

# The shape of a generic translation surface

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A translation structure equips a Riemann surface with a singular flat metric. Not much is known about the shape of a generic translation surface. We consider the stratum  $\mathcal{H}_1(2g - 2)$  of translation surfaces of genus  $g$  with one singularity and show that the expected diameter of a surface is bounded above by a uniform multiple of  $\sqrt{\frac{\log g}{g}}$ . This is smaller than what one would expect by analogy from the result of Mirzakhani about the expected diameter of a hyperbolic metric on a Riemann surface. In fact, more generally, we compute the expected value of the covering radius of a translation surface in any stratum  $\mathcal{H}_1(\kappa)$ . To prove our result, we need an estimate for the volume of the thin part of  $\mathcal{H}_1(\kappa)$  which is given in the appendix.

## 1 Introduction

Translation surfaces have been studied in depth for many years. However, there is no clear picture for the shape of a generic translation surface. The goal of this paper is to study the asymptotic growth rate of the diameter of a generic translation surface in the minimal stratum. A motivation for this study is a paper of Mirzakhani [Mir13] in which she computed the expected value of several geometric functions (such as systole, Cheeger constant, etc.) on  $\mathcal{M}_g$ , the moduli space of Riemann surfaces of genus  $g$ , equipped with the Weil–Petersson volume measure  $\nu_{\text{wp}}$ . For example, she proved that the expected value of the diameter of a generic hyperbolic surface of genus  $g$  grows like  $\log g$  as  $g \rightarrow \infty$ . Specifically

$$\mathbb{E}_{\mathcal{M}_g}(\text{diam}) = \frac{\int_{\mathcal{M}_g} \text{diam} \, d\nu_{\text{wp}}}{\text{Vol}_{\text{wp}}(\mathcal{M}_g)} \asymp \log g$$

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where  $\asymp$  means that the two sides are equal up to uniform multiplicative constants that are independent of  $g$ .

The space of all translation surfaces is naturally stratified by the number and the type of the singularities they can have and the expected shape of a translation surface may be different depending on the stratum.

To get some ideas about possible shapes, we consider three different approaches to understand a generic translation surface: one by explicit construction, one by thick-thin decomposition, and one by comparison to hyperbolic geometry. For the first approach, we take a  $4g$ -sided polygon of area 1 and diameter comparable to 1 with opposite sides that are parallel and of the same length. When we identify these opposite edges by translations, we obtain a translation surface whose diameter is also comparable to 1. By varying the lengths and the directions of the edges slightly, we obtain an open subset of some stratum  $\mathcal{H}_1(\kappa)$  of translation surfaces with diameter comparable to 1. One may wonder whether this is a good model for a generic translation surface.

As a second consideration, one can think of a translation surface as a union of *thick pieces* (containing no curves of small extremal length) connected to each other by potentially very long cylinders (see [Raf07]). One may then wonder how large a cylinder can be in a typical surface. So, a priori, the average diameter could go to infinity with genus.

A third idea would be to translate the result of Mirzakhani directly. Namely, a hyperbolic surface  $x$  of genus  $g$  has an area comparable to  $g$ . To make  $x$  have area 1, one needs to scale  $x$  down by a factor comparable to  $\frac{1}{\sqrt{g}}$ . Then, the result of Mirzakhani would suggest that the expected value of the diameter should be comparable to  $\frac{\log g}{\sqrt{g}}$ .

However, the answer we find is different from all these models. Let  $\nu$  be the normalized Lebesgue measure on  $\mathcal{H}_1(2g-2)$  as in [Mas82, Vee82].

**Theorem A (Expected diameter)**

*For large values of  $g$ , we have*

$$\mathbb{E}_{\mathcal{H}_1(2g-2)}(\text{diam}) = \frac{\int_{\mathcal{H}_1(2g-2)} \text{diam}(X) d\nu(X)}{\nu(\mathcal{H}_1(2g-2))} \leq 20 \cdot \sqrt{\frac{\log g}{g}}.$$

This shows in particular that, unlike the first suggested model, the expected value of the diameter goes to zero as  $g \rightarrow \infty$ . In fact, it is even smaller than what you would get from scaling a hyperbolic surface (by a factor  $\frac{1}{\sqrt{g}}$ ) to have area 1. So, the situation is different from that of hyperbolic surfaces.

Note that for  $(X, \omega) \in \mathcal{H}(2g-2)$ , the diameter of  $X$  equals (up to a factor of 2) the maximum distance from a point in  $X$  to the singularity of  $(X, \omega)$ . This number, which also equals to the maximum radius of an immersed disk in  $(X, \omega)$ , is called the *covering radius* of  $(X, \omega)$ . We denote the covering radius of  $(X, \omega)$  by  $\text{c-rad}(X, \omega)$ . Theorem A can be interpreted as estimating the expected value of the covering radius of a translation surface

in  $\mathcal{H}_1(2g-2)$ . In fact, we can obtain such estimates for every stratum. Let  $\kappa = (k_1, \dots, k_\ell)$  be a tuple of positive integers and let  $\mathcal{H}_1(\kappa)$  be the stratum of unit area translation surfaces with  $\ell$  singularities of degrees  $k_1, \dots, k_\ell$ .

**Theorem B (Expected covering radius)**

Let  $\mathcal{H}(\kappa)$  be a stratum of translation surfaces of genus  $g$ . Then, for large values of  $g$ , we have

$$\mathbb{E}_{\mathcal{H}_1(\kappa)}(\text{c-rad}) = \frac{\int_{\mathcal{H}_1(\kappa)} \text{c-rad}(X) d\nu(X)}{\nu(\mathcal{H}_1(\kappa))} \leq 20 \cdot \sqrt{\frac{\log g}{g}}.$$

It would be interesting to compute the expected value of the diameter of surfaces in  $\mathcal{H}_1(\kappa)$  but that can not be achieved with our current methods. At the moment, it is not even clear that the expected value of the diameter in every stratum goes to zero as the genus of the underlying surface goes to infinity.

In contrast with Theorem A and Theorem B, we have the following absolute lower bound for the covering radius of any elements in  $\mathcal{H}_1(\kappa)$ .

**Proposition C (Lower bound on diameter)**

For every  $(X, \omega) \in \mathcal{H}_1(\kappa)$ , we have

$$\text{c-rad}(X) \geq \sqrt{\frac{2}{3\sqrt{3} \cdot (2g + \ell - 2)}}.$$

*Proof.* Let  $\kappa = (k_1, \dots, k_\ell)$  and let  $(X, \omega) \in \mathcal{H}_1(\kappa)$ . Consider a Delaunay triangulation of  $(X, \omega)$ . The number of triangles is  $2(2g + \ell - 2)$ . Hence, the area of the largest triangle is greater than or equal to  $\frac{1}{2(2g + \ell - 2)}$ . This triangle is inscribed in a circle of radius at least  $\sqrt{\frac{2}{3\sqrt{3} \cdot (2g + \ell - 2)}}$ . As the triangulation is Delaunay, the corresponding disk is an immersed Euclidean disk in  $X$ . Hence its radius is a lower bound for the covering radius of the translation surface.  $\square$

*Remark 1.1 (Non-connectedness of  $\mathcal{H}_1(\kappa)$ ).* The stratum  $\mathcal{H}_1(\kappa)$  is not always connected. For  $\kappa = (k_1, \dots, k_\ell)$ , when every  $k_i$  is even, there are different components corresponding to even and odd spin structure. Also,  $\mathcal{H}_1(2g-2)$  and  $\mathcal{H}_1(g-1, g-1)$  have a component consisting of hyperelliptic surfaces. So, a stratum may have up to three connected components (see [KZ03, Theorem 1] for exact statement). In our formula for the expected value of the diameter or the covering radius, we will not consider each component separately. In fact, it is known that the volume of the hyperelliptic component of  $\mathcal{H}_1(2g-2)$  is approximately of order  $(2g)^{-2g}$  [AEZ16, Theorem 1.1]. So we cannot even conclude that the expected value of the diameter for the hyperelliptic component goes to zero as  $g \rightarrow \infty$ . It was recently shown that the components associated to even and odd spin structures have asymptotically equal volumes as  $g \rightarrow \infty$  [CMS]. Hence, the statement of Theorem A does hold for these two components with a different constant.

We now outline the proof of Theorem B which immediately implies Theorem A. For every  $(X, \omega) \in \mathcal{H}_1(\kappa)$ , we find either an embedded disk or a cylinder that approximates the covering radius. When there exists a large embedded disk, we take out a parallelogram whose area is proportional to the area of the disk and glue the opposite sides to build a new translation surface. The resulting translation surface is in the stratum  $\mathcal{H}(\kappa, 2)$  and its area is smaller than  $(X, \omega)$  by a definite amount. We then renormalize this translation surface to have unit area. We call this process *taxing*. Because of the renormalization process, the Jacobian of the taxing map is very large but the volume of  $\mathcal{H}(\kappa)$  and  $\mathcal{H}(\kappa, 2)$  are comparable. Hence, the volume of the subset of  $\mathcal{H}(\kappa)$  where there is a large embedded disk is small. This allows us to show that the integral of the covering radius on these sets is small.

When there is a cylinder of large height, then either the cylinder has large area or a small circumference. In these cases, we bound the measure of the set of translation surfaces that have such cylinders by bounding the associated Siegel–Veech constant. Namely, for given length  $\delta > 0$ , area  $A \in [0, 1)$  and a translation surface  $(X, \omega) \in \mathcal{H}_1(\kappa)$ , let  $N_{\text{cyl}}(X, \delta, A)$  be the number of cylinders in  $X$  where the circumference is at most  $\delta$  and the area is at least  $A$ .

**Theorem D (Expected number of cylinders)**

*For large values of  $g$ , we have*

$$\mathbb{E}_{\mathcal{H}_1(\kappa)}(N_{\text{cyl}}(\cdot, \delta, A)) = \frac{\int_{\mathcal{H}_1(\kappa)} N_{\text{cyl}}(X, \delta, A) d\nu(X)}{\nu(\mathcal{H}_1(\kappa))} \leq 30\pi g \cdot \delta^2 \cdot (1 - A)^{2g+\ell-3}.$$

*In particular, for  $\mathcal{H}_{\text{thin-cyl}}(\delta, A)$  the set of translation surfaces  $(X, \omega) \in \mathcal{H}_1(\kappa)$  for which  $N_{\text{cyl}}(X, \delta, A)$  is not zero, we have*

$$\frac{\nu(\mathcal{H}_{\text{thin-cyl}}(\delta, A))}{\nu(\mathcal{H}_1(\kappa))} \leq 30\pi g \cdot \delta^2 \cdot (1 - A)^{2g+\ell-3}.$$

For  $A = 0$ , this is analogous to the estimate given by Mirzakhani [Mir13, Theorem 4.2] for the Weil–Petersson volume of the set  $\mathcal{M}_g^\delta$  of Riemann surfaces of genus  $g$  with at least one closed curve of length less than or equal to  $\delta$ . Namely

$$\frac{\text{Vol}_{\text{wp}}(\mathcal{M}_g^\delta)}{\text{Vol}_{\text{wp}}(\mathcal{M}_g)} \asymp \delta^2.$$

In the setting of translation surfaces, another notion of thin part is the set of translation surfaces that have a short saddle connection. In fact, for a stratum  $\mathcal{H}(\kappa)$  of translation surfaces and  $\mathcal{H}_{\text{thin}}(\delta)$  the set of translation surfaces in  $\mathcal{H}(\kappa)$  that have a saddle connection of length at most  $\delta$ , Masur and Smillie showed (compare equation (7) in the proof of Theorem 10.3 in [MS91])

$$\nu(\mathcal{H}_{\text{thin}}(\delta)) = O(\delta^2).$$

However, the dependence of the constant on the genus or more generally on the stratum was not known. For completion, we also find an estimate for the number  $N_{\text{sc}}(X, \delta)$  of saddle connections of length at most  $\delta$  in  $X$ .

