i(\alpha, \beta).$$

Also, let $i(\mu) = i(\mu, \mu)$ be the self-intersection number. A set μ is a k*–marking* if μ fills S and $i(\mu) \leq k$. Two sets μ, ν are ℓ*–close* if $i(\mu, \nu) \leq \ell$.

Define $k_0 = \max_\tau i(V(\tau))$, where τ ranges over tracks with vertex cycles $V(\tau)$ filling S. Define $\ell_0 = \max_{\tau,\sigma} i(V(\tau), V(\sigma))$, where σ ranges over tracks obtained from τ by a single splitting. Referring to [15] for the necessary definitions, we define $k_1 = \max_\mu i(\mu)$, where μ ranges over *complete clean markings* of S. Define $\ell_1 = \max_{\mu,\nu} i(\mu, \nu)$, where ν ranges over markings obtained from μ by a single *elementary move*. Define $\ell_2 = \max_\tau \min_\mu i(V(\tau), \mu)$.

Note that there are only finitely many tracks τ and finitely many complete clean markings μ, up to the action of $\mathcal{MCG}(S)$. Since $|\mathcal{B}(\tau)| \leq 6 \cdot \xi(S)$, the number of splittings of τ is also bounded. Lemma 2.4 of [15] bounds the number of elementary moves for μ. Thus the quantities k_0, k_1, ℓ_0, ℓ_1 are well-defined. An upper bound for ℓ_2 can be obtained by surgering $V(\tau)$ to obtain a complete clean marking: see the discussion preceding Lemma 6.1 in [1]. Now define $k = \max\{k_0, k_1\}$ and $\ell = \max\{\ell_0, \ell_1, \ell_2\}$. Define $\mathcal{M}(S)$ to be the *marking graph*: the vertices are

k–markings and the edges are given by ℓ–closeness. (When S is an annulus we take $\mathcal{M}(S) = \mathcal{A}(S)$. Recall that $\mathcal{A}(S)$ is quasi-isometric to $\mathcal{MCG}(S, \partial) \cong \mathbb{Z}$.)

That $\mathcal{M}(S)$ is connected now follows from the discussion at the beginning of [15, Section 6.4]. Accordingly, define $d_{\mathcal{M}(S)}(\mu, \nu)$ to be the length of the shortest edge-path between the markings μ and ν.

Since the above definitions are stated in terms of geometric intersection number, the mapping class group $\mathcal{MCG}(S)$ acts via isometry on $\mathcal{M}(S)$. Counting the appropriate set of ribbon graphs proves that the action has finitely many orbits of vertices and edges. The Alexander method [5, Section 2.4] proves that vertex stabilizers are finite and hence the action is proper. It now follows from the Milnor-Švarc Lemma [2, Proposition I.8.19] that any Cayley graph for $\mathcal{MCG}(S)$ is quasi-isometric to $\mathcal{M}(S)$.

Define $\mathcal{T}(S)$, the *train track graph* as follows: vertices are isotopy classes of birecurrent train tracks $\tau \subset S$ so that $V(\tau)$ fills S. Connect vertices τ and σ by an edge exactly when σ is a slide or split of τ. Let $d_{\mathcal{T}(S)}(\tau, \upsilon)$ be the minimal number of edges in a path in $\mathcal{T}(S)$ connecting τ to υ, if such a path exists. Note that the map $\tau \mapsto V(\tau)$ from $\mathcal{T}(S)$ to $\mathcal{M}(S)$ sends edges to edges (or to vertices) and thus is distance non-increasing. For further discussion of graphs tightly related to $\mathcal{T}(S)$ see [10].

We adopt the following notations. If $\{\tau_i\}_{i=0}^N$ is a sliding and splitting sequence in $\mathcal{T}(S)$ and $I = [p, q] \subset [0, N]$ is a subinterval then $|I| = q - p$ and $d_{\mathcal{T}(S)}(I) = d_{\mathcal{T}(S)}(\tau_p, \tau_q)$. If $\tau, \sigma \in \mathcal{T}(S)$ then define

$$d_{\mathcal{M}(X)}(\tau, \sigma) = d_{\mathcal{M}(X)}\big(V(\tau|X), V(\sigma|X)\big).$$

Also take $d_{\mathcal{M}(X)}(I) = d_{\mathcal{M}(X)}(\tau_p, \tau_q)$.

