SIEGEL-VEECH TRANSFORMS ARE IN L^2

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Dedicated to the memory of William Veech.

Abstract. Let $\mathcal{H}$ denote a connected component of a stratum of translation surfaces. We show that the Siegel-Veech transform of a bounded compactly supported function on $\mathbb{R}^2$ is in $L^2(\mathcal{H}, \mu)$, where μ is Lebesgue measure on $\mathcal{H}$, and give applications to bounding error terms for counting problems for saddle connections. We also propose a new invariant associated to $SL(2, \mathbb{R})$-invariant measures on strata satisfying certain integrability conditions.

1. Introduction

Motivated by counting problems for polygonal billiards and more generally for linear flows on surfaces, Veech [24] introduced what is now known as the *Siegel-Veech transform* on the moduli space of abelian differentials (in analogy with the Siegel transform arising from the space of unimodular lattices in $\mathbb{R}^n$). The main result of [24] is an integration (L^1) formula for this transform (see §1.3.1), a version of the classical Siegel integral formula.

Our main result Theorem 1.1 is that the Siegel-Veech transform $\widehat{f}$ of any bounded compactly supported function f on $\mathbb{R}^2$ satisfies

$$\widehat{f} \in L^2(\mathcal{H}, \mu)$$

with respect to the natural Lebesgue measure μ on any (connected component of a) stratum $\mathcal{H}$ of abelian differentials.

1.1. Translation surfaces.

A *translation surface* S is a pair (X, ω) where X is a Riemann surface and ω is a holomorphic 1-form. The terminology is motivated by the fact that integrating ω (away from its zeros) gives an atlas of charts to $\mathbb{C}$ whose transition maps are translations. These can be viewed as singular flat metrics with trivial rotational holonomy, with isolated cone-type singularities corresponding to

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zeros of ω. An *saddle connection* γ on S is a geodesic segment connecting two zeros of ω with none in its interior. Associated to each saddle connection is its *holonomy vector*

$$z_\gamma = \int_\gamma \omega \in \mathbb{C}$$

and its length

$$|\gamma| = \int_\gamma |\omega|.$$

We denote the set of holonomy vectors by Λ_ω. Λ_ω is a discrete subset of the plane $\mathbb{C} \sim \mathbb{R}^2$.

1.2. **Strata.** The moduli space Ω_g of genus g translation surfaces is the bundle over the moduli space M_g of genus g Riemann surfaces with fiber over each Riemann surface X given by $\Omega(X)$, the vector space of holomorphic 1-forms on X. Ω_g decomposes into *strata* depending on the combinatorics of the differentials.

Since the orders of the zeros of ω must sum to $2g-2$, there is a stratum associated to each integer partition of $2g-2$. Each of these strata has at most three connected components [13].

The flat metric associated to a one-form ω also gives a notion of area on the surface. We consider the subset of *area 1* surfaces of a connected component of a stratum, and denote it by $\mathcal{H}$. We will, by abuse of notation, often simply refer to this as a stratum, and we will denote elements of it by (X, ω).

1.2.1. *Lebesgue measure.* The group $GL(2, \mathbb{R})$ acts on Ω_g via linear post-composition with charts, preserving combinatorics of differentials. The subgroup $SL(2, \mathbb{R})$ preserves each area 1 subset, so acts on each stratum $\mathcal{H}$. On each stratum, there is a natural measure μ, known as *Masur-Veech* or Lebesgue measure, constructed using *period coordinates* on strata (see, e.g. [8] or [28] for a nice exposition of the construction of this measure). A crucial result, independently shown by W. Veech and the third-named author, is

Theorem. [15, 26] *μ is a finite $SL(2, \mathbb{R})$-invariant ergodic measure on each stratum $\mathcal{H}$.*

1.3. **Siegel-Veech transforms.** Fix a stratum $\mathcal{H}$. Let $B_c(\mathbb{R}^2)$ denote the space of bounded compactly supported functions on $\mathbb{R}^2 \sim \mathbb{C}$. Given $(X, \omega) \in \mathcal{H}$ and $f \in B_c(\mathbb{R}^2)$, Veech [24] introduced the *Siegel-Veech transform*

$$\widehat{f}(X, \omega) = \sum_{v \in \Lambda_\omega} f(v).$$

Note that this is a *finite* sum for any fixed f and ω, since Λ_ω is discrete. Our main result is:

Theorem 1.1. *Let $f \in B_c(\mathbb{R}^2)$. Then $\widehat{f} \in L^2(\mathcal{H}, \mu)$.*

This corrects a mistake in [2], which claimed that if f was the indicator function of the unit disk, that $\widehat{f} \notin L^2$. In fact, the proof in [2] only shows $\widehat{f} \notin L^3$.