**Theorem E (Expected number of saddle connections)**

For large values of  $g$ , we have

$$\mathbb{E}_{\mathcal{H}_1(\kappa)}(N_{\text{sc}}(\cdot, \delta)) = \frac{\int_{\mathcal{H}_1(\kappa)} N_{\text{sc}}(X, \delta) d\nu(X)}{\nu(\mathcal{H}_1(\kappa))} \leq 90\pi g^2 \cdot \delta^2.$$

In particular, for  $\mathcal{H}_{\text{thin}}(\delta)$  the set of translation surfaces  $(X, \omega) \in \mathcal{H}_1(\kappa)$  for which  $N_{\text{sc}}(X, \delta)$  is not zero, we have

$$\frac{\nu(\mathcal{H}_{\text{thin}}(\delta))}{\nu(\mathcal{H}_1(\kappa))} \leq 90\pi g^2 \cdot \delta^2.$$

*Remark 1.2 (Siegel–Veech constants and non-connectedness).* Theorem D and Theorem E are proven in the appendix and in the proof, we make use of Siegel–Veech constants. Explicit formulas for values of various Siegel–Veech constants were computed by Eskin–Masur–Zorich in [EMZ03] in terms of combinatorial data and volumes of related strata with lower complexity. However, their methods give a precise answer only for connected strata. Hence, we do not compute exact values, rather we find upper bounds for Siegel–Veech constants that suffice for our purposes. Recently, Aggarwal in [Agg18b] used the recursive formula for volume of strata given by Eskin–Okounkov [EO01] to compute the asymptotic growth rate of these volumes (see also [CMZ18] for the principal stratum and [Sau18] for the minimal stratum).

One should also compare Theorem D and Theorem E to computations for values of various Siegel–Veech constants given in the appendix in [Agg18b] written by Anton Zorich. For example, Theorem D is very similar to [Agg18b, Corollary 5]. The difference is that in [Agg18b, Corollary 5], only saddle connections bounding a cylinder of multiplicity 1 are counted. However, higher multiplicity saddle connections do not pose a problem; this has been made precise by Aggarwal in [Agg18a]. Also, the assumption on area being at least  $A$  contributes a factor of  $(1 - A)^{2g-2}$  to the estimate which is consistent with a result of Vorobets [Vor05, Theorem 1.8]. Hence, Theorem D and Theorem E essentially follow from a combination of these results. However, we write a details proof in the case of  $\mathcal{H}_1(2g - 2)$ , namely, we show that by a careful reading of [EMZ03] and incorporating the estimates given in [Agg18b] one can obtain these theorems.

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## 2 Four types of translation surfaces in $\mathcal{H}_1(\kappa)$

For this paper, a *translation surface*  $(X, \omega)$  is defined by a compact connected Riemann surface  $X$ , a finite set  $\Sigma \subseteq X$ , and a translation structure on  $X \setminus \Sigma$ , i.e. a maximal atlas on  $X \setminus \Sigma$  such that the transition maps are translations. The second parameter  $\omega$  refers to the unique Abelian differential that is associated to a given translation structure on  $X \setminus \Sigma$ . The elements of  $\Sigma$  correspond to zeros of  $\omega$  and are called *singularities* of  $(X, \omega)$ . Every singularity  $\sigma \in \Sigma$  is a cone point of the translation structure with *cone angle*  $2\pi(k+1)$  where  $k$  is the order of  $\sigma$  as a zero of  $\omega$ . The total sum of the orders of the zeros is equal to  $2g-2$  where  $g$  is the genus of  $X$ . The translation structure defines also a metric  $d$  on  $X$ . With this metric, the *diameter* of  $X$  is defined to be  $\text{diam}(X) := \max\{d(x, y) : x, y \in X\}$  and the *covering radius* of  $X$  is defined to be  $\text{c-rad} := \max\{d(x, \sigma) : x \in X, \sigma \in \Sigma\}$ . See [Str84] and [Zor06] for background information on translation surfaces.

Given a partition of  $2g-2$  as a sum of integers  $k_i \geq 1$ , the *stratum*  $\mathcal{H}(k_1, \dots, k_\ell)$  is defined to be the set of all translation surfaces  $(X, \omega)$  of genus  $g$  with  $\ell$  singularities of orders  $k_1, \dots, k_\ell$ . The subsets of translation surfaces of area 1 and area at most 1 in  $\mathcal{H}(k_1, \dots, k_\ell)$  are denoted by  $\mathcal{H}_1(k_1, \dots, k_\ell)$  and  $\mathcal{H}_{\leq 1}(k_1, \dots, k_\ell)$ , respectively.

A *saddle connection* of  $(X, \omega)$  is a geodesic segment from one singularity of  $(X, \omega)$  to another (not necessarily different) singularity that is disjoint from  $\Sigma$  in its interior. The translation structure of  $(X, \omega)$  associates a vector in  $\mathbb{C}$ , called the *holonomy vector*, to a given oriented saddle connection.

A saddle connection can also be thought of as an element of the relative homology group of  $(X, \omega)$  relative to  $\Sigma$ . If  $\mathcal{B}$  is a set of saddle connections that form a basis for the relative homology, then the holonomy vectors of the elements of  $\mathcal{B}$  determine  $(X, \omega)$ . For every  $(X, \omega)$  and  $\mathcal{B}$ , there is a neighborhood  $U$  of  $(X, \omega)$  in  $\mathcal{H}(k_1, \dots, k_\ell)$  such that for every  $(X', \omega')$  in  $U$ , all elements of  $\mathcal{B}$  (thought of as elements in the relative homology group) can still be represented in  $(X', \omega')$  as saddle connections. Then the set of holonomy vectors of saddle connections in  $\mathcal{B}$  give coordinates for points in  $U$ . We refer to this set of holonomy vectors as *period coordinates* for  $\mathcal{H}(k_1, \dots, k_\ell)$  around  $(X, \omega)$ . (See [Mas82] for details.)

The period coordinates give an embedding from  $U$  to  $\mathbb{C}^{2g+l-1}$ . The associated pullback measure in  $\mathcal{H}(k_1, \dots, k_\ell)$  is called the *normalized Lebesgue measure* denoted by  $\nu$  and was studied by Masur [Mas82] and Veech [Vee82]. This also defines a measure on  $\mathcal{H}_1(k_1, \dots, k_\ell)$  in the following way. For an open set  $V \subseteq \mathcal{H}_1(k_1, \dots, k_\ell)$ , the measure is defined to be  $\nu(U)$  where  $U$  is the cone over  $V$  of translation surfaces of area at most 1. We abuse notation

and denote the measure on  $\mathcal{H}_1(k_1, \dots, k_\ell)$  also by  $\nu$ .

In the following, let  $\kappa = (k_1, \dots, k_\ell)$  be a fixed partition of  $2g - 2$  and  $\mathcal{H}(\kappa)$  the corresponding stratum. We show the main theorem by dividing the stratum  $\mathcal{H}_1(\kappa)$  into four (not necessarily disjoint) parts and investigate the behaviour of the expected value of the covering radius separately for every part.

**Definition 2.1** ( $\mathcal{H}_{\text{small-diam}}$ ,  $\mathcal{H}_{\text{poor-cyl}}$ ,  $\mathcal{H}_{\text{rich-cyl}}$ , and  $\mathcal{H}_{\text{rich-disk}}$ )

For  $g \geq 2$ , define the following four subsets of  $\mathcal{H}_1(\kappa)$ :

- Let  $\mathcal{H}_{\text{small-diam}}$  be the subset of points  $(X, \omega)$  where we have  $\text{c-rad}(X) < 18 \cdot \sqrt{\frac{\log g}{g}}$ .
- Let  $\mathcal{H}_{\text{poor-cyl}}$  be the subset of points  $(X, \omega)$  where  $\text{c-rad}(X) \geq \frac{1}{\sqrt{g}}$  and where there exists a cylinder  $C(X)$  of height at least  $\text{c-rad}(X)$  and  $\text{area}(C(X)) \leq \frac{1}{g}$ . We call this the poor cylinder case.
- Let  $\mathcal{H}_{\text{rich-cyl}}$  be the subset of points  $(X, \omega)$  where  $\text{c-rad}(X) \geq \frac{1}{\sqrt{g}}$  and where there exists a cylinder  $C(X)$  of height at least  $\text{c-rad}(X)$  and  $\text{area}(C(X)) \geq \frac{1}{g}$ . We call this the rich cylinder case.
- Let  $\mathcal{H}_{\text{rich-disk}}$  be the subset of points  $(X, \omega)$  where  $\text{c-rad}(X) \geq 18 \cdot \sqrt{\frac{\log g}{g}}$  and where there exists an embedded disk  $D(X)$  of diameter at least  $\text{c-rad}(X)$ . We call this the rich disk case.

Note that for a given translation surface  $(X, \omega)$ , we can be in the cylinder case and in the rich disk case. Moreover, the choice of  $C(X)$  in the cylinder case and  $D(X)$  in the rich disk case is not canonical. However, for every translation surface  $(X, \omega)$  in one of  $\mathcal{H}_{\text{poor-cyl}}$ ,  $\mathcal{H}_{\text{rich-cyl}}$ , or  $\mathcal{H}_{\text{rich-disk}}$ , we fix  $C(X)$  or  $D(X)$ , respectively. In particular, we fix  $D(X)$  such that the holonomy vector defining the location of the center of  $D(X)$  is locally constant in  $\mathcal{H}_{\text{rich-disk}}$  (see the proof of Lemma 5.3 for details on this choice).

*Lemma 2.2* (The four cases cover  $\mathcal{H}_1(\kappa)$ ). For  $g \geq 2$ , we have

$$\mathcal{H}_1(\kappa) = \mathcal{H}_{\text{small-diam}} \cup \mathcal{H}_{\text{poor-cyl}} \cup \mathcal{H}_{\text{rich-cyl}} \cup \mathcal{H}_{\text{rich-disk}}.$$

*Proof.* Let  $(X, \omega) \in \mathcal{H}_1(\kappa) \setminus \mathcal{H}_{\text{small-diam}}$ . Then there exists a point  $x \in X$  such that  $d(x, \Sigma) = \text{c-rad}(X) \geq 18 \cdot \sqrt{\frac{\log g}{g}}$ . In particular, there exists an immersed, locally flat, open disk around this point with radius  $\text{c-rad}(X)$ .

Consider an immersed disk with the same center but radius  $\frac{1}{2} \text{c-rad}(X)$ . If this disk is not embedded, then two points in the disk have to be identified. This defines a closed geodesic and hence a core curve of a cylinder. The circumference of this cylinder is the length of the core curve which is at most  $\text{c-rad}(X)$ . The height of this cylinder then has to be at least  $\text{c-rad}(X, \omega)$  (see Figure 2.1). Furthermore, we have  $\text{c-rad}(X) \geq 18 \cdot \sqrt{\frac{\log g}{g}} \geq \frac{1}{\sqrt{g}}$ .

So, if we are not in the rich disk case then we are in the (poor or rich) cylinder case.  $\square$

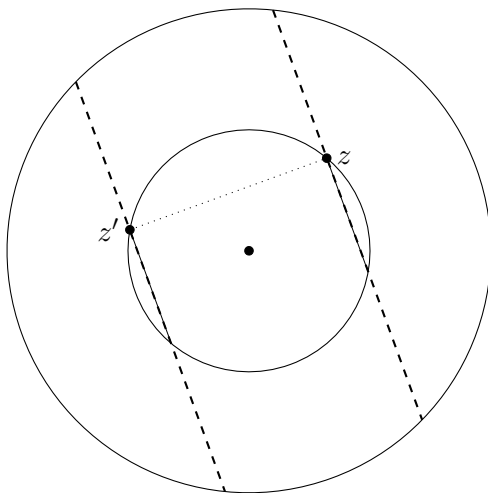


Figure 2.1: If  $z$  and  $z'$  are identified in the smaller disk, then the dotted line between them is a core curve of a cylinder. Hence, the two dashed lines are identified and give a lower bound on the height of the cylinder.

### 3 The poor cylinder case

We will deal with the poor cylinder case and the rich cylinder case similarly by estimating the measure of the subset of  $\mathcal{H}_{\text{poor-cyl}}$  (or  $\mathcal{H}_{\text{rich-cyl}}$ ) where the height of the cylinder is in a certain given range. Essentially, we use a Riemann sum argument to estimate the integral.