Theorem 6.1. *For any surface S with $\xi(S) \geq 1$ there is a constant $Q = Q(S)$ with the following property: Suppose that $\{\tau_i\}_{i=0}^N$ is a sliding and splitting sequence in $\mathcal{T}(S)$. Then the sequence $\{V(\tau_i)\}_{i=0}^N$, as parameterized by splittings, is a Q–quasi-geodesic in the marking graph.*

Before proving this we give our final generalization of [10, Corollary 3], which follows from Theorem 6.1.

Theorem 6.2. *For any surface S with $\xi(S) \geq 1$ there is a constant $Q = Q(S)$ with the following property: If $\{\tau_i\}_{i=0}^N$ is a sliding and splitting sequence in $\mathcal{T}(S)$, injective on slide subsequences, then $\{\tau_i\}$ is a Q–quasi-geodesic.*

Notice that here, unlike Theorem 6.1, the parameterization is by index. We will use:

Lemma 6.3. *There is a constant $A = A(S)$ so that if $\{\tau_i\}_{i=1}^N$ is an injective sliding sequence in $\mathcal{T}(S)$ then $N + 1 \leq A$.* □

Proof of Theorem 6.2. Let $\{\tau_i\}$ be the given sliding and splitting sequence in $\mathcal{T}(S)$. Let $I = [p, q] \subset [0, N]$ be a subinterval. Note that $d_{\mathcal{T}(S)}(I) \leq |I|$ because $\{\tau_i\}$ is an edge-path in $\mathcal{T}(S)$.

Define $\mathcal{S}(I)$ to be the set of indices $r \in I$ where $r + 1 \in I$ and τ_{r+1} is a splitting of τ_r. Thus $|I| \leq_A |\mathcal{S}(I)|$, where A is the constant of Lemma 6.3. Now, Theorem 6.1 implies $|\mathcal{S}(I)| \leq_Q d_{\mathcal{M}(S)}(I)$. Finally, since $\tau \mapsto V(\tau)$ is distance non-increasing we have $d_{\mathcal{M}(S)}(I) \leq d_{\mathcal{T}(S)}(I)$. Deduce that $|I| \leq_Q d_{\mathcal{M}(S)}(I)$, for a somewhat larger value of Q. □

Remark 6.4. Note that we have not used the connectedness of $\mathcal{T}(S)$, an issue that appears to be difficult to approach combinatorially. For a proof of connectedness see [10, Corollary 3.7].

6.2. **Hyperbolicity and the distance estimate.** To prove Theorem 6.1 we will need:

Theorem 6.5. [14, Theorem 1.1] *For every connected compact orientable surface X there is a constant δ_X so that $\mathcal{C}(X)$ is δ_X–hyperbolic.* □

An important consequence of the Morse Lemma [2, Theorem III.H.1.7] is a reverse triangle inequality.

Lemma 6.6. *For every δ and Q, there is a constant $\mathsf{R}_0 = \mathsf{R}_0(\delta, Q)$ with the following property: For any δ–hyperbolic space $\mathcal{X}$, for any Q–unparameterized quasi-geodesic $f\colon [m, n] \to \mathcal{X}$, and for any $a, b, c \in [m, n]$, if $a \leq b \leq c$ then*

$$d_{\mathcal{X}}(\alpha, \beta) + d_{\mathcal{X}}(\beta, \gamma) \leq d_{\mathcal{X}}(\alpha, \gamma) + \mathsf{R}_0$$

where $\alpha, \beta, \gamma = f(a), f(b), f(c)$. □

We now take $\mathsf{R}_0 = \mathsf{R}_0(\delta, Q(S))$ where $\delta = \max\{\delta_X \mid X \subset S\}$, as provided by Theorem 6.5, and $Q(S)$ is the constant of Theorem 5.5. The next central result needed is the *distance estimate* for $\mathcal{M}(S)$. Let $[\cdot]_C$ be the *cut-off function*:

$$[x]_C = \left\{ \begin{array}{ll} 0, & \text{if } x < C \\ x, & \text{if } x \geq C \end{array} \right\}.$$

We may now state the distance estimate.