1.3.1. *Siegel-Veech formulas.* Veech [24] showed $\widehat{f} \in L^1(\mathcal{H}, \mu)$, and using the $SL(2,\mathbb{R})$-invariance of μ and a classification of the $SL(2,\mathbb{R})$-invariant measures on $\mathbb{R}^2$, showed

Theorem. [24] *There is a constant $c = c(\mu)$ so that*

$$\int_{\mathcal{H}} \widehat{f} d\mu = c \int_{\mathbb{R}^2} f dm,$$

where m is Lebesgue measure on $\mathbb{R}^2$.

This is a generalization of the (two-dimensional version of) *Siegel integral formula* [22], which applies to averages of similar transforms over spaces of unimodular lattices.

1.3.2. *Siegel-Veech constants.* In fact, Veech showed that for *any* $SL(2,\mathbb{R})$-invariant ergodic finite measure λ, that $\widehat{f} \in L^1(\mathcal{H}, \lambda)$, and that there is $c = c_{SV}(\lambda)$ so that

$$\int_{\mathcal{H}} \widehat{f} d\lambda = c \int_{\mathbb{R}^2} f dm.$$

These constants $c_{SV}(\lambda)$ are known as *Siegel-Veech constants* and are important numerical invariants associated to $SL(2,\mathbb{R})$-invariant measures.

1.4. **Siegel-Veech measures.** For any measure λ with $\widehat{f} \in L^2(\mathcal{H}, \lambda)$, we can define two measure-valued invariants. First, we extend the notion of Siegel-Veech transform to $B_c(\mathbb{R}^4)$, viewing $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$.

1.4.1. *Generalized Siegel-Veech transforms.* Given

$$h \in B_c(\mathbb{R}^4) = B_c(\mathbb{R}^2 \times \mathbb{R}^2).$$

define the *Siegel-Veech transform*

$$\widehat{h}(\omega) = \sum_{v_1, v_2 \in \Lambda_\omega} h(v_1, v_2).$$

Note that if $h(x, y) = f(x)f(y)$ for $f \in B_c(\mathbb{R}^2)$,

$$\widehat{h} = \widehat{f}^2.$$

1.4.2. *Measure-valued invariants.*

Theorem 1.2. *Let λ denote an $SL(2,\mathbb{R})$-invariant measure on $\mathcal{H}$ so that for any $f \in B_c(\mathbb{R}^2)$, $\widehat{f} \in L^2(\mathcal{H},\lambda)$. Then there exist* Siegel-Veech measures $\nu = \nu(\lambda)$ *on* $\mathbb{R}\backslash\{0\}$ *and* $\eta = \eta(\lambda)$ *on* $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \infty$ *such that for any* $h \in B_c(\mathbb{R}^4)$,

$$\begin{aligned}\int_{\mathcal{H}} \widehat{h}(\omega)d\lambda(\omega) &= \int_{\mathbb{R}\backslash\{0\}} \left(\int_{SL(2,\mathbb{R})} h(tx,y)d\kappa(x,y)\right) d\nu(t) \\ &+ \int_{\mathbb{P}^1(\mathbb{R})} \left(\int_{\mathbb{R}^2} h(x,sx)dx\right) d\eta(s).\end{aligned}$$

1.5. **Counting bounds.** Veech introduced the Siegel-Veech transform to understand counting problems. Given a translation surface (X,ω) (or simply ω), let

$$N(\omega,R) = \#\left(\Lambda_\omega \cap B(0,R)\right)$$

denote the number of saddle connections of length at most R. The third-named author showed that there are constants

$$0 < c_1 = c_1(\omega) \leq c_2 = c_2(\omega)$$

so that

$$c_1R^2 \leq N(\omega,R) \leq c_2R^2.$$

The Siegel-Veech formula computes the *mean* of $N(\omega,R)$,

$$\int_{\mathcal{H}} N(\omega,R)d\mu(\omega) = c_{SV}(\mu)\pi R^2.$$

Eskin and the third-named author [9] showed that for μ-almost every $\omega \in \mathcal{H}$,