Recall that for  $(X, \omega) \in \mathcal{H}_{\text{poor-cyl}}$ , the height of  $C(X)$  is at least  $c\text{-rad}(X) \geq \frac{1}{\sqrt{g}}$  and the area of  $C(X)$  is at most  $\frac{1}{g}$ . We divide  $\mathcal{H}_{\text{poor-cyl}}$  into subsets based on the height of the cylinder. For every  $n \geq 1$ , define

$$\mathcal{H}_{\text{poor-cyl}}(n) = \left\{ (X, \omega) \in \mathcal{H}_{\text{poor-cyl}} : \text{height}(C(X)) \in \left( 2^{n-1} \cdot \frac{1}{\sqrt{g}}, 2^n \cdot \frac{1}{\sqrt{g}} \right) \right\}.$$

In particular, if  $(X, \omega)$  is a translation surface in  $\mathcal{H}_{\text{poor-cyl}}(n)$  then we have that  $(X, \omega)$  contains a cylinder whose circumference is at most  $\frac{1}{2^{n-1} \cdot \sqrt{g}}$ . With Corollary 7.2 and the estimate  $30\pi < 100$ , we can calculate the measure of  $\mathcal{H}_{\text{poor-cyl}}(n)$ .

**Corollary 3.1 (Measure of  $\mathcal{H}_{\text{poor-cyl}}(n)$ )**

*For large values of  $g$ , we have*

$$\nu(\mathcal{H}_{\text{poor-cyl}}(n)) \leq \nu\left(\mathcal{H}_{\text{thin-cyl}}\left(\frac{1}{2^{n-1} \cdot \sqrt{g}}\right)\right) \leq \frac{400}{2^{2n}} \cdot \nu(\mathcal{H}_1(\kappa)).$$



**Theorem 3.2 (Expected covering radius on  $\mathcal{H}_{\text{poor-cyl}}$ )**

For large values of  $g$ , we have

$$\frac{\int_{\mathcal{H}_{\text{poor-cyl}}} \text{c-rad } d\nu}{\nu(\mathcal{H}_1(\kappa))} \leq 400 \cdot \frac{1}{\sqrt{g}}.$$

*Proof.* Note that the height of  $C(X)$  cannot be larger than twice the covering radius of  $X$ . Hence, for an  $(X, \omega) \in \mathcal{H}_{\text{poor-cyl}}(n)$ , we have  $\text{c-rad}(X) \in \left(2^{n-2} \cdot \frac{1}{\sqrt{g}}, 2^n \cdot \frac{1}{\sqrt{g}}\right)$ . We can now calculate the integral by using  $\mathcal{H}_{\text{poor-cyl}} = \cup_{n \geq 1} \mathcal{H}_{\text{poor-cyl}}(n)$ . For the third inequality below we use Corollary 3.1.

$$\begin{aligned} \int_{\mathcal{H}_{\text{poor-cyl}}} \text{c-rad } d\nu &\leq \sum_{n=1}^{\infty} \int_{\mathcal{H}_{\text{poor-cyl}}(n)} \text{c-rad } d\nu \\ &\leq \sum_{n=1}^{\infty} \nu(\mathcal{H}_{\text{poor-cyl}}(n)) \cdot 2^n \cdot \frac{1}{\sqrt{g}} \\ &\leq \sum_{n=1}^{\infty} \frac{400}{2^{2n}} \cdot \nu(\mathcal{H}_1(\kappa)) \cdot 2^n \cdot \frac{1}{\sqrt{g}} \\ &= 400 \cdot \frac{1}{\sqrt{g}} \cdot \nu(\mathcal{H}_1(\kappa)) \cdot \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= 400 \cdot \frac{1}{\sqrt{g}} \cdot \nu(\mathcal{H}_1(\kappa)) \end{aligned}$$

This finishes the proof of the statement.  $\square$

## 4 The rich cylinder case

We can do a similar approach for the rich cylinder case as in the poor cylinder case. Recall that for  $(X, \omega) \in \mathcal{H}_{\text{rich-cyl}}$ , the height of the cylinder  $C(X)$  is at least  $\text{c-rad}(X) \geq \frac{1}{\sqrt{g}}$  and the area of  $C(X)$  is at least  $\frac{1}{g}$ .

For  $n, m \geq 1$ , consider  $\mathcal{H}_{\text{rich-cyl}}(n, m)$  to be the subset of  $\mathcal{H}_{\text{rich-cyl}}$  where the height of  $C(X)$  is in  $\left(2^{n-1} \cdot \frac{1}{\sqrt{g}}, 2^n \cdot \frac{1}{\sqrt{g}}\right)$  and the area is in  $\left(m \cdot \frac{1}{g}, (m+1) \cdot \frac{1}{g}\right)$ . This implies that for a translation surface in  $\mathcal{H}_{\text{rich-cyl}}(n, m)$ , the circumference of  $C(X)$  is bounded from above by  $\frac{m+1}{2^{n-1}} \cdot \frac{1}{\sqrt{g}}$ .

With Theorem 7.1, we can calculate the measure of  $\mathcal{H}_{\text{rich-cyl}}(n, m)$ .

**Corollary 4.1 (Measure of  $\mathcal{H}_{\text{rich-cyl}}(n, m)$ )**

For large values of  $g$ , we have

$$\nu(\mathcal{H}_{\text{rich-cyl}}(n, m)) \leq \frac{400 \cdot (m+1)^2}{2^{2n}} \cdot e^{-m} \cdot \nu(\mathcal{H}_1(\kappa)).$$

*Proof.* This follows directly from Theorem 7.1 with the following calculation:

$$\begin{aligned}
\nu(\mathcal{H}_{\text{rich-cyl}}(n, m)) &\leq \mathcal{H}_{\text{thin-cyl}}\left(\frac{m+1}{2^{n-1}} \cdot \frac{1}{\sqrt{g}}, \frac{m}{g}\right) \\
&\leq 100 \cdot g \cdot \frac{(m+1)^2}{2^{2n-2}} \cdot \frac{1}{g} \cdot \left(1 - \frac{m}{g}\right)^{2g+\ell-3} \cdot \nu(\mathcal{H}_1(\kappa)) \\
&\leq \frac{400 \cdot (m+1)^2}{2^{2n}} \cdot \left(e^{-m/g}\right)^{2g+\ell-3} \cdot \nu(\mathcal{H}_1(\kappa)) \\
&\leq \frac{400 \cdot (m+1)^2}{2^{2n}} \cdot e^{-m} \cdot \nu(\mathcal{H}_1(\kappa)) \quad \square
\end{aligned}$$

**Theorem 4.2 (Expected covering radius on  $\mathcal{H}_{\text{rich-cyl}}$ )**

For large values of  $g$ , we have

$$\frac{\int_{\mathcal{H}_{\text{rich-cyl}}} \text{c-rad } d\nu}{\nu(\mathcal{H}_1(\kappa))} \leq 4400 \cdot \frac{1}{\sqrt{g}}.$$

*Proof.* Note that the height of  $C(X)$  cannot be larger than twice the covering radius of  $X$ . Hence, for an  $(X, \omega) \in \mathcal{H}_{\text{rich-cyl}}(n, m)$ , we have  $\text{c-rad}(X) \in \left(2^{n-2} \cdot \frac{1}{\sqrt{g}}, 2^n \cdot \frac{1}{\sqrt{g}}\right)$ .

We can now calculate the integral by using  $\mathcal{H}_{\text{rich-cyl}} = \cup_{m \geq 1} \cup_{n \geq 1} \mathcal{H}_{\text{rich-cyl}}(n, m)$ . For the third inequality below we use Corollary 4.1.

$$\begin{aligned}
\int_{\mathcal{H}_{\text{rich-cyl}}} \text{c-rad } d\nu &\leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{\mathcal{H}_{\text{rich-cyl}}(n, m)} \text{c-rad } d\nu \\
&\leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \nu(\mathcal{H}_{\text{rich-cyl}}(n, m)) \cdot 2^n \cdot \frac{1}{\sqrt{g}} \\
&\leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{400 \cdot (m+1)^2}{2^{2n}} \cdot e^{-m} \cdot \nu(\mathcal{H}_1(\kappa)) \cdot 2^n \cdot \frac{1}{\sqrt{g}} \\
&= 400 \cdot \frac{1}{\sqrt{g}} \cdot \nu(\mathcal{H}_1(\kappa)) \cdot \sum_{m=1}^{\infty} (m+1)^2 \cdot e^{-m} \cdot \sum_{n=1}^{\infty} \frac{1}{2^n} \\
&= 400 \cdot \frac{1}{\sqrt{g}} \cdot \nu(\mathcal{H}_1(\kappa)) \cdot \sum_{m=1}^{\infty} (m+1)^2 \cdot e^{-m}
\end{aligned}$$

We now find a bound for the sum in this term. For every  $m \geq 6$ , we have

$$\frac{(m+2)^2 \cdot e^{-(m+1)}}{(m+1)^2 \cdot e^{-m}} = \left(\frac{m+2}{m+1}\right)^2 \cdot e^{-1} \leq \left(\frac{8}{7}\right)^2 \cdot \frac{1}{e} < \frac{1}{2}.$$

Hence, the sum is bounded by seven times the first term; that is, we have

$$\sum_{m=1}^{\infty} (m+1)^2 \cdot e^{-m} \leq 7 \cdot 2^2 \cdot e^{-1} < 11.$$

This finally shows the statement. □

## 5 The rich disk case

Recall that  $\mathcal{H}_{\text{rich-disk}}$  is the subset of translation surfaces  $(X, \omega)$  in  $\mathcal{H}_1(\kappa)$  where  $\text{c-rad}(X) \geq 18 \cdot \sqrt{\frac{\log g}{g}}$  and where there exists an embedded disk of diameter at least  $\text{c-rad}(X)$ .

Note that the area of an embedded disk can never be larger than 1, hence the diameter of the disk has to be smaller than  $\frac{2}{\sqrt{\pi}}$ . Hence on  $\mathcal{H}_{\text{rich-disk}}$ , the covering radius of a translation surface is globally bounded by  $\frac{2}{\sqrt{\pi}}$ . In particular, for small genera we have  $\frac{2}{\sqrt{\pi}} \leq 18 \cdot \sqrt{\frac{\log g}{g}}$  and hence  $\mathcal{H}_{\text{rich-disk}}$  is empty.

The idea in the rich disk case is to take some area from the large embedded disk and to distribute it to the rest of the surface. Note that when doing so, we leave the stratum and change the topology of the surface. To define this map, let  $\mathcal{H}_{\leq 1, \text{rich-disk}}$  be the cone over  $\mathcal{H}_{\text{rich-disk}}$  of translation surface in  $\mathcal{H}_{\leq 1}(\kappa)$  that are obtained from some  $(X, \omega) \in \mathcal{H}_{\text{rich-disk}}$  by scaling by a factor  $\sqrt{\lambda}$  with  $0 < \lambda \leq 1$ .

### Definition 5.1 (Taxing map)

For  $g \geq 2$  and  $\xi = \frac{3\sqrt{2}}{2} \cdot \sqrt{\frac{\log g}{g}}$ , define

$$T: \mathcal{H}_{\leq 1, \text{rich-disk}} \times (-\xi, \xi)^6 \rightarrow \mathcal{H}_{\leq 1}(\kappa, 2)$$

in the following way.

Let  $(X', \omega') \in \mathcal{H}_{\leq 1, \text{rich-disk}}$ ,  $\lambda = \text{area}(X')$ , and  $(x_1, x_2, x_3) \in (-\xi, \xi)^6 \subseteq \mathbb{C}^3$ . Let  $(X, \omega)$  be a scaled version of  $(X', \omega')$  with area 1, i.e., we multiply every saddle connection by a factor of  $\frac{1}{\sqrt{\lambda}}$ . By definition, there is a choice of an embedded disk  $D(X)$  in  $X$  with diameter

$$d \geq \text{c-rad}(X) \geq 18 \cdot \sqrt{\frac{\log g}{g}} = 6\sqrt{2}\xi$$

and center  $c$ . We consider a parallelogram  $P = P(X, x_1, x_2, x_3)$  with center  $c' = c + x_1$  and edges  $(2\xi, 0) + x_2$  and  $(0, 2\xi) + x_3$ . This parallelogram  $P$  is entirely contained in a disk around  $c$  with radius  $3\sqrt{2}\xi$ . Hence,  $P$  is contained in the embedded disk and therefore is embedded itself (see Figure 5.1). Furthermore, let  $P' = \sqrt{\lambda}P$  be the image of  $P$  in  $(X', \omega')$  under scaling by  $\sqrt{\lambda}$ .