Theorem 6.7. [15, Theorem 6.12] *For any surface S, there is a constant $\mathsf{C}(S)$ so that for every $C \geq \mathsf{C}(S)$ there is an $\mathsf{E} \geq 1$ so that for all $\mu, \nu \in \mathcal{M}(S)$*

$$d_{\mathcal{M}(S)}(\mu, \nu) =_{\mathsf{E}} \sum [d_X(\mu, \nu)]_C$$

where the sum ranges over essential subsurfaces $X \subset S$. □

6.3. **Marking distance.** Suppose that $\{\tau_i\}_{i=0}^N \subset \mathcal{T}(S)$ is a sliding and splitting sequence. Let $V_i = V(\tau_i)$ be the set of vertex cycles of τ_i. As $i(V_i) \leq k_0$ and $i(V_i, V_{i+1}) \leq \ell_0$ the map $i \mapsto V_i$ gives rise to an edge-path in $\mathcal{M}(S)$.

Suppose that $[p, q] \subset [0, N]$. Let $\mathcal{S}_X(p, q)$ be the set of indices $r \in [p, q-1]$ so that $\tau_{r+1}|X$ is a splitting of $\tau_r|X$. (When X is an annulus $\tau_{r+1}|X$ may also differ from $\tau_r|X$ by a pair of splits.)

Remark 6.8. We do not place indices r onto S_X where $\tau_{r+1}|X$ is a subtrack of $\tau_r|X$; the number of such indices is bounded by a constant depending only on X.

Recall that we omit the subscript from $\mathcal{S}_X$ when $X = S$. As a piece of notation, set $I_S = [0, N]$. When $X \subset S$ is essential take $V(\tau|X)$ to be the vertex cycles of the induced track. Recall that $I_X \subset I_S$, defined in Section 5.2, is the accessible interval for $X \subset S$. If $J = [m, n] \subset [0, N]$ is an interval then define $\mathcal{S}_X(J) = \mathcal{S}_X(m, n)$, $d_X(J) = d_X(\tau_m, \tau_n)$, and so on. Here is the crucial tool for the proof of Theorem 6.1.

Proposition 6.9. *Suppose that $X \subset S$ is an essential subsurface and $J_X \subset I_X$ is a subinterval. There is a constant $A = A(X)$, independent of the sequence $\{\tau_i\}$, so that $|\mathcal{S}_X(J_X)| \leq_A d_{\mathcal{M}(X)}(J_X)$.*

Proof of Theorem 6.1. Suppose that $\{\tau_i\}_{i=0}^N$ is sliding and splitting sequence. Define $V_i = V(\tau_i)$. By our definition of $\mathcal{M}(S)$, and since slides do not effect $P(\tau)$ [20, Proposition 2.2.2], the map $i \mapsto V_i$ gives an edge-path in $\mathcal{M}(S)$. Thus $d_{\mathcal{M}}(V_i, V_j) \leq |\mathcal{S}(i, j)|$. The opposite inequality, with multiplicative and additive error at most $A(S)$, follows from Proposition 6.9. □

It remains to prove Proposition 6.9. This occupies the reminder of the paper. We first sketch the proof.

We induct on the complexity of the subsurface $X \subset S$. Partition the interval J_X into two kinds of subintervals. The first are *inductive* intervals. They arise from proper subsurfaces $Y \subset X$ for which the projection $d_Y(J_X)$ is above a given threshold. The second are *straight* intervals; those for which the diameter of the projection to all proper subsurfaces is below another threshold.