$$\lim_{R\to\infty} \frac{N(\omega,R)}{\pi R^2} = c_{SV}(\mu).$$

1.5.1. *Error terms.* Our results can be viewed as showing that Siegel-Veech transforms have finite *variance*. Variance bounds in turn yield *concentration bounds*, bounding the probability of large discrepancy from the mean. Finally, combining concentration bounds with the Borel-Cantelli lemma yield almost everywhere error term bounds for $N(\omega,R)$. Suppose we write

$$L(R) = \|N(\omega,R)\|_2^2,$$

and let $e(R)$ denote an *error* function. Define

$$V(R) = L(R) - \|N(\omega,R)\|_1^2 = L(R) - c_{SV}(\mu)^2\pi^2R^4.$$

Theorem 1.3. *Let $R_k \to \infty$ be a sequence, and e be a function such that*

$$\sum_{k=1}^{\infty} \frac{V(R_k)}{e(R_k)^2} < \infty.$$

Then for μ-almost every $\omega \in \mathcal{H}$, there is a k_0 so that for all $k \geq k_0$

$$\left|N(\omega, R_k) - c_{SV}(\mu)\pi R_k^2\right| < e(R_k).$$

Currently, this theorem is ineffective, as we do not have an explicit computation of $L(R)$. Recently, Nevo-Ruhr-Weiss [17], using exponential mixing of Teichmüller flow, give error bounds with a power savings (along the full sequence).

1.6. **Organization of the paper.** In §2, we prove Theorem 1.1 via intermediate results Theorem 2.1 and Theorem 2.2, which prove the result for the indicator function of the unit disk and of a ball of radius R respectively. In §3, we prove Theorem 1.2 and discuss several explicit computations in special cases.

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2. Measure bounds and decompositions

In this section we prove Theorem 1.1, via first proving them for the indicator function of a small disk (Theorem 2.1) and then for the disk of radius R (Theorem 2.2).

2.1. **Tail bounds.** Given $f \in B_c(\mathbb{R}^2)$, to prove $\widehat{f} \in L^2(\mathcal{H}, \mu)$, we need to show that

$$\sum_{k=1}^{\infty} \mu(X, \omega) : \widehat{f}(X, \omega) > \sqrt{k}) < \infty. \tag{2.1}$$

2.2. **A fixed disc.** The first iteration of our main result is:

Theorem 2.1. *Fix a small ϵ_0 and let $f : \mathbb{R}^2 \to \mathbb{R}$ be the indicator function of the disc of radius ϵ_0. Then*

$$\widehat{f} \in L^2(\mathcal{H}, \mu).$$

Proof. We will divide the stratum $\mathcal{H}$ into a finite number of subsets $\mathcal{H}_i$ and prove

$$\int_{\mathcal{H}_i} \widehat{f}^2 d\mu < \infty$$

for each $\mathcal{H}_i$.

2.3. **Thick part.** We begin by noting that by definition in the thick part $\mathcal{H}_1 \subset \mathcal{H}$ where every saddle connection has length at least ϵ_0, then $\widehat{f}(X,\omega) = 0$.

2.4. **No short loops.** Next let $\mathcal{H}_2$ denote the set of (X,ω) for which there are saddle connections of length smaller than ϵ_0 but there are no homotopically nontrivial closed curves of length less than ϵ_0.

Take the L^1 Delaunay triangulation of (X,ω). There are no loops with length shorter than ϵ_0. By Theorem 1 of Chew [7] any saddle connection is homotopic to a path in the edges of the Delaunay triangulation of length at most a fixed multiple of the length of the saddle connection. Since there are no loops shorter than ϵ_0, any such path in the edges traverses successively at most a fixed number of edges shorter than ϵ_0 before traversing one of length at least ϵ_0. Thus a saddle connection of length at most 1 can be written as a union of at most $O(1/\epsilon_0)$ edges of the triangulation and therefore expressed in terms of a fixed basis for $H_1(X,\omega,\Sigma)$ as a linear combination with coefficients that are $O(1/\epsilon_0)$. Thus there are $O(1/\epsilon_0^N)$ saddle connections, where N is the dimension of $H_1(X,\omega,\Sigma)$. Since ϵ_0 is fixed, this is a bounded function and so again

$$\int_{\mathcal{H}_2} \widehat{f}^2 d\mu < \infty.$$

2.5. **Short loops.** Now we treat the case that there are short loops of length smaller than some fixed ϵ_0. Let $\mathcal{H}_3$ be the set of (X,ω) with a short curve of length at most ϵ_0. Next let N be the maximum number of edges in any triangulation of (X,ω) and set $p = \frac{2}{3N}$.