Now, we define  $(Y', \zeta')$  to be the translation surface where we remove  $P'$  from  $(X', \omega')$  and glue the two pairs of parallel boundaries of  $P'$ . This introduces a new singularity with cone angle  $3 \cdot 2\pi$ , hence  $(Y', \zeta') \in \mathcal{H}_{\leq 1}(\kappa, 2) := \mathcal{H}_{\leq 1}(k_1, \dots, k_\ell, 2)$ . As shown in Lemma 5.2 below,

$$\text{area}(Y') \leq \left(1 - \frac{9}{2} \cdot \frac{\log g}{g}\right) \cdot \text{area}(X').$$

Hence we can rescale  $(Y', \zeta')$  by this factor, and obtain a surface  $T((X', \omega'), (x_1, x_2, x_3))$  that still has area less than or equal to 1.

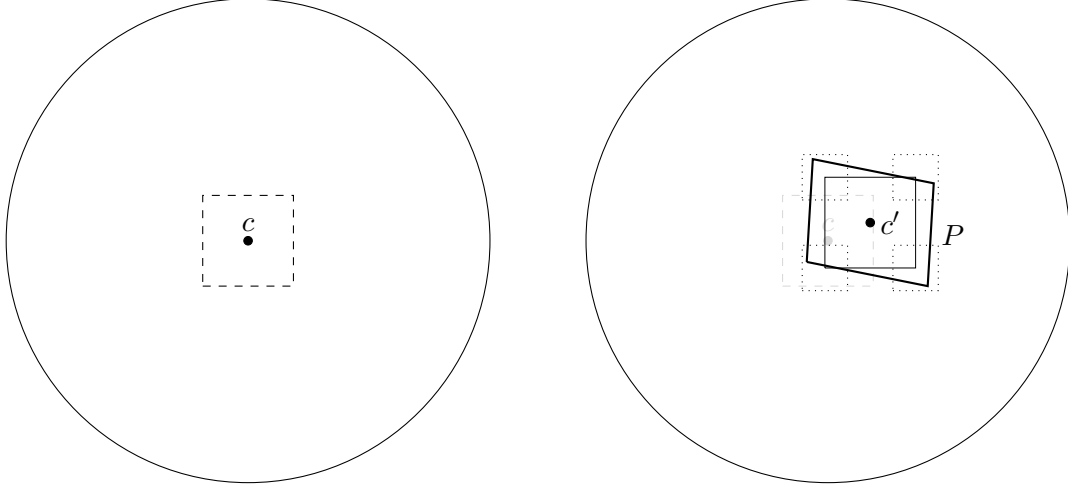


Figure 5.1: Let  $c$  be the center of the embedded disk. The center  $c'$  of the parallelogram could be anywhere in the dashed square. For a chosen  $c'$ , the corners of the parallelogram can vary from the corners of the standard square within the dotted squares.

*Lemma 5.2 (Area of  $P(X, x_1, x_2, x_3)$ ).* We have

$$\frac{9}{2} \cdot \frac{\log g}{g} \leq \text{area}(P(X, x_1, x_2, x_3)) \leq \frac{81}{2} \cdot \frac{\log g}{g}.$$

*Proof.* The sides of the parallelogram in  $(X, \omega)$  can be at most of length  $2\xi + \xi = \frac{9\sqrt{2}}{2} \cdot \sqrt{\frac{\log g}{g}}$ . Furthermore, the parallelogram contains a square with edges of length  $2\xi - \xi = \frac{3\sqrt{2}}{2} \cdot \sqrt{\frac{\log g}{g}}$ .  $\square$

We study the map  $T$  by considering it as the concatenation of the maps

$$T_P: \mathcal{H}_{\leq 1, \text{rich-disk}} \times (-\xi, \xi)^6 \rightarrow \mathcal{H}(\kappa, 2)$$

(removing  $P'$ ) and

$$T_S: \mathcal{H}(\kappa, 2) \rightarrow \mathcal{H}(\kappa, 2)$$

(scaling the surface by  $(1 - \frac{9}{2} \cdot \frac{\log g}{g})^{-\frac{1}{2}}$ ).

*Lemma 5.3 (Properties of  $T_P$ ).*

- (i) Every translation surface in  $\mathcal{H}(\kappa, 2)$  has at most  $g$  preimages under  $T_P$ .
- (ii) We have  $\text{Jac}(T_P, ((X', \omega'), (x_1, x_2, x_3))) = (\text{area}(X', \omega'))^3$  on  $\mathcal{H}_{\leq 1, \text{rich-disk}} \times (-\xi, \xi)^6$ .

*Proof.* Let  $\ell$  be the number of singularities of  $\mathcal{H}(\kappa)$ . Then the complex dimension of  $\mathcal{H}(\kappa)$  is  $2g + \ell - 1$ . Hence,  $\mathcal{H}_{\leq 1, \text{rich-disk}} \times (-\xi, \xi)^6$  has complex dimension  $2g + \ell + 2$ . Also, the complex dimension of  $\mathcal{H}(\kappa, 2)$  is  $2(g + 1) + (\ell + 1) - 1 = 2g + \ell + 2$ . Hence the image and the domain have the same dimension.

We now show that a translation surface  $(Y', \zeta')$  in  $\mathcal{H}(\kappa, 2)$  has at most  $g$  preimages under  $T_P$ . Recall first that  $(Y', \zeta')$  has genus  $g + 1$ , hence it can have at most  $g$  singularities with cone angle  $6\pi$ . We now fix a singularity  $\tau$  of  $(Y', \zeta')$  with cone angle  $6\pi$  and call the set of the other singularities  $\Sigma$ . If the two shortest saddle connections in  $(Y', \zeta')$  with one end point in  $\tau$  are disjoint and have in fact both their end points at  $\tau$ , then  $(Y', \zeta')$  is possibly contained in the image of  $T_P$  where  $\tau$  is the singularity that is newly introduced by  $T_P$ . We construct the only possible preimage  $(X', \omega')$  of  $(Y', \zeta')$  under  $T_P$ .

Cutting open along these two short saddle connections as above gives us the possibility to glue in a parallelogram  $P'$ . Doing so, all corners of the former parallelogram will be regular points (that is, their cone angle is  $2\pi$ ) and we obtain a translation surface  $(X', \omega')$  of area  $\lambda \in (0, 1]$ , that is  $(X', \omega') \in \mathcal{H}_{\leq 1}(\kappa)$ . Let  $(X, \omega) = \left(\frac{1}{\sqrt{\lambda}}X', \frac{1}{\sqrt{\lambda}}\omega'\right)$ , that is  $\text{area}(X, \omega) = 1$ .

Let  $c$  be the center of the embedded disk  $D(X)$  in  $(X, \omega)$  and let  $v$  be a vector connecting a singularity  $\sigma \in \Sigma$  to  $c$  that can be represented as a geodesic segment. We can choose the rich disks in the translation surfaces in  $\mathcal{H}_{\text{rich-disk}}$  such that the vector  $v$  is constant in a neighborhood of  $X$  in  $\mathcal{H}(\kappa)$ . Let  $c' \in X$  be the preimage of the point in  $(X', \omega')$  which is the center of the parallelogram that was glued in to obtain  $(X', \omega')$ .

Let  $y_2$  be the holonomy vector of the edge  $e_2$  of the parallelogram  $P'$  with the smaller imaginary part and  $y_3$  be the holonomy vector of the edge  $e_3$  of  $P'$  with the larger imaginary part. Define  $x_1 = c' - c$ ,  $x_2 = \frac{1}{\sqrt{\lambda}}y_2 - (2\xi, 0)$ , and  $x_3 = \frac{1}{\sqrt{\lambda}}y_3 - (0, 2\xi)$ . Then  $(\sqrt{\lambda}(X, \omega), (x_1, x_2, x_3))$  is the only possible preimage of  $(Y', \zeta')$  when  $\tau$  is a fixed singularity with angle  $6\pi$ .

In particular,  $T_P$  is locally injective. The equality of dimensions and the local injectivity imply that the image of  $T_P$  is a subset of  $\mathcal{H}(\kappa, 2)$  with non-empty interior.

To compare the measure in the domain and in the image, we locally (around the points  $(X', \omega')$  and  $(Y', \zeta')$ ) choose *compatible period coordinates* on  $\mathcal{H}_{\leq 1, \text{rich-disk}} \subseteq \mathcal{H}(\kappa)$  and on  $\mathcal{H}(\kappa, 2)$ .

For any triangulation  $\Delta$  of  $(Y', \zeta')$  by saddle connections, there is an open set  $U_\Delta$  in  $\mathcal{H}(\kappa, 2)$  where the edges of  $\Delta$  (as a topological triangulation) can still be represented by saddle connections. Also, any subset  $\mathcal{E}$  of edges of  $\Delta$  that does not contain a contractible loop can be extended to a set  $\mathcal{B}$  of edges of  $\Delta$  that form a basis for the relative homology of  $Y'$  relative to  $\Sigma \cup \{\tau\}$ . Let  $\mathcal{E}$  be the set of the following three saddle connections in  $(Y', \zeta')$ : the two shortest saddle connections  $e_2$  and  $e_3$  that connect  $\tau$  to itself and a saddle connection  $e_1$  that connects a singularity in  $\Sigma$  to  $\tau$  and is disjoint from the other two such that the holonomy vector of  $e_1$  is  $y_1 = v + x_1 - (x_2 + x_3)/2 - (\xi, \xi)$ . (That is,  $e_1$  lands on the lower left corner of the parallelogram spanned by  $e_2$  and  $e_3$ .)

Complete  $\mathcal{E}$  to a triangulation  $\Delta$  of  $(Y', \zeta')$  by successively adding edges that are disjoint from previous edges. Let  $\mathcal{E}_\Sigma$  be the set of edges in  $\Delta$  that start and end in singularities in  $\Sigma$ . Then  $\mathcal{E} \cup \mathcal{E}_\Sigma$  spans the relative homology. This is because any edge  $e$  connecting a singularity in  $\Sigma$  to  $\tau$  is either in  $\mathcal{E}$  or there is a contractible loop in  $\Delta$  consisting of  $e$ , the edge  $e_1$  in  $\mathcal{E}$  connecting a singularity in  $\Sigma$  to  $\tau$ , and an edge path in  $\mathcal{E}_\Sigma$ . Let  $\mathcal{B}_\Sigma$  be a subset of  $\mathcal{E}_\Sigma$  so that  $\mathcal{B} = \mathcal{B}_\Sigma \cup \mathcal{E}$  forms a basis for the relative homology of  $Y'$  relative to  $\Sigma \cup \{\tau\}$ .

By construction, edges in  $\mathcal{E}_\Sigma$  can also be represented as saddle connections in  $(X', \omega')$ . In fact, the edges associated to  $\mathcal{B}_\Sigma$  form a basis for the relative homology of  $X'$  relative to  $\Sigma$  as  $|\mathcal{E}| = 3$  and the complex dimension of  $\mathcal{H}(\kappa)$  is by 3 smaller than the dimension of  $\mathcal{H}(\kappa, 2)$ . By making  $U_\Delta$  smaller, we can ensure that any point in  $\mathcal{H}_{\leq 1, \text{rich-disk}}$  that is a preimage of a point in  $U_\Delta$  is contained in an open set  $U_X$  where edges in  $\mathcal{B}_\Sigma$  can still be represented by saddle connections. We call the period coordinates given by  $\mathcal{B}_\Sigma$  for points in  $U_X$  and the period coordinates given by  $\mathcal{B}$  for points in  $U_\Delta$  a pair of compatible period coordinates.