Lemma 6.12 uses the structure theorem (5.3) to show that intervals disjoint from the inductive intervals are straight. The main technical step of the proof of Proposition 6.9 is Lemma 6.13: the number of splittings in a straight interval $I \subset J_X$ is bounded by the diameter $d_X(I)$, measured in the curve complex of X. This again uses our structure theorem and also the distance estimate (6.7): the latter saying that if all strict subsurface projections are small then marking distance is quasi-equal to distance in $\mathcal{C}(X)$.

We then divide the straight intervals into the *long* and the *short*: those which are longer than a threshold defined by the reverse triangle inequality (6.6) and those which are shorter. Lemma 6.18 bounds the total number of splittings in long straight intervals by $d_X(J_X)$. The sum in short straight intervals is bounded by the number of inductive intervals (Lemma 6.19). In Lemma 6.22 we apply the inductive hypothesis provided by Proposition 6.9 to prove that the number of splittings in all inductive intervals is bounded by a sum of marking distances. The distance estimate, in turn, implies that the sum of marking distances is bounded by a sum of subsurface projections.

Adding these estimates, on the number of splittings in straight intervals and in inductive intervals, produces the the desired bound of Proposition 6.9.

6.4. **Inductive and straight intervals.** We fix two thresholds $\mathsf{T}_0, \mathsf{T}_1$ so that:

$$\max\left\{6\mathsf{N}_1 + 2\mathsf{N}_2 + 2\mathsf{K}_0(X) + 2, 2\mathsf{R}_0, M_2(X), \mathsf{C}(X)\right\} \leq \mathsf{T}_0(X)$$
$$\max\left\{\mathsf{T}_0(X) + 2\mathsf{R}_0, \mathsf{B}_0\mathsf{N}_2\right\} \leq \mathsf{T}_1(X)$$

Here N_1 is an upper bound for $d_Y(\alpha, \beta)$ where $Y \subset S$ is any essential subsurface, τ is a track, and α and β are wide with respect to τ. The constant B_0 is an upper bound for the number of branches in any induced track. The constant N_2 is an upper bound for the distance (in any subsurface projection) between the vertices of τ (or $\tau|X$) and the vertices of a single splitting or subtrack of τ. Also, $M_2(X)$ is the constant provided by Lemma 6.1 of [15].

Recall that the interval $J_X \subset I_X$ is given in Proposition 6.9.

Definition 6.10. Suppose that $Y \subset X$ is an essential subsurface with $\xi(Y) < \xi(X)$. If $d_Y(J_X) \geq \mathsf{T}_0(X)$ then we call Y an *inductive* subsurface of X and take $J_Y = I_Y \cap J_X$ as the associated *inductive subinterval* of J_X. If $d_Y(J_X) < \mathsf{T}_0(X)$ then we set $J_Y = \emptyset$.

Suppose I is a subinterval of J_X. Define $\operatorname{diam}_Y(I)$ to be the diameter, in $\mathcal{AC}(Y)$, of the union $\cup_{i \in I} \pi_Y(\tau_i)$.

Definition 6.11. A subinterval $I \subset J_X$ is a *straight subinterval* for X if for all essential subsurfaces $Y \subset X$, with $\xi(Y) < \xi(X)$, we have $\operatorname{diam}_Y(I) \leq \mathsf{T}_1(X)$.

Lemma 6.12. *If $I \subset J_X$ is disjoint from all inductive subintervals of J_X then I is straight for X.*

Proof. Fix an essential $Y \subset X$ with $\xi(Y) < \xi(X)$. It suffices to show, for every subinterval $J \subset I$, that $d_Y(J) \leq \mathsf{T}_1(X)$.

If $J \cap I_Y = \emptyset$ then Theorem 5.3 implies $d_Y(J) \leq \mathsf{K}_0$. Suppose that J meets I_Y; thus $J_Y = \emptyset$ by hypothesis and so Y is not inductive. It follows that $d_Y(I) < \mathsf{T}_0(X)$. By Lemma 6.6 we have $d_Y(J) < \mathsf{T}_0(X) + 2\mathsf{R}_0$. □

Lemma 6.13. *There is a constant $A = A(X)$, independent of $\{\tau_i\}$, so that if $I \subset J_X$ is straight then $|\mathcal{S}_X(I)| \leq_A d_X(I)$.*