Choose $\delta > 0$ small enough so

$$\frac{1+p}{1+\delta} \geq 1 + \frac{1}{2N}$$

and

$$\frac{3}{2(1+\delta)} \geq \frac{4}{3}.$$

Let $|\gamma|$ be the length of shortest saddle connection on (X,ω), and again let $\widehat{f}(X,\omega)$ count the number of saddle connections that lie in a disc of radius ϵ_0. By Theorem 5.1 of [9], for some fixed ϵ_0, there is C (depending on δ but not $|\gamma|$) such that

$$\widehat{f}(X,\omega) \leq \frac{C}{|\gamma|^{1+\delta}}.$$

If $\widehat{f}(X,\omega) \geq \sqrt{k}$, then the above bound says there is $c > 0$ such that

$$|\gamma| \leq ck^{-\frac{1}{2(1+\delta)}}.$$

We break the set of $(X,\omega) \in \mathcal{H}_3$ such that $\widehat{f} \geq \sqrt{k}$ into three sets $\Omega_0(k) \cup \Omega_1(k) \cup \Omega_3(k)$. It suffices to prove

$$\sum_k \mu(\Omega_i(k)) < \infty$$

for each i. Let

$$\Omega_0(k) = \{(X,\omega) : \widehat{f}(X,\omega) \geq \sqrt{k} \text{ and there is flat cylinder homotopic to } \gamma\}.$$

We can assume Λ_ω contains a saddle connection crossing the cylinder. Then the height of the cylinder is at most ϵ_0. The shortest saddle connection β crossing the cylinder has a component in the direction of the cylinder of length at most $|\gamma|$ and an orthogonal component of length at most ϵ_0. If we include γ and β as part of a collection of saddle connections whose holonomy vectors (or period coordinates) define the Lebesgue measure μ we see

$$\mu(\Omega_0(k)) = O(|\gamma|^3) = O(k^{-\frac{3}{2(1+\delta)}}) = O(k^{-\frac{4}{3}}),$$

by the choice of δ. (We are citing [16] Theorem 10.1 for the first equality). Thus

$$\sum_k \mu(\Omega_0(k)) = \sum_k O(k^{-\frac{4}{3}}) < \infty.$$

Next let $\epsilon(X,\omega)$ be the length of the second shortest loop on (X,ω). Let

$$\Omega_1(k) = \{(X,\omega) \in \mathcal{H}_3 : \widehat{f}(X,\omega) \geq \sqrt{k} \text{ and } \epsilon(X,\omega) \leq |\gamma|^p\}.$$

Then since we have a saddle connection of length $|\gamma|$ and one of length $|\gamma|^p$ we have

$$\mu(\Omega_1(k)) = O(|\gamma|^{2+2p}) = O(k^{-\frac{1+p}{1+\delta}}) = O(k^{-1-\frac{1}{N}}),$$

by the choice of δ, p above, Thus

$$\sum_k \mu(\Omega_1(k)) < \infty.$$

Finally let $\Omega_2(k)$ be the set of surfaces (X,ω) with no cylinder homotopic to γ and

$$\widehat{f}(X,\omega) \geq \sqrt{k},\ \epsilon(X,\omega) > |\gamma|^p.$$

Since there is no cylinder any saddle connection when written as a composition of edges in the Delaunay triangulation does not follow the circumference of γ multiple times as it might if there were a cylinder. Then by the exact same argument as before when considering Delaunay triangulation,

$$\widehat{f}(X,\omega) = O\left(\epsilon^{-N}\right) = O\left(|\gamma|^{-Np}\right).$$