On a pair of compatible period coordinates (together with the three vectors  $x_1, x_2, x_3$ ), we therefore have that  $T_P$  is affine: it is the identity on all but the last three vectors and the coefficients of  $e_1, e_2$  and  $e_3$  depend only on  $x_1, x_2$  and  $x_3$ . That is,  $T_P$  can be represented in the following form

$$\begin{pmatrix} I_{2g} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} \mathcal{B}_\Sigma \\ \mathcal{E} \end{pmatrix} + \begin{pmatrix} 0 \\ B \end{pmatrix}.$$

Recalling that

$$\begin{aligned} y_1 &= \sqrt{\lambda} \left( x_1 + v - (\xi, \xi) - \frac{x_2}{2} - \frac{x_3}{2} \right) \\ y_2 &= \sqrt{\lambda} (x_2 + (2\xi, 0)) \\ y_3 &= \sqrt{\lambda} (x_3 + (0, 2\xi)) \end{aligned}$$

we have

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = A\mathcal{E} + B = \sqrt{\lambda} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \sqrt{\lambda} \begin{pmatrix} -(\xi, \xi) + v \\ (2\xi, 0) \\ (0, 2\xi) \end{pmatrix}$$

The transformation above should be thought of as a map from  $\mathbb{R}^6$  to  $\mathbb{R}^6$ . Hence,

$$\text{Jac}(T_P, ((X', \omega'), (x_1, x_2, x_3))) = \sqrt{\lambda}^6 = (\text{area}(X', \omega'))^3. \quad \square$$

*Lemma 5.4 (Jacobian of the scaling map).* We have  $\text{Jac}(T_S) \geq g^{9/2}$  on  $T_P(\mathcal{H}_{\text{rich-disk}} \times (0, 1])$ .

*Proof.* Let  $(X, \omega) \in \mathcal{H}(\kappa, 2)$  be in the image of  $T_P$ . The map  $T_S$  changes all of the  $2 \cdot (2g + \ell + 2) \geq 2 \cdot 2g$  real period coordinates uniformly by the factor  $\left(1 - \frac{\frac{9}{2} \cdot \log g}{g}\right)^{-\frac{1}{2}}$ . For the calculation, we use that for every  $x \leq 1$ , we have  $1 \geq 1 - x^2$  and hence  $\frac{1}{1-x} \geq 1 + x$ .

$$\begin{aligned}
\text{Jac}(T_S) &= \left( \left( 1 - \frac{\frac{9}{2} \cdot \log g}{g} \right)^{-\frac{1}{2}} \right)^{2 \cdot 2g} \\
&= \left( \frac{1}{1 - \frac{\frac{9}{2} \cdot \log g}{g}} \right)^{2g} \\
&\geq \left( 1 + \frac{\frac{9}{2} \cdot \log g}{g} \right)^{2g} \\
&\geq \left( 1 + \frac{1}{\frac{9/2 \cdot \log g}{g}} \right)^{\left( \frac{g}{9/2 \cdot \log g} + 1 \right) \cdot \frac{2g}{2 \cdot \frac{9/2 \cdot \log g}{g}}} \\
&\geq e^{9/2 \cdot \log g} = g^{9/2}. \quad \square
\end{aligned}$$

Combining Lemma 5.3 and Lemma 5.4, we get the following bound for the measure.

**Corollary 5.5 (Measure of  $\mathcal{H}_{\text{rich-disk}}$ )**

We have

$$\nu(\mathcal{H}_{\text{rich-disk}}) \leq \frac{2}{3^6} \cdot (\log g)^{-3} \cdot g^{-1/2} \cdot \nu(\mathcal{H}_1(\kappa, 2)).$$

*Proof.* Recall that we use the same notation  $\nu$  for the measure on the whole stratum  $\mathcal{H}(\kappa)$  and the measure on  $\mathcal{H}_1(\kappa)$ . By definition, we have

$$\nu(\mathcal{H}_{\text{rich-disk}}) = \nu(\mathcal{H}_{\leq 1, \text{rich-disk}}) \quad \text{and} \quad \nu(\mathcal{H}_1(\kappa, 2)) = \nu(\mathcal{H}_{\leq 1}(\kappa, 2)).$$

Denote the image of  $\mathcal{H}_{\leq 1, \text{rich-disk}} \times (-\xi, \xi)^6$  under  $T$  (or  $T_P$ ) with  $\text{im}(T)$  (or  $\text{im}(T_P)$ ), respectively). From Lemma 5.4 and from the fact that  $\text{im}(T) \subseteq \mathcal{H}_{\leq 1}(\kappa, 2)$ , we get

$$\nu(\text{im}(T_P)) \cdot g^{9/2} \leq \nu(\text{im}(T)) \leq \nu(\mathcal{H}_{\leq 1}(\kappa, 2)).$$

Let  $\mathcal{H}_{\geq \frac{1}{2}, \text{rich-disk}}$  be the set of translation surfaces in  $\mathcal{H}_{\leq 1, \text{rich-disk}}$  that have area at least  $\frac{1}{2}$ . As the complex dimension of  $\mathcal{H}_{\leq 1, \text{rich-disk}}$  is  $2g + \ell - 1$ , we have

$$\nu\left(\mathcal{H}_{\geq \frac{1}{2}, \text{rich-disk}}\right) = \left(1 - \left(\frac{1}{2}\right)^{2(2g+\ell-1)}\right) \cdot \nu(\mathcal{H}_{\leq 1, \text{rich-disk}}) \geq \frac{1}{2} \cdot \nu(\mathcal{H}_{\text{rich-disk}}).$$

By Lemma 5.3, we have

$$g \cdot \nu\left(T_P\left(\mathcal{H}_{\geq \frac{1}{2}, \text{rich-disk}} \times (-\xi, \xi)^6\right)\right) \geq \left(\frac{1}{2}\right)^3 \cdot \nu\left(\mathcal{H}_{\geq \frac{1}{2}, \text{rich-disk}}\right) \cdot (2\xi)^6.$$

This implies

$$\begin{aligned}
\nu(T_P(\mathcal{H}_{\leq 1, \text{rich-disk}} \times (-\xi, \xi)^6)) &\geq \nu(T_P(\mathcal{H}_{\geq \frac{1}{2}, \text{rich-disk}} \times (-\xi, \xi)^6)) \\
&\geq 8\xi^6 \cdot g^{-1} \cdot \nu(\mathcal{H}_{\geq \frac{1}{2}, \text{rich-disk}}) \\
&\geq 4\xi^6 \cdot g^{-1} \cdot \nu(\mathcal{H}_{\text{rich-disk}}).
\end{aligned}$$

Combining these measure comparisons and inserting  $\xi = \frac{3\sqrt{2}}{2} \cdot \sqrt{\frac{\log g}{g}}$ , we can now deduce

$$\begin{aligned}
\nu(\mathcal{H}_{\text{rich-disk}}) &\leq \frac{1}{4} \xi^{-6} \cdot g \cdot \nu(T_P(\mathcal{H}_{\leq 1, \text{rich-disk}} \times (-\xi, \xi)^6)) \\
&\leq \frac{2}{3^6} \cdot \frac{g^3}{(\log g)^3} \cdot g \cdot g^{-9/2} \cdot \nu(\mathcal{H}_{\leq 1}(\kappa, 2)). \quad \square
\end{aligned}$$

**Theorem 5.6 (Expected covering radius on  $\mathcal{H}_{\text{rich-disk}}$ )**

For large values of  $g$ , we have

$$\frac{\int_{\mathcal{H}_{\text{rich-disk}}} \text{c-rad } d\nu}{\nu(\mathcal{H}_1(\kappa))} \leq 2^{-9} \cdot \sqrt{\frac{\log g}{g}}.$$

*Proof.* Recall that on  $\mathcal{H}_{\text{rich-disk}}$ , the covering radius of a translation surface is globally bounded by  $\frac{2}{\sqrt{\pi}}$ . This gives us the following calculation.

$$\begin{aligned}
\int_{\mathcal{H}_{\text{rich-disk}}} \text{c-rad } d\nu &\leq \frac{2}{\sqrt{\pi}} \cdot \nu(\mathcal{H}_{\text{rich-disk}}) \\
&\leq \frac{2}{\sqrt{\pi}} \cdot \frac{2}{3^6} \cdot (\log g)^{-3} \cdot g^{-1/2} \cdot \nu(\mathcal{H}_1(\kappa, 2)) \\
&\leq \frac{4}{3^6 \cdot \sqrt{\pi}} \cdot \sqrt{\frac{\log g}{g}} \cdot (\log g)^{-7/2} \cdot \nu(\mathcal{H}_1(\kappa, 2))
\end{aligned}$$

By [Agg18b, Theorem 1.4], we have the estimate  $\nu(\mathcal{H}_1(\kappa)) = \frac{4}{\prod_{i=1}^{\ell} (k_i+1)} \cdot (1 + \mathcal{O}(\frac{1}{g}))$  for a stratum  $\mathcal{H}_1(\kappa)$  with  $\kappa = (k_1, \dots, k_{\ell})$ . In particular, this also gives us  $\nu(\mathcal{H}_1(\kappa, 2)) = \frac{4}{3 \cdot \prod_{i=1}^{\ell} (k_i+1)} \cdot (1 + \mathcal{O}(\frac{1}{g}))$ . Hence, for large values of  $g$ , we have

$$\frac{\nu(\mathcal{H}_1(\kappa, 2))}{\nu(\mathcal{H}_1(\kappa))} \leq \frac{1}{2}.$$

The calculation  $\frac{2}{3^6 \cdot \sqrt{\pi}} \leq 2^{-9}$  finishes the proof. □



## 6 Proof of the main theorem

Now we can put together the ingredients for the proof of our main theorem.

We have shown in the previous three sections that the expected value of the covering radius goes to zero on  $\mathcal{H}_{\text{poor-cyl}}$ , on  $\mathcal{H}_{\text{rich-cyl}}$ , and on  $\mathcal{H}_{\text{rich-disk}}$ . When looking at the explicit statements in Theorem 3.2 and Theorem 4.2, it is also clear that the rates  $400 \cdot \frac{1}{\sqrt{g}}$  and  $4400 \cdot \frac{1}{\sqrt{g}}$  are smaller than  $\frac{1}{2} \cdot \sqrt{\frac{\log g}{g}}$  for large values of  $g$ . In Theorem 5.6, the rate is also smaller than  $\sqrt{\frac{\log g}{g}}$ .

The only missing part is  $\mathcal{H}_{\text{small-diam}}$ . However, this set is defined so that the covering radius is smaller than  $18 \cdot \sqrt{\frac{\log g}{g}}$ , hence the expected value of the covering radius is also bounded by  $18 \cdot \sqrt{\frac{\log g}{g}}$ . Therefore, by summing up these four expected values, we have proven Theorem B that we state here again.

### Theorem 6.1 (Expected covering radius)

For large values of  $g$ , we have

$$\mathbb{E}_{\mathcal{H}_1(\kappa)}(\text{c-rad}) = \frac{\int_{\mathcal{H}_1(\kappa)} \text{c-rad}(X) d\nu(X)}{\nu(\mathcal{H}_1(\kappa))} \leq 20 \cdot \sqrt{\frac{\log g}{g}}.$$

## 7 Appendix

In this appendix, we find an upper bound for the the number of cylinders with an upper bound on the circumference and a lower bound on the area. We also give an upper bound for the number of saddle connections with an upper bound on the length. We then use these bounds to give upper bounds for the measure of the thin part of the stratum  $\mathcal{H}_1(\kappa) = \mathcal{H}_1(k_1, \dots, k_\ell)$ .

Here, we understand the  $\delta$ -thin part in three different ways. First, we consider the set  $\mathcal{H}_{\text{thin-cyl}}(\delta)$  of translation surfaces on which there exists a cylinder that has circumference at most  $\delta$ . Second, we do a similar computation for  $\mathcal{H}_{\text{thin-cyl}}(\delta, A)$  under the additional assumption that the area of this cylinder is bounded from below by some constant  $A \in [0, \frac{1}{2}]$ . Third, for the sake of completion, we consider the usual thin part, namely the set  $\mathcal{H}_{\text{thin}}(\delta)$  of translation surfaces that contain a saddle connection of length at most  $\delta$  that do not necessarily bound a cylinder.