Proof. If X is an annulus then, by Theorem 5.3, for every $r \in I$ the core curve $\alpha \subset X$ is carried by and wide in τ_r. It follows that the number of switches in $\alpha \subset \tau_r|X$ is bounded by some constant $K = K(S)$. Let $q = \max I$ and pick any $\beta \in V(\tau_q|X)$. As in the proof of Theorem 5.5 let $\sigma_r \subset \tau_r|X$ be the minimal subtrack carrying β. Thus σ_r has either exactly four branches and two switches, or is an embedded arc. It follows that every $K^2/4$ consecutive splittings in $S_X(I)$ induces at least one splitting in the sequence of tracks $\{\sigma_r\}$. Therefore the singleton sets $V(\sigma_r)$ form a quasi-geodesic in $\mathcal{A}(X)$. Since $V(\sigma_r) \subset V(\tau_r|X)$ the proof is complete when X is an annulus.

We assume for the rest of the proof that X is not an annulus. The map $i \mapsto V(\tau_i|X)$, taking tracks to their vertex cycles, is generally not injective. (For example, see [20, Proposition 2.2.2].) However:

Claim 6.14. There is a constant $\mathsf{N}_0 = \mathsf{N}_0(X)$, independent of $\{\tau_i\}$, so that if $V(\tau_r|X) = V(\tau_s|X)$ then $|\mathcal{S}_X(r, s)| \leq \mathsf{N}_0$.

Proof. Let $\mu = V(\tau_r|X)$. Our hypothesis on $\tau_s|X$ and induction proves that $V(\tau_t|X) = \mu$ for all $t \in [r, s]$. Recurrence and uniqueness of carrying [19, Proposition 3.7.3] implies that $\tau_{t+1}|X$ is a split or a slide of $\tau_t|X$, and not a subtrack, for all $t \in [r, s-1]$.

If $t \in [r, s]$ and $b \in \mathcal{B}(\tau_t|X)$ then define $w_\mu(b) = \sum_{\alpha \in \mu} w_\alpha(b)$. Let

$$M(t) = \big(w_\mu(b) : b \text{ is a large branch of } \tau_t|X\big)$$

be the sequence of given numbers, arranged in non-increasing order. Note that if $\tau_{t+1}|X$ is a slide of $\tau_t|X$ then $M(t+1) = M(t)$. However, if $t \in \mathcal{S}_X(r, s)$ then the recurrence of $\tau_t|X$ implies that $M(t+1) < M(t)$, in lexicographic order. Since there are only finitely many possibilities for an induced track $\tau|X$, up to the action of $\mathcal{MCG}(X)$, the claim follows. □

Notice that if $V(\tau_{i+1}|X) \neq V(\tau_i|X)$ then $V(\tau_{i+1}|X) \neq V(\tau_j|X)$ for $j \leq i$. This is because $P(\tau_{k+1}|X) \subset P(\tau_k|X)$ for all k. Using $C = 1 + \max\{\mathsf{C}(X), \mathsf{T}_1(X)\}$ as the cut-off in Theorem 6.7 gives some constant of quasi-equality, say E. Define $\mathsf{R}_1 = \mathsf{E} + 1$.

Suppose that $[p, q] = I$, the straight subinterval of J_X given by Lemma 6.13. We define a function $\rho\colon [0, M] \to I$ as follows: let $\rho(0) = p$ and let $\rho(n+1)$ be the smallest element in $[\rho(n), q]$ with $d_{\mathcal{M}(X)}(\tau_{\rho(n)}, \tau_{\rho(n+1)}) = \mathsf{R}_1$. (If $\rho(n+1)$ is undefined then take $M = n+1$ and $\rho(M) = q$.) Let $B(\mu)$ be the ball of radius R_1 about the marking $\mu \in \mathcal{M}(X)$. Define

$$\mathsf{V} = \max\left\{|B(\mu)| : \mu \in \mathcal{M}(X)\right\}.$$

Deduce from Claim 6.14 and the remark immediately following that, for all $n \in [0, M-1]$,

$$|\mathcal{S}_X(\rho(n), \rho(n+1))| \leq \mathsf{N}_0\mathsf{V}.$$

Thus

$$|\mathcal{S}_X(I)| \leq \mathsf{N}_0\mathsf{V} \cdot M.$$

So to prove Lemma 6.13 it suffices to bound M from above in terms of $d_X(I)$.