For this to be bigger than $\sqrt{k}$ have

$$|\gamma| = O\left(k^{-\frac{1}{2Np}}\right) = O\left(k^{-\frac{3}{4}}\right),$$

by the choice of p. Thus

$$\mu(\Omega_2(k)) = O(|\gamma|^2) = O\left(k^{-\frac{3}{2}}\right),$$

and again we have

$$\sum_k \mu(\Omega_2(k)) < \infty.$$

□

Theorem 2.2. *Suppose f is the characteristic function of a disc of radius R. Then $\widehat{f} \in L^2(\mathcal{H},\mu)$.*

Proof. We cover the disc of radius R with sectors of angle $\frac{\epsilon_0^2}{R^2}$. It is enough to show that for f the characteristic function of any of these sectors the function $\widehat{f}$ is in $L^2(\mathcal{H},\mu)$. Let θ_0 the center angle of this sector. Let $t_0 = \log\frac{R}{\epsilon_0}$. Let (X,ω) any translation surface and consider $(Y,\omega') = g_{t_0} r_{-\theta_0}(X,\omega)$. That is, we rotate so direction θ_0 is vertical and flow time t_0. Then since the angle is $\frac{\epsilon_0^2}{R^2}$, every saddle connection of (X,ω) in that sector has length at most ϵ_0 on (Y,ω'). Let h be the characteristic function of the disc of radius ϵ_0. Then since the flow is μ measure preserving,

$$\int \widehat{f}^2(X,\omega)d\mu(X) \leq \int \widehat{h}^2(Y,\omega')d\mu(Y) < \infty.$$

□

2.6. **Proof of Theorem 1.1.** Let $f \in B_c(\mathbb{R}^2)$. then there is an $R > 0$ so that the support of f is contained in $B(0, R)$, and letting $C = \max f$, we have

$$f \leq C\chi_{B(0,R)},$$

so

$$\widehat{f} \leq C\widehat{\chi}_{B(0,R)}.$$

Applying Theorem 2.2, we have our result. □

Remark: We remark that Wright [27] showed that any (X, ω) in a rank 1 orbit closure in $\mathcal{H}$ is completely periodic. This means that for any direction with a cylinder γ, the surface can be written as a union of cylinders in that direction, and there are always such cylinder directions (actually a dense set). Thus the set of (X, ω) with $\widehat{f} \geq \sqrt{k}$ coincides with the set $\Omega_0(k)$. Nguyen in Proposition 4.3 [18] proved that for any ergodic $SL(2, \mathbb{R})$ invariant measure ν on a rank 1 orbit closure, and any such cylinder γ,

$$\nu(\Omega_0(k)) = O(|\gamma|^3).$$

Together with the discussion in the cylinder case in the proof of Theorem 2.1, this gives that $\widehat{f} \in L^2(\nu)$ for any such ν.

3. Siegel-Veech measures

In this section, we prove Theorem 1.2, and give examples of the resulting Siegel-Veech measures in some special cases.

3.1. **Transforms and bounds.** Let τ denote an $SL(2, \mathbb{R})$ invariant measure on a stratum $\mathcal{H}$ of abelian differentials, and suppose that for any $f \in B_c(\mathbb{R}^2)$, $\widehat{f} \in L^2(\mathcal{H}, \tau)$. Then, for any $h \in B_c(\mathbb{R}^4)$, $\widehat{h} \in L^1(\mathcal{H}, \tau)$, since we can dominate

$$\widehat{h}(\omega) = \sum_{v_1, v_2 \in \Lambda_\omega} h(v_1, v_2)$$

by $\widehat{f}^2$ where $f = \max(h)\chi_H$, where H denotes the union of the projections of the support of h via the maps $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$

$$(x, y) \longmapsto x \text{ and } (x, y) \longmapsto y.$$

3.2. **Haar measures.** Our claim in Theorem 1.2 is that there are *Siegel-Veech measures* $\nu = \nu(\tau)$ on $\mathbb{R}\backslash\{0\}$ and $\eta = \eta(\tau)$ on $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \infty$ such that

$$\begin{aligned}\int_{\mathcal{H}} \widehat{h}(\omega) d\mu(\omega) &= \int_{\mathbb{R}\backslash\{0\}} \left(\int_{SL(2,\mathbb{R})} h(tx, y) d\kappa(x, y) \right) d\nu(t) \\ &+ \int_{\mathbb{P}^1(\mathbb{R})} \left(\int_{\mathbb{R}^2} h(x, sx) dx \right) d\eta(s).\end{aligned}$$

Here, κ is Haar measure on $SL(2,\mathbb{R})$ (with a fixed normalization), and the integral over $SL(2,\mathbb{R})$ is taken over pairs $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ with $\det(x, y) = 1$, that is, we view $SL(2,\mathbb{R})$ as a subset of $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$.