Recall that the complex dimension  $d$  of  $\mathcal{H}(\kappa)$  is equal to  $2g + \ell - 1$ .

### Theorem 7.1 (Expected number of cylinders)

Consider a stratum  $\mathcal{H}(\kappa)$  of complex dimension  $d$ . Given  $\delta > 0$ ,  $0 < A < 1$ , and  $(X, \omega) \in \mathcal{H}_1(\kappa)$ , let  $N_{\text{cyl}}(X, \delta, A)$  be the number of cylinders in  $X$  where the circumference is at

most  $\delta$  and the area is at least  $A$ . For large values of  $g$ , we have

$$\mathbb{E}_{\mathcal{H}_1(\kappa)}(N(\bullet, \delta, A)) = \frac{\int_{\mathcal{H}_1(\kappa)} N_{\text{cyl}}(X, \delta, A) d\nu(X)}{\nu(\mathcal{H}_1(\kappa))} \leq 30\pi g \cdot \delta^2 \cdot (1 - A)^{d-2}.$$

In particular, for  $\mathcal{H}_{\text{thin-cyl}}(\delta)$  the set of translation surfaces  $(X, \omega) \in \mathcal{H}_1(\kappa)$  for which  $N_{\text{cyl}}(X, \delta, A)$  is not zero, we have

$$\frac{\nu(\mathcal{H}_{\text{thin-cyl}}(\delta, A))}{\nu(\mathcal{H}_1(\kappa))} \leq 30\pi g \cdot \delta^2 \cdot (1 - A)^{d-2}.$$

From Theorem 7.1, we can directly deduce the measure of  $\mathcal{H}_{\text{thin-cyl}}(\delta) = \mathcal{H}_{\text{thin-cyl}}(\delta, 0)$  where we do not have any restriction on the area of the cylinder. This is used in the proof of Corollary 3.1.

**Corollary 7.2 (Measure of  $\mathcal{H}_{\text{thin-cyl}}(\delta)$ )**

For  $\mathcal{H}_{\text{thin-cyl}}(\delta)$  the set of translation surfaces  $(X, \omega) \in \mathcal{H}_1(\kappa)$  for which  $N_{\text{cyl}}(X, \delta, 0)$  is not zero, we have

$$\frac{\nu(\mathcal{H}_{\text{thin-cyl}}(\delta))}{\nu(\mathcal{H}_1(\kappa))} \leq 30\pi g \cdot \delta^2.$$

For the sake of completion, we also give the corresponding statement on the number of saddle connections.

**Theorem 7.3 (Expected number of saddle connections)**

For given  $\delta > 0$  and a translation surface  $(X, \omega) \in \mathcal{H}_1(\kappa)$ , let  $N_{\text{sc}}(X, \delta)$  be the number of saddle connections in  $X$  of length at most  $\delta$ . For large values of  $g$ , we have

$$\mathbb{E}_{\mathcal{H}_1(\kappa)}(N(\bullet, g, \delta)) = \frac{\int_{\mathcal{H}_1(\kappa)} N_{\text{sc}}(X, \delta) d\nu(X)}{\nu(\mathcal{H}_1(\kappa))} \leq 90\pi g^2 \cdot \delta^2.$$

In particular, for  $\mathcal{H}_{\text{thin}}(\delta)$  the set of translation surfaces  $(X, \omega) \in \mathcal{H}_1(\kappa)$  for which  $N_{\text{sc}}(X, \delta)$  is not zero, we have

$$\frac{\nu(\mathcal{H}_{\text{thin}}(\delta))}{\nu(\mathcal{H}_1(\kappa))} \leq 90\pi g^2 \cdot \delta^2.$$

The key tools for these computations is to compute upper bounds for the Siegel–Veech constants associated to the two different situations. Given a counting problem, that we are interested in, and a translation surface  $(X, \omega)$  in  $\mathcal{H}_1(\kappa)$ , we can consider the set  $V = V(X) \subset \mathbb{R}^2$  of the holonomy vectors of the collection of saddle connections of interest. (see Subsection 7.2 for details). There is an action of  $\text{SL}(2, \mathbb{R})$  on each (connected component of a) stratum as well as on  $\mathbb{R}^2$ . One has that  $V(X)$  is a set of vectors equivariant under these actions. The Siegel–Veech formula (2) below holds for any  $\text{SL}(2, \mathbb{R})$ -invariant measure; in particular for the normalized Lebesgue measure that we consider in this paper. Now,

for a connected component  $\mathcal{H}$  of a stratum, a compactly supported continuous function  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  (for example a characteristic function of a ball) and a set of vectors  $V(X)$  for every  $(X, \omega) \in \mathcal{H}$  (for example, the set of holonomy vectors of saddle connections of multiplicity 1) define  $\hat{f}: \mathcal{H} \rightarrow \mathbb{R}$  by

$$\hat{f}(X, \omega) = \sum_{v \in V(X)} f(v). \quad (1)$$

Then there is a constant  $c(V)$  [Vee98, Theorem 0.5], called the *Siegel–Veech constant*, which is independent of  $f$  and such that

$$\frac{1}{\nu(\mathcal{H})} \int_{\mathcal{H}} \hat{f} d\nu = c(V) \int_{\mathbb{R}^2} f dx dy. \quad (2)$$

We will not compute these Siegel–Veech constants but determine upper bounds which suffice to calculate upper bounds for the counting problems of interest. The plan for the appendix is as follows. We carry out the computations in detail in the (non-connected) minimal stratum  $\mathcal{H}_1(2g-2)$ . For the minimal stratum, we follow the arguments in [EMZ03], but consider additionally the area condition for cylinders. Then we show that the Siegel–Veech constant for saddle connections of multiplicity greater than 1 is dominated by the one for multiplicity 1 as  $g$  goes to  $\infty$ . Then we consider general strata. In this case, in the Appendix to [Agg18b], Zorich gave the Siegel–Veech constant without the area constraint in the case of multiplicity 1. Adding the area condition is exactly the same as in the minimal stratum. Then we quote Aggarwal in [Agg18a] which says that for these connected strata the higher multiplicity constants are also negligible in comparison to multiplicity 1 as  $g \rightarrow \infty$  so they can be ignored.

## 7.1 Setting to calculate Siegel–Veech constants in the minimal stratum

We now begin the computations in the case of the minimal stratum  $\mathcal{H}_1(2g-2)$ . The key idea of the proof from [EMZ03] is that a set of  $p$  homologous short saddle connections give a decomposition of the translation surface into several pieces that are themselves surfaces with boundary. In our setting, there are three different types of surfaces with boundary than can appear in this construction (see Figure 7.1):

- figure eight type: this has one singularity and one boundary component which consists of two saddle connections, it can have any genus  $g \geq 1$
- two holes type: this has two boundary components that consist each of one saddle connection, it has two singularities and can have any genus  $g \geq 1$
- cylinder: this is a special case of the two holes type where we have two boundary components but no genus

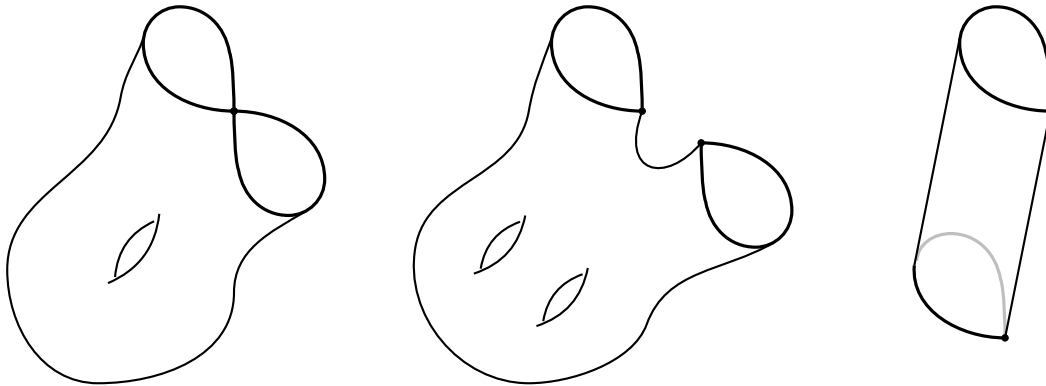


Figure 7.1: The three types of surfaces with boundary: figure eight type, two holes type, cylinder (from left to right).

Choosing any sequence of such pieces and gluing them together in a cyclic way, gives us a new surface. Note that it is not possible to use only figure eight type surfaces without at least one surface of another type. If we would do so, the result is not a surface as the singularity becomes a double point of the translation structure. Also, we are in the special situation that the obtained surfaces should have only one singularity. Because of the cyclic gluing, we can have only one surface with two singularities in the sequence, that is, either a two hole type or a cylinder.

## 7.2 Upper bound for Siegel–Veech constants for a given configuration with one cylinder in the minimal stratum

In this section, we assume that we have a cylinder, so there is no surface with two holes. We will then recover Formula 13.1 from [EMZ03] but posing additionally the condition on the area. The case of a saddle connection that does not bound a cylinder and hence where there is a surface with two holes, will be discussed later in Subsection 7.4.

We first need to deal with each combinatorial type separately. Recall that the minimal stratum  $\mathcal{H}_1(2g-2)$  has three connected components which we denote by  $\mathcal{H}^j$  for  $j = 1, 2, 3$  from now on. The Siegel–Veech constant for a given connected component  $\mathcal{H}^j$  will be denoted by  $c^j(V)$ . We study a specific combinatorial type, described as follows. Let  $p$  be the multiplicity of the saddle connection in its homology class; that is, the number of pieces in which we cut the surface (not counting the cylinder as the two boundary components of the cylinder are not counted both). Hence, we have that the surface is divided into  $p$  surfaces with boundary  $(X_i, \omega_i)$  of genera  $g_i \geq 1$  with  $\sum_{i=1}^p g_i = g-1$  and additionally a cylinder. Suppose that the first surface in the cyclic order is the cylinder. For the other  $p$  surfaces, let  $a_i = 2g_i - 2$  be the order of the corresponding singularity. Note that we can also choose

how much of the cone angle is on one side of the figure eight and how much is on the other side. We refer to this data by  $(p, \mathcal{C})$ , that is, a saddle connection has the combinatorial type  $(p, \mathcal{C})$  if the  $p$  saddle connections in its homology class decompose the surface  $(X, \omega)$  with the data described above. Let  $V_{\text{cyl}}(p, \mathcal{C}, A)$  be the set of holonomy vectors of saddle connections of combinatorial type  $(p, \mathcal{C})$  where the corresponding cylinder has area at least  $A \in [0, 1)$ . Respectively, there is a Siegel–Veech constant  $c^j(V_{\text{cyl}}(p, \mathcal{C}, A))$  on the connected component  $\mathcal{H}^j$  which corresponds to counting saddle connections of combinatorial type  $(p, \mathcal{C})$  where the cylinder has area at least  $A$ . We proceed to find this  $c^j(V_{\text{cyl}}(p, \mathcal{C}, A))$  by mimicking the calculations from [EMZ03, Section 13].