Claim 6.15. Fix $n \in [0, M-2]$. Let $\tau, \sigma = \tau_{\rho(n)}, \tau_{\rho(n+1)}$. Then $d_X(\tau, \sigma) \geq \mathsf{R}_0 + 1$.

Proof. We use Theorem 6.7. Note that $d_{\mathcal{M}(X)}(\tau, \sigma) = \mathsf{R}_1$. Since R_1 is greater than the additive error there is at least one non-vanishing term in the sum $\sum_{Y \subset X} [d_Y(\tau, \sigma)]_C$.

However, since $[\rho(n), \rho(n+1)] \subset [p, q]$ and $[p, q] = I$ is straight we have $d_Y(\tau, \sigma) \leq \mathsf{T}_1(X)$ for all $Y \subset X$ with $\xi(Y) < \xi(X)$. Thus $d_X(\tau, \sigma)$ is the only term of the sum greater than the cut-off C. Since $C > \mathsf{T}_1(X) \geq \mathsf{R}_0$, we have $d_X(\tau, \sigma) \geq \mathsf{R}_0 + 1$ and the claim is proved. □

Thus we have

$$\begin{aligned} d_X(I) &\geq -(M-1) \cdot \mathsf{R}_0 + \sum_{n=0}^{M-1} d_X(\tau_{\rho(n)}, \tau_{\rho(n+1)}) \\ &\geq M - 1 + d_X(\tau_{\rho(M-1)}, \tau_{\rho(M)}) \\ &\geq M - 1 \end{aligned}$$

where the first and second lines follow from Lemma 6.6 and Claim 6.15 respectively. This completes the proof of Lemma 6.13. □

Lemma 6.16. *There is a constant $A = A(X)$ with the following property: Suppose that $J_Y \subset J_X$ is an inductive interval. Suppose that $I \subset J_Y$ is a straight subinterval for X. Then $|\mathcal{S}_X(I)| \leq A$.*

Proof. Let $[p, q] = I$. Applying Theorem 5.3, as $p \in J_Y \subset I_Y$, the multicurve ∂Y is wide with respect to τ_p. It follows that ∂Y is also wide with respect to τ_p^X. Note that the curves of $V(\tau_p|X)$ are also wide with respect to τ_p^X. Repeating this discussion for q, and then applying Lemma 5.2 and the triangle inequality gives a uniform bound for $d_X(\tau_p, \tau_q)$. The lemma now follows from Lemma 6.13. □

6.5. **Long and short intervals.**

Definition 6.17. A straight subinterval I for X is *short* if $d_X(I) \leq 4\mathsf{R}_0$. Otherwise I is *long*.

By Lemma 6.13, if I is a short straight interval then $|\mathcal{S}_X(I)|$ is uniformly bounded by a constant depending only on X.

Let Ind be the set of inductive subsurfaces $Y \subset X$. Define $\mathsf{Ind}' = \mathsf{Ind} \cup \{X\}$. Note that every maximal subinterval of $J_X \setminus \cup_{Y \in \mathsf{Ind}} J_Y$ is straight, by Lemma 6.12. We partition these maximal subintervals into the sets Long and Short as the given interval is long or short respectively.