This follows by the $SL(2,\mathbb{R})$-invariance of μ and the classification of $SL(2,\mathbb{R})$-orbits on $\mathbb{R}^4$. By the invariance of τ, and our integrability condition, we have that

$$h \longmapsto \int_{\mathcal{H}} \widehat{h}(\omega) d\mu(\omega)$$

is a $SL(2,\mathbb{R})$-invariant linear functional on $B_c(\mathbb{R}^4)$, the set of bounded compactly supported functions on $\mathbb{R}^4$. Therefore, there is an $SL(2,\mathbb{R})$-invariant measure $m = m(\tau)$ (a *Siegel-Veech measure*) on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 = M_2(\mathbb{R})$ so that

$$\int_{\mathcal{H}} \widehat{h}(\omega) d\mu(\omega) = \int_{\mathbb{R}^4} h dm$$

3.3. $SL(2,\mathbb{R})$**-invariant measures on** $\mathbb{R}^4$**.** To describe $SL(2,\mathbb{R})$-invariant measures on $\mathbb{R}^4$, we need to understand $SL(2,\mathbb{R})$-orbits on $\mathbb{R}^4$. For $t \in \mathbb{R}$, let

$$D_t = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \det(x, y) = t\}$$

For $t \neq 0$, D_t is an $SL(2,\mathbb{R})$-orbit. D_0 decomposes further. For $s \in \mathbb{P}^1(\mathbb{R})$, let

$$L_s = \{(x, sx) : x \in \mathbb{R}^2, x \neq 0\},$$

with

$$L_\infty = \{(0, y) : y \in \mathbb{R}^2, y \neq 0\}$$

3.3.1. *Orbits and measures.* D_t and L_s are the non-trivial $SL(2,\mathbb{R})$ orbits on $\mathbb{R}^2 \times \mathbb{R}^2$, and each carries a unique (up to scaling) $SL(2,\mathbb{R})$-invariant measure. These are the (non-atomic) ergodic invariant measures for $SL(2,\mathbb{R})$ action on $\mathbb{R}^2 \times \mathbb{R}^2$. On D_t, the measure is Haar measure on $SL(2,\mathbb{R})$, and on L_s it is Lebesgue on $\mathbb{R}^2$. Thus, associated to any $SL(2,\mathbb{R})$ invariant measure m on $\mathbb{R}^4$ we have measures $\nu = \nu(m)$ and $\eta = \eta(m)$ so that

$$\begin{aligned}\int_{\mathbb{R}^4} hdm &= \int_{\mathbb{R}\setminus\{0\}} \left(\int_{SL(2,\mathbb{R})} h(tx,y)d\kappa(x,y) \right) d\nu(t) \\ &+ \int_{\mathbb{P}^1(\mathbb{R})} \left(\int_{\mathbb{R}^2} h(x,sx)dx \right) d\eta(s).\end{aligned}$$

3.4. **Siegel-Veech measures from measures on strata.** Putting $\nu(\tau) = \nu(m(\tau))$ and $\eta(\tau) = \eta(m(\tau))$, we have our *Siegel-Veech measures.* These measures are interesting invariants associated to $SL(2,\mathbb{R})$-invariant measures τ on $\mathcal{H}$. A natural question is:

Question. *Let μ denote Lebesgue measure on the stratum $\mathcal{H}$. What are the spectral measures $\nu(\mu)$ and $\eta(\mu)$?*

3.4.1. *Virtual Triangles.* Smillie-Weiss [23] introduced the notion of *virtual triangles* on a surface. A virtual triangle is simply a pair of (distinct) saddle connections, and the *area* of a virtual triangle is the (absolute value of the) determinant of the matrix given by the holonomy vectors.

They showed that there is a positive lower bound on the area of virtual triangles on the surface ω if and only if the surface ω is an *lattice surface*, that is, its stabilizer $SL(X,\omega)$ under $SL(2,\mathbb{R})$ is a lattice. In this case the $SL(2,\mathbb{R})$ orbit is *closed*, and the Haar measure τ on $SL(2,\mathbb{R})/SL(X,\omega)$ is finite.