For this, let  $\gamma$  be the holonomy vector of a saddle connection of combinatorial type  $(p, \mathcal{C})$  and  $h$  be the height of the cylinder that  $\gamma$  bounds. Note that the surface  $(X, \omega)$  that we build can have any area less than or equal to 1. However, it always corresponds to a surface of area 1 with a cylinder where the circumference is bounded from above by  $\delta$  and the area is bounded from below by  $A$ . Hence, we have

$$|\gamma| \leq \delta \cdot \sqrt{\text{area}(X, \omega)} \quad \text{and} \quad h \cdot |\gamma| \geq A \cdot \text{area}(X, \omega).$$

As  $\text{area}(X, \omega) = \sum_{i=1}^p \text{area}(X_i, \omega_i) + h|\gamma|$ , the first inequality is equivalent to

$$h \geq \frac{|\gamma|}{\delta^2} - \frac{\sum_{i=1}^p \text{area}(X_i, \omega_i)}{|\gamma|} \tag{3}$$

whereas the second inequality is equivalent to

$$h \geq \frac{A}{1-A} \cdot \frac{\sum_{i=1}^p \text{area}(X_i, \omega_i)}{|\gamma|}. \tag{4}$$

Both of the lower bounds on  $h$  have to be fulfilled and together they are sufficient to obtain a translation surface as desired. We distinguish two cases now, depending on whether the bound from (3) or the bound from (4) is larger and implies the other inequality. Note that

$$\frac{|\gamma|}{\delta^2} - \frac{\sum_{i=1}^p \text{area}(X_i, \omega_i)}{|\gamma|} \geq \frac{A}{1-A} \cdot \frac{\sum_{i=1}^p \text{area}(X_i, \omega_i)}{|\gamma|},$$

is equivalent to

$$|\gamma| \geq \delta \sqrt{\frac{\sum_{i=1}^p \text{area}(X_i, \omega_i)}{1-A}}.$$

For the following calculation, let  $d = 4g$  be the real dimension of  $\mathcal{H}(2g-2)$  and  $d_i = 4g_i$  be the real dimension of  $\mathcal{H}(a_i)$ . Note that we have  $\sum_{i=1}^p d_i = d - 4$ . Furthermore, we set

$$r_i = \sqrt{\text{area}(X_i, \omega_i)} \quad \text{and} \quad D(z) = \left\{ (r_1, \dots, r_p) : \sum_{i=1}^p r_i^2 \leq z \right\}.$$

Define  $\mathcal{H}_{\text{thin-cyl}}(\delta, \mathcal{C}, A)$  to be the set of translation surfaces in  $\mathcal{H}_{\text{thin-cyl}}(\delta, A)$  where the saddle connection of length at most  $\delta$  that bounds a cylinder of area  $A$  has the combinatorial type  $(p, \mathcal{C})$ . Similar to [EMZ03, Section 13.1], we have

$$\begin{aligned} \nu(\mathcal{H}_{\text{thin-cyl}}(\delta, \mathcal{C}, A)) &= \\ &= WM \cdot \left( \int_{D(1-A)} \prod_{i=1}^p r_i^{d_i-1} dr_i \left( \int_{|\gamma| \leq \delta} \sqrt{\frac{\sum r_i^2}{1-A}} \int_{\frac{A}{1-A} \frac{\sum r_i^2}{|\gamma|} \leq h \leq \frac{1-\sum r_i^2}{|\gamma|}} \int_0^{|\gamma|} dt dh d\gamma + \right. \right. \\ &\quad \left. \left. + \int_{\delta \sqrt{\frac{\sum r_i^2}{1-A}} \leq |\gamma| \leq \delta} \int_{\frac{|\gamma|}{\delta^2} - \frac{\sum r_i^2}{|\gamma|} \leq h \leq \frac{1-\sum r_i^2}{|\gamma|}} \int_0^{|\gamma|} dt dh d\gamma \right) \right) + o(\delta^2) \end{aligned}$$

where  $t$  is a twist parameter of the cylinder,  $W = \prod \nu(\mathcal{H}(a_i))$ , and  $M$  is the combinatorial constant that counts how many different surfaces of area 1 can be built for the fixed data  $\delta$ ,  $(p, \mathcal{C})$ , and  $A$ .

We first integrate over  $t$  and  $h$  and then over  $\gamma$  to obtain

$$\begin{aligned} \nu(\mathcal{H}_{\text{thin-cyl}}(\delta, \mathcal{C}, A)) &= WM \cdot \int_{D(1-A)} \prod_{i=1}^p r_i^{d_i-1} \left( \int_{|\gamma| \leq \delta} \sqrt{\frac{\sum r_i^2}{1-A}} \left( 1 - \frac{1}{1-A} \sum r_i^2 \right) d\gamma \right. \\ &\quad \left. + \int_{\delta \sqrt{\frac{\sum r_i^2}{1-A}} \leq |\gamma| \leq \delta} \left( 1 - \frac{|\gamma|^2}{\delta^2} \right) d\gamma \right) \prod_{i=1}^p dr_i + o(\delta^2) \\ &= WM \cdot \delta^2 \cdot \pi \cdot \left( \int_{D(1-A)} \prod_{i=1}^p r_i^{d_i-1} \right) \left( 1 - \frac{1}{1-A} \sum r_i^2 \right) \left( \frac{1}{1-A} \sum r_i^2 \right) \prod_{i=1}^p dr_i \\ &\quad + WM \cdot \delta^2 \cdot \pi \cdot \left( \int_{D(1-A)} \prod_{i=1}^p r_i^{d_i-1} \right) \left( 1 - \frac{1}{1-A} \sum r_i^2 \right) \prod_{i=1}^p dr_i \\ &\quad - WM \cdot \pi \cdot \left( \int_{D(1-A)} \prod_{i=1}^p r_i^{d_i-1} \right) \int_{\delta \sqrt{\frac{\sum r_i^2}{1-A}}}^{\delta} \frac{s^2}{\delta^2} \cdot 2s ds \prod_{i=1}^p dr_i + o(\delta^2). \end{aligned}$$

With the substitution  $u = \frac{s^2}{\delta^2}$ , we can write the inner most integral in the last summand as

$$\delta^2 \cdot \int_{\frac{1}{1-A} \sum r_i^2}^1 u du,$$

giving

$$\begin{aligned}
& \nu(\mathcal{H}_{\text{thin-cyl}}(\delta, \mathcal{C}, A)) = \\
& = WM \cdot \delta^2 \cdot \pi \left( \int_{D(1-A)} \prod_{i=1}^p r_i^{d_i-1} \right) \left( 1 - \frac{1}{1-A} \sum r_i^2 \right) \left( 1 + \frac{1}{1-A} \sum r_i^2 \right) \prod_{i=1}^p dr_i \\
& \quad - WM \cdot \delta^2 \cdot \pi \left( \int_{D(1-A)} \prod_{i=1}^p r_i^{d_i-1} \right) \left( \frac{1}{2} - \frac{1}{2(1-A)^2} \left( \sum r_i^2 \right)^2 \right) \prod_{i=1}^p dr_i + o(\delta^2) \\
& = \pi \delta^2 \cdot WM \int_{D(1-A)} \prod_{i=1}^p r_i^{d_i-1} \\
& \quad \cdot \left( 1 - \frac{1}{(1-A)^2} \left( \sum r_i^2 \right)^2 - \frac{1}{2} + \frac{1}{2(1-A)^2} \left( \sum r_i^2 \right)^2 \right) \prod_{i=1}^p dr_i + o(\delta^2) \\
& = \pi \delta^2 \cdot WM \int_{D(1-A)} \prod_{i=1}^p r_i^{d_i-1} \left( \frac{1}{2} - \frac{1}{2} \left( \sum \left( \frac{r_i}{\sqrt{1-A}} \right)^2 \right)^2 \right) \prod_{i=1}^p dr_i + o(\delta^2) \\
& = \pi \delta^2 \cdot WM \int_{D(1-A)} \prod_{i=1}^p r_i^{d_i-1} \\
& \quad \cdot \left( \left( 1 - \sum \left( \frac{r_i}{\sqrt{1-A}} \right)^2 \right) - \frac{1}{2} \left( 1 - \sum \left( \frac{r_i}{\sqrt{1-A}} \right)^2 \right)^2 \right) \prod_{i=1}^p dr_i + o(\delta^2).
\end{aligned}$$

Now make the substitution  $s_i = \frac{r_i}{\sqrt{1-A}}$  and since  $\sum_{i=1}^p (d_i - 1) = d - 4 - p$  and  $d = 4g$ , we get

$$\begin{aligned}
& \nu(\mathcal{H}_{\text{thin-cyl}}(\delta, \mathcal{C}, A)) \\
& = (1-A)^{\frac{p}{2}} \cdot \prod_{i=1}^p (1-A)^{\frac{(d_i-1)}{2}} \cdot \pi \delta^2 \cdot WM \\
& \quad \cdot \int_{D(1)} \prod_{i=1}^p s_i^{d_i-1} \left( \left( 1 - \sum s_i^2 \right) - \frac{1}{2} \left( 1 - \sum s_i^2 \right)^2 \right) \prod_{i=1}^p ds_i \\
& = (1-A)^{2g-2} \cdot \pi \delta^2 \cdot WM \\
& \quad \cdot \int_{D(1)} \prod_{i=1}^p s_i^{d_i-1} \left( \left( 1 - \sum s_i^2 \right) - \frac{1}{2} \left( 1 - \sum s_i^2 \right)^2 \right) \prod_{i=1}^p ds_i + o(\delta^2).
\end{aligned}$$

This formula for the volume is precisely the formula in [EMZ03] for  $q = 1$ , multiplied by the factor  $(1-A)^{2g-2}$  which accounts for the area bound. This factor is the same for all combinatorial types  $(p, \mathcal{C})$ , in particular it is independent of  $p$ . The role of the factor  $(1-A)^{2g-2}$  was also shown by Vorobets in [Vor05, Theorem 1.8].

Now we find the combinatorial constant  $M$  and investigate its dependence on  $g$ . Here we may exactly use the formulae from [EMZ03, Section 13.3]. As we are in the minimal stratum, most of the factors in the formula are equal to 1. In this particular case, we can use the following estimate.

$$M \leq \prod_{i=1}^p (a_i + 1)$$

The above expression for  $\nu(\mathcal{H}_{\text{thin-cyl}}(\delta, \mathcal{C}, A))$  can now be used to calculate  $\int_{\mathcal{H}_1(2g-2)} \hat{f} d\nu$  where  $\hat{f}$  is the function associated to  $V_{\text{cyl}}(p, \mathcal{C}, A)$  which Eskin–Masur–Zorich use to find the associated Siegel–Veech constant. In our setting, the above integral is still an upper bound for  $\int_{\mathcal{H}^j} \hat{f} d\nu$  and we get an upper bound for  $c^j(V_{\text{cyl}}(p, \mathcal{C}, A))$ . Following the reasoning in [EMZ03, page 135], we have

$$\begin{aligned} c^j(V_{\text{cyl}}(p, \mathcal{C}, A)) &\leq M \cdot (1-A)^{2g-2} \cdot \frac{1}{2^{p-1}} \cdot \frac{\prod_i (\frac{d_i}{2} - 1)!}{(\frac{d}{2} - 2)!} \cdot \frac{\prod_{i=1}^p \nu(\mathcal{H}(a_i))}{\nu(\mathcal{H}^j)} \\ &\leq (1-A)^{2g-2} \cdot \frac{1}{2^{p-1}} \cdot \prod_{i=1}^p (a_i + 1) \cdot \frac{\prod_i (\frac{d_i}{2} - 1)!}{(\frac{d}{2} - 2)!} \cdot \frac{\prod_{i=1}^p \nu(\mathcal{H}(a_i))}{\nu(\mathcal{H}^j)}. \end{aligned}$$

In this formula, the term  $\prod_{i=1}^p \nu(\mathcal{H}(a_i))$  is the previous  $W$  and the factor  $\frac{(1-A)^{2g-2}}{2^{p-1}} \cdot \frac{\prod_i (\frac{d_i}{2} - 1)!}{(\frac{d}{2} - 2)!}$  comes from the integral. We now replace the exact values  $a_i = 2g_i - 2$ ,  $d_i = 4g_i$ , and  $d = 4g$ . Furthermore, we have from [Sau18, Theorem 1.9] or the more general [Agg18b, Theorem 1.4] that  $\nu(\mathcal{H}(a_i)) = \frac{4}{2g_i-1} \cdot (1 + \mathcal{O}(\frac{1}{g}))$  and  $\nu(\mathcal{H}_1(2g-2)) = \frac{4}{2g-1} \cdot (1 + \mathcal{O}(\frac{1}{g}))$ . To avoid keeping track of the error term, we use the weaker bounds  $\nu(\mathcal{H}(a_i)) \leq \frac{4.1}{2g_i-1}$  and  $\nu(\mathcal{H}_1(2g-2)) \leq \frac{4.1}{2g-1}$  for large values of  $g$ . Then we have

$$\begin{aligned} c^j(V_{\text{cyl}}(p, \mathcal{C}, A)) &\leq (1-A)^{2g-2} \cdot \frac{1}{2^{p-1}} \cdot \prod_{i=1}^p (2g_i - 1) \cdot \frac{\prod_{i=1}^p (2g_i - 1)!}{(2g - 2)!} \cdot \frac{4.1^p}{\nu(\mathcal{H}^j) \cdot \prod_{i=1}^p (2g_i - 1)} \\ &\leq 2 \cdot 2.05^p \cdot (1-A)^{2g-2} \cdot \frac{\prod_{i=1}^p (2g_i - 1)!}{(2g - 2)!} \cdot \frac{1}{\nu(\mathcal{H}^j)}. \end{aligned}$$