Lemma 6.18. *There is a constant $A = A(X)$, independent of $\{\tau_i\}$, so that*

$$\sum_{I \in \mathsf{Long}} |\mathcal{S}_X(I)| \leq_A d_X(J_X).$$

Proof. From Lemma 6.13 we deduce

$$\sum_{I \in \mathsf{Long}} |\mathcal{S}_X(I)| \leq_A |\mathsf{Long}| + \sum_{I \in \mathsf{Long}} d_X(I)$$

where the first term on the right arises from addition of additive errors. By the definition of a long straight interval and from Lemma 6.6 deduce

$$4\mathsf{R}_0|\mathsf{Long}| \leq \sum_{I \in \mathsf{Long}} d_X(I) \leq d_X(J_X) + 2\mathsf{R}_0|\mathsf{Long}|.$$

Thus $2\mathsf{R}_0|\mathsf{Long}| \leq d_X(J_X)$. These inequalities combine to prove the lemma, for a somewhat larger value of $A = A(X)$. □

Lemma 6.19. *There is a constant $A = A(X)$, independent of $\{\tau_i\}$, so that*

$$\sum_{I \in \mathsf{Short}} |\mathcal{S}_X(I)| \leq_A |\mathsf{Ind}'|.$$

Proof. By Lemma 6.13 the number of splittings in any short straight interval is a priori bounded (depending only on X). Since $|\mathsf{Short}| \leq |\mathsf{Ind}'|$ the lemma follows. □

Lemma 6.20. *If $Z \in \mathsf{Ind}$ then*

$$\text{card}\left\{Y \in \mathsf{Ind} \mid Z \subset Y,\, \xi(Z) < \xi(Y)\right\} \leq 2(\xi(X) - \xi(Z) - 1).$$

This follows from and is strictly weaker than Theorem 4.7 and Lemma 6.1 of [15]. We give a proof, using our structure theorem, to extract the necessary lower bound for $\mathsf{T}_0(X)$.

Proof of Lemma 6.20. Suppose that $U \in \mathsf{Ind}$ contains Z. Suppose $J_Z = [p, q]$ and $J_X = [m, n]$. Thus ∂Z is wide with respect to τ_p. So $d_U(\tau_p, \partial Z) \leq \mathsf{N}_1$, by the definition of N_1, and the same holds at the index q. Thus $d_U(\tau_p, \tau_q) \leq 2\mathsf{N}_1$. The subsurface U *precedes* or *succeeds* Z if $d_U(\tau_m, \tau_p)$ or $d_U(\tau_q, \tau_n)$, respectively, is greater than or equal to $2\mathsf{N}_1 + \mathsf{N}_2 + \mathsf{K}_0(X) + 1$. Note that U must precede or succeed Z (or both) as otherwise $d_U(\tau_m, \tau_n) < 6\mathsf{N}_1 + 2\mathsf{N}_2 + 2\mathsf{K}_0(X) + 2 \leq \mathsf{T}_0(X)$, a contradiction.

It now suffices to consider subsurfaces U and V that both succeed and both contain Z. If $\max J_U \leq \max J_V$ then $U \subset V$. For, if not, ∂V cuts U while missing Z. Since ∂V is wide at the index $r = \max J_V$ we deduce that

$$\begin{aligned} d_U(\tau_q, \tau_n) &\leq d_U(\tau_q, \partial Z) + d_U(\partial Z, \partial V) + d_U(\partial V, \tau_r) + \\ &\quad + d_U(\tau_r, \tau_{r+1}) + d_U(\tau_{r+1}, \tau_n) \\ &\leq 2\mathsf{N}_1 + \mathsf{N}_2 + \mathsf{K}_0(X) + 1 \end{aligned}$$

and this is a contradiction. Thus the surfaces in Ind that strictly contain Z, and succeed Z, are nested. □

Definition 6.21. *Assign* an index $r \in \mathcal{S}_X(J_X)$ to a subsurface $Y \subset X$ if $Y \in \mathsf{Ind}'$, $r \in J_Y$, $\tau_{r+1}|Y$ is a splitting of $\tau_r|Y$ and there is no subsurface $Z \subset Y, \xi(Z) < \xi(Y)$ with those three properties.