This condition, known as *no small virtual triangles* (NSVT) can be summarized as saying that $\nu(\tau)$ has no support in a neighborhood of 0. More generally, given an arbitrary $SL(2,\mathbb{R})$-invariant measure τ, the support of $\nu(\tau)$ is the collection of virtual triangle areas for surfaces ω in the support of τ.

3.5. **Lattice surfaces and covering loci.** For some lattice surfaces and loci of covers, we have examples where we can compute these measures explicitly.

3.5.1. *Flat tori.* For

$$\mathcal{H}(\emptyset) = SL(2,\mathbb{R})/SL(2,\mathbb{Z}),$$

the moduli space of abelian differentials on flat tori, these measures were implicitly computed by Schmidt [21], see [3] for an explicit computation with full proofs (which correct a mistake in a paper of Rogers [20]).

Precisely, if the Haar measure κ on $SL(2,\mathbb{R})$ is normalized so that

$$\kappa(SL(2,\mathbb{R})/SL(2,\mathbb{Z})) = \zeta(2),$$

we have that

$$\nu = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{\zeta(2)}\phi(n)\delta_n$$

and

$$\eta = \delta_1 + \delta_{-1}.$$

3.5.2. *Covering loci and affine lattices.* In the stratum $\mathcal{H}(1,1)$ we have the subvariety $\mathcal{V}$ of two identical tori glued along a slit. Equivalently, these are double covers of a flat torus branched over two points. We can thus identify $\mathcal{V}$ with $\mathcal{H}(0,0)$, the space of tori with two marked points (the slit is a segment connecting the two points). In turn, we have

$$\mathcal{H}(0,0) = ASL(2,R)/ASL(2,\mathbb{Z}),$$

where

$$ASL(2,\mathbb{R}) = SL(2,\mathbb{R}) \ltimes \mathbb{R}^2, ASL(2,\mathbb{Z}) = SL(2,\mathbb{Z}) \ltimes Z^2.$$

We will identify $\omega \in \mathcal{V}$ with cosets $[g,v]$, $g \in SL(2,\mathbb{R})$, $v \in \mathbb{R}^2/g\mathbb{Z}^2$. We have [6] that

$$\Lambda_{[g,v]} = g\mathbb{Z}^2_{\text{prim}} \cup (g\mathbb{Z}^2 + v),$$

where $\mathbb{Z}^2_{\text{prim}}$ is the set of *primitive* integer vectors. Thus, we can break the Siegel-Veech measures up into the measures associated to each piece. For the first piece $g\mathbb{Z}^2_{\text{prim}}$, the computation in §3.5.1 yields the measures, and for the second, these were computed in [1].

3.5.3. *Lattice surfaces.* More generally, for lattice surfaces ω, it seems possible to use the fact that there are vectors $v_1, \dots v_k \in \mathbb{R}^2$ so that

$$\Lambda_\omega = \bigcup_{j=1}^{k} SL(X,\omega)v_j$$

to turn to algebraic techniques to compute these measures, which will depend on the action of $SL(X,\omega)$ on $\mathbb{R}^2 \times \mathbb{R}^2$.

4. L^2 bounds and error terms

4.1. **Notation.** We prove Theorem 1.3. Recall we write

$$L(R) = \|N(\omega,R)\|_2^2,$$

and

$$V(R) = L(R) - \|N(\omega,R)\|_1^2 = L(R) - c_{SV}(\mu)^2\pi^2R^4.$$

4.2. **Expectation and variance.** Then

$$\begin{aligned}\mu\left(\omega \in \mathcal{H} : \left|N(\omega, R) - c_{SV}(\mu)\pi R^2\right| > e(R)\right) &= \\ \mu\left(\omega \in \mathcal{H} : \left|N(\omega, R) - c_{SV}(\mu)\pi R^2\right|^2 > e(R)^2\right) &\leq \\ \frac{\int_{\mathcal{H}} \left|N(\omega, R) - c_{SV}(\mu)\pi R^2\right|^2 d\mu}{e(R)^2} &= \frac{V(R)}{e(R)^2}.\end{aligned}$$

4.3. **Borel-Cantelli.** Theorem 1.3 then follows from applying the easy part of the Borel-Cantelli lemma to the sequence of sets

$$A_k = \{\omega \in \mathcal{H} : \left|N(\omega, R) - c_{SV}(\mu)\pi R^2\right| > e(R_k).$$

□

References

[1] J. S. Athreya, *Random Affine Lattices*, Contemporary Mathematics, volume 639, 160-174, 2015.