Lemma 6.22. *There is a constant $A = A(X)$, independent of $\{\tau_i\}$, so that the number of splittings contained in inductive intervals is quasi-bounded by $|\mathsf{Ind}| + \sum_{Y \in \mathsf{Ind}} d_Y(J_X)$.*

Proof. Fix $Y \in \mathsf{Ind}$. Consider an index $r \in J_Y$ that is assigned to X. Let $I \subset J_Y$ be the maximal interval containing r so that all indices in $\mathcal{S}_X(I)$ are assigned to X. We now show that I is straight. Let Z be any essential subsurface of X with $\xi(Z) < \xi(X)$ and let $[r, s] = J \subset I$ be any subinterval. If $J \cap I_Z = \emptyset$ then Theorem 5.3 implies that $d_Z(J) \leq \mathsf{K}_0$. If J meets I_Z then, as no splittings of J are assigned to Z we deduce that $\tau_s|Z$ is obtained from $\tau_r|Z$ by sliding and taking subtracks only. Thus $d_Z(J) \leq \mathsf{B}_0\mathsf{N}_2 \leq \mathsf{T}_1(X)$, as desired.

By Lemma 6.16 we find that $|\mathcal{S}_X(I)|$ is bounded. It follows that the number of splittings in the inductive intervals is quasi-bounded by $\sum_{Y \in \mathsf{Ind}} |\mathcal{S}_Y(J_Y)|$.

By induction, Proposition 6.9 gives

$$|\mathcal{S}_Y(J_Y)| \leq_A d_{\mathcal{M}(Y)}(J_Y).$$

Taking a cutoff of $C = 1 + \max\{\mathsf{C}(Y), \mathsf{T}_0(X) + 2\mathsf{R}_0\}$ and applying the distance estimate Theorem 6.7 we have a quasi-inequality

$$d_{\mathcal{M}(Y)}(J_Y) \ \leq_{\mathsf{E}} \sum_{Z \subset Y} [d_Z(J_Y)]_C.$$

Since $d_Z(J_Y) \leq d_Z(J_X) + 2\mathsf{R}_0$ for all $Z \subset Y$, it follows that non-zero terms in the sum only arise for subsurfaces in $\mathsf{Ind}'(Y) = \{Z \in \mathsf{Ind}' \mid Z \subset Y\}$. Since $2\mathsf{R}_0 \leq \mathsf{T}_0(X)$ we have $[d_Z(J_Y)]_C \leq 2 \cdot d_Z(J_X)$. Making $A = A(X)$ larger if necessary we have

$$|\mathcal{S}_Y(J_Y)| \leq_A \sum_{Z \in \mathsf{Ind}'(Y)} d_Z(J_X).$$

Thus

$$\begin{aligned} \sum_{Y \in \mathsf{Ind}} |\mathcal{S}_Y(J_Y)| &\leq_A |\mathsf{Ind}| + \sum_{Y \in \mathsf{Ind}} \sum_{Z \in \mathsf{Ind}'(Y)} d_Z(J_X) \\ &\leq_A |\mathsf{Ind}| + \sum_{Y \in \mathsf{Ind}} d_Y(J_X) \end{aligned}$$

where the final quasi-inequality follows from Lemma 6.20, taking A larger as necessary. Note that the term $|\mathsf{Ind}|$ on the middle line arises by adding additive errors. This proves Lemma 6.22. □

Since every index in $\mathcal{S}_X(J_X)$ is either in a long or short straight interval or in an inductive interval, from Lemmas 6.18, 6.19, and 6.22 and increasing A slightly, we have:

$$|\mathcal{S}_X(J_X)| \leq_A d_X(J_X) + |\mathsf{Ind}'| + \sum_{Y \in \mathsf{Ind}} d_Y(J_X).$$

Note that $|\mathsf{Ind}'| \leq_A d_{\mathcal{M}(X)}(J_X)$; this follows from the hierarchy machine (in particular Lemma 6.2 and Theorem 6.10 of [15]) and because $\mathsf{T}_0(X) \geq M_2(X)$, the constant of Lemma 6.1 in [15]. Finally,

$$\sum_{Y \in \mathsf{Ind}'} d_Y(J_X) \leq_A d_{\mathcal{M}(X)}(J_X)$$

follows from the distance estimate (Theorem 6.7) and because $\mathsf{T}_0(X) \geq \mathsf{C}(X)$. This completes the proof of Proposition 6.9. □

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