[2] J. S. Athreya and J. Chaika, *The distribution of gaps for saddle connection directions.* Geometric and Functional Analysis,Volume 22, Issue 6, 1491-1516, 2012.

[3] J. S. Athreya and S. Fairchild, in preparation.

[4] J. S. Athreya and G. A. Margulis, *Logarithm laws for unipotent flows, I*, Journal of Modern Dynamics, volume 3, number 3, pages 359-378, July 2009.

[5] J. S. Athreya and G. Margulis, *Values of Random Polynomials at Integer Points*, preprint.

[6] Y. Cheung, P. Hubert and H. Masur, *Dichotomy for the Hausdorff dimension of the set of nonergodic directions.* Invent. Math. 183 (2011), no. 2, 337-383.

[7] P. Chew, *There is a Planar Graph Almost as Good as the Complete Graph*, Proceedings of 2nd Symposium on Computational Geometry, Yorktown Heights, NY, 1986.

[8] A. Eskin, *Counting Problems in moduli space.* Handbook of dynamical systems. Vol. 1B, 581–595, Elsevier B. V., Amsterdam, 2006.

[9] A. Eskin and H. Masur, *Asymptotic Formulas on Flat Surfaces*, Ergodic Theory and Dynam. Systems, v.21, 443-478, 2001.

[10] A. Eskin, M. Mirzakhani, and A. Mohammadi, *Isolation, equidistribution, and orbit closures for the $SL(2,\mathbb{R})$ action on moduli space*, Volume 182, 673-721 Volume 182 (2015).

[11] A. Eskin, M. Mirzakhani and K. Rafi, *Counting geodesics in a stratum.* To appear, Inventiones Mathematicae.

[12] S. Kerckhoff, H. Masur and J. Smillie, *Ergodicity of billiard flows and quadratic differentials.* Ann. of Math. (2) 124 (1986), no. 2, 293-311.

[13] M. Kontsevich, A. Zorich, *Connected components of the moduli spaces of Abelian differen- tials with prescribed singularities*, Invent. Math., 153 (2003), no.3, 631-678.

[14] H. Masur, *The growth rate of trajectories of a quadratic differential*, Ergodic Theory Dynam. Systems 10 (1990), no. 1, 151–176.

[15] H. Masur, *Interval exchange transformations and measured foliations*, Ann. of Math. (2) 115 (1982), no. 1, 169–200.
[16] H. Masur and J. Smillie, *Hausdorff dimension of sets of nonergodic measured foliations.* Annals of Mathematics, (2)134 (1991), no. 3, 455-543.
[17] A. Nevo, R. Ruhr, and B. Weiss, *Effective counting on translation surfaces*, preprint, `arXiv:1708.06263`
[18] D. Nguyen, *Volume of the set of surfaces with small saddle connection in rank one affine manifolds*, preprint, `arXiv:1211.7314`
[19] K. Rafi Hyperbolicity in Teichmller space. Geometry Topology 18-5 (2014) 3025–3053.
[20] C. A. Rogers, *The number of lattice points in a set*, Proc. London Math. Soc. (3) 6 (1956), 305–320.
[21] W. Schmidt, A metrical theorem in geometry of numbers. Transactions of the American Mathematical Society (1960), pp. 516-529.
[22] C. L. Siegel, *A mean value theorem in geometry of numbers*, Ann. Math. 46, 340–347 (1945).
[23] J. Smillie and B. Weiss, *Characterizations of lattice surfaces*, Invent. Math. 180 (2010), no. 3, 535–557.
[24] W. Veech, *Siegel Measures*, Annals of Mathematics, 148, (1998), 895-944.
[25] W. Veech, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. math. 97, 553-583 (1989).
[26] W. A. Veech. Gauss measures for transformations on the space of interval exchange maps. *Ann. of Math. (2)*, 115(1):201–242, 1982.
[27] A. Wright Cylinder deformations in orbit closures of translation surfaces, *Geometry Topology* (19) 413-438, 2015.
[28] A. Zorich, *Flat surfaces*, Frontiers in number theory, physics, and geometry. I, 437–583, Springer, Berlin, 2006.

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