### 7.3 Counting of cylinders

Now we want to calculate upper bounds for the Siegel–Veech constant for saddle connections that bound a cylinder. For a given multiplicity  $p$ , we consider all ways how to decompose the surface into a cylinder and  $p$  surfaces with boundary and how to decompose the angle of the singularity of order  $a_i$  in each of the surfaces. That is, for a connected component  $\mathcal{H}^j$ , we define  $V_{\text{cyl}}(p, A)$  to be the set of holonomy vectors for all saddle connections bounding a cylinder of area at least  $A$  with multiplicity  $p$ . Let  $c^j(V_{\text{cyl}}(p, A))$  be the associated Siegel–Veech constant. Then,  $c^j(V_{\text{cyl}}(p, A))$  is calculated as sum over all possible combinatorial



types  $(p, \mathcal{C})$  for fixed  $p$ :

$$\begin{aligned}
c^j(V_{\text{cyl}}(p, A)) &= \sum_{\mathcal{C}} c^j(V_{\text{cyl}}(p, \mathcal{C}, A)) \\
&\leq \sum_{g_1+\dots+g_p=g-1} \prod_{i=1}^p (a_i - 1) \cdot 2 \cdot 2.05^p \cdot (1 - A)^{2g-2} \cdot \frac{\prod_{i=1}^p (2g_i - 1)!}{(2g - 2)! \cdot \nu(\mathcal{H}^j)} \\
&\leq \frac{2 \cdot 2.05^p \cdot (1 - A)^{2g-2}}{\nu(\mathcal{H}^j)} \cdot \sum_{g_1+\dots+g_p=g-1} \frac{\prod_{i=1}^p (2g_i)!}{(2g - 2)!}
\end{aligned}$$

We use  $a_i - 1 \leq 2g_i$  here to obtain the third line.

Note that  $\prod_{i=1}^p (2g_i)!$  and  $(2g - 2)!$  have the same number of factors. We have that  $\prod_{i=1}^p (2g_i)!$  is the largest when all but one  $g_i$  are equal to 1. In this situation, let  $g_1$  be the largest one, that is  $g_1 = g - 1 - (p - 1)$ . Therefore, we have

$$\prod_{i=1}^p (2g_i)! \leq 2^{p-1} \cdot (2g - 2p)!$$

which implies

$$\begin{aligned}
\frac{\prod_{i=1}^p (2g_i)!}{(2g - 2)!} &\leq 2^{p-1} \cdot \frac{(2g - 2p)!}{(2g - 2)!} \\
&= \frac{2^{p-1}}{(2g - 2)(2g - 3) \cdots (2g - 2p + 1)} \\
&\leq \frac{1}{(2g - 2)(2g - 3) \cdots (2g - p)}.
\end{aligned}$$

This upper bound is independent of the choice of the  $g_i$ . There are  $\binom{g-2}{p-1}$  choices of the  $g_i$ . This is because we can consider an ordered set with  $g - 1$  elements and divide it into  $p$  subsets of cardinality  $\geq 1$  by specifying which elements are the last in their corresponding subsets. The last one in the whole ordered set has to be the last one of a subset. Apart from that, we can choose any  $p - 1$  elements out of the remaining  $g - 2$  elements to be last ones.

Therefore, we can calculate:

$$\begin{aligned}
\sum_{g_1+\dots+g_p=g-1} \frac{\prod_{i=1}^p (2g_i)!}{(2g - 2)!} &\leq \binom{g-2}{p-1} \cdot \frac{1}{(2g - 2)(2g - 3) \cdots (2g - p)} \\
&= \frac{1}{(p-1)!} \cdot \frac{(g-2)(g-3) \cdots (g-p)}{(2g - 2)(2g - 3) \cdots (2g - p)} \\
&\leq \frac{1}{(p-1)!}
\end{aligned}$$

Still fixing a connected component  $\mathcal{H}^j$ , we define  $V_{\text{cyl}}(A)$  to be the set of holonomy vectors for all saddle connections bounding a cylinder of area at least  $A$ . Let  $c^j(V_{\text{cyl}}(A))$  be the associated Siegel–Veech constant. To compute  $c^j(V_{\text{cyl}}(A))$ , we have to sum over all  $p$ . This gives us:

$$\begin{aligned}
c^j(V_{\text{cyl}}(A)) &= \sum_{p=1}^{g-1} c^j(V_{\text{cyl}}(p, A)) \\
&\leq \frac{(1-A)^{2g-2}}{\nu(\mathcal{H}^j)} \cdot 2 \cdot \sum_{p=1}^{g-1} 2.05^p \cdot \frac{1}{(p-1)!} \\
&\leq \frac{(1-A)^{2g-2}}{\nu(\mathcal{H}^j)} \cdot 4.1 \cdot \sum_{p=0}^{g-2} \frac{2.05^p}{p!} \\
&\leq \frac{4.1e^{2.05} \cdot (1-A)^{2g-2}}{\nu(\mathcal{H}^j)} \leq \frac{32(1-A)^{2g-2}}{\nu(\mathcal{H}^j)}. \tag{5}
\end{aligned}$$

We are now ready to prove Theorem 7.1 for the minimal stratum.

*Proof.* Let  $V_{\text{cyl}}(A)$  be defined as above,  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  be the characteristic function of the ball of radius  $\delta$  and  $\hat{f}: \mathcal{H}^j \rightarrow \mathbb{R}$  be the associated function defined in Equation 1.

We argue that  $\hat{f}(X) = N_{\text{cyl}}(X, \delta, A)$  outside of a measure zero set. This is because, in the minimal stratum, the relative homology is the same as the absolute homology. Hence, the set  $\mathcal{H}_{\text{parallel}}$  of translation surfaces that have two non-homologous saddle connections whose holonomy vectors have the same direction has measure zero. This has two consequences. First, there is only a measure zero set of translation surfaces that has a cylinder with a boundary component that contains more than one saddle connections. That is, generically, for every cylinder of area  $A$ , we have a vector in  $V_{\text{cyl}}(A)$ . Secondly, the set of translation surfaces that have more than one cylinder giving the same vector in  $V_{\text{cyl}}(A)$  has also measure zero.

We also have

$$\int_{\mathcal{H}^j} \hat{f} d\nu = c^j(V_{\text{cyl}}(A)) \cdot \int_{\mathbb{R}^2} f dx dy \cdot \nu(\mathcal{H}^j) \leq 32\pi \delta^2 (1-A)^{2g-2}.$$

We can now estimate the desired integral by using the bounds on the integrals for all three connected components  $\mathcal{H}^j$  on the stratum.

$$\begin{aligned}
\frac{\int_{\mathcal{H}_1(2g-2)} N_{\text{cyl}}(X, \delta, A) d\nu(X)}{\nu(\mathcal{H}_1(2g-2))} &\leq \frac{1}{\nu(\mathcal{H}_1(2g-2))} \cdot \sum_{i=1}^3 \int_{\mathcal{H}^i} \hat{f} d\nu \\
&\leq \frac{2g}{3.9} \cdot \sum_{i=1}^3 32\pi \delta^2 (1-A)^{2g-2} \\
&\leq 160g \delta^2 (1-A)^{2g-2}.
\end{aligned}$$

For the second to last line, we use again [Agg18b, Theorem 1.4], this time in the form that  $\nu(\mathcal{H}_1(2g-2)) \geq \frac{3.9}{2g}$  for large values of  $g$ .

The estimate for  $\nu(\mathcal{H}_{\text{thin-cyl}}(\delta, A))$  follows from the fact that  $X \in \mathcal{H}_{\text{thin-cyl}}(\delta, A)$  if and only if  $N_{\text{cyl}}(X, \delta, A) \geq 1$ .  $\square$

Note that the constant in front of  $g \delta^2 (1-A)^{2g-2}$  is larger than claimed in the statement of Theorem 7.1 as the bounds in the proof are not sharp. We will show the better bound in the general case, using results of [Agg18b, Agg18a].

## 7.4 Counting of saddle connections

For the sake of completion, we now turn to the case of saddle connections that do not bound a cylinder. Let  $p$  again be the multiplicity of the saddle connection. Then the surface decomposes into  $p$  surfaces with boundary of which  $p-1$  are of figure eight type and exactly one is of two holes type. In particular, there is no cylinder.

Suppose that the first surface is the surface of two holes type. Let  $b'_1, b''_1 \geq 0$  be integers such that the interior angle at the holes is  $(2b'_1 + 3)\pi$  and  $(2b''_1 + 3)\pi$  with  $b'_1 + b''_1 = 2g_1 - 2$ . Then the real dimension of the stratum  $\mathcal{H}(b'_1, b''_1)$  of the first surface is  $d_1 = 4g_1 + 2$  and the volume of the stratum  $\mathcal{H}(b'_1, b''_1)$  is approximately  $\frac{4}{(b'_1+1)(b''_1+1)}$  (see [Agg18b, Theorem 1.4]). We use again the weaker bound  $\nu(\mathcal{H}(b'_1, b''_1)) \leq \frac{4.1}{(b'_1+1)(b''_1+1)}$  for large values of  $g$ .

Similarly to before, this data defines the combinatorial type  $(p, \mathcal{C})$ . We consider the corresponding set  $V_{\text{loop}}(p, \mathcal{C})$  of holonomy vectors of saddle connections of combinatorial type  $(p, \mathcal{C})$  that do not bound a cylinder. Then Formula 13.1 from [EMZ03] gives us that in the situation of having exactly one surface of two holes type and no cylinder, the Siegel–Veech constant  $c^j(V_{\text{loop}}(p, \mathcal{C}))$  for this data is bounded in the following way.

$$\begin{aligned}
& c^j(V_{\text{loop}}(p, \mathcal{C})) \\
& \leq \frac{1}{2^{p-1}} \cdot (b'_1 + 1)(b''_1 + 1) \cdot \prod_{i=2}^p (a_i + 1) \cdot \frac{\prod_i (\frac{d_i}{2} - 1)!}{(\frac{d}{2} - 2)!} \cdot \frac{\nu(\mathcal{H}(b'_1, b''_1)) \cdot \prod_{i=2}^p \nu(\mathcal{H}(a_i))}{\nu(\mathcal{H}^j)} \\
& \leq \frac{1}{2^{p-1}} \cdot (b'_1 + 1)(b''_1 + 1) \\
& \quad \cdot \prod_{i=2}^p (2g_i - 1) \cdot \frac{(2g_1)! \cdot \prod_{i=2}^p (2g_i - 1)!}{(2g - 2)!} \cdot \frac{4.1^p}{(b'_1 + 1)(b''_1 + 1) \cdot \prod_{i=2}^p (2g_i - 1) \cdot \nu(\mathcal{H}^j)} \\
& = 2 \cdot 2.05^p \cdot \frac{2g_1 \cdot \prod_{i=1}^p (2g_i - 1)!}{(2g - 2)!} \cdot \frac{1}{\nu(\mathcal{H}^j)}.
\end{aligned}$$

Note that the bound for this Siegel–Veech constant  $c^j(V_{\text{loop}}(p, \mathcal{C}))$  differs exactly by a factor of  $2g_1$  from the bound for the Siegel–Veech constant  $c^j(V_{\text{cyl}}(p, \mathcal{C}, 0))$  where we have a cylinder without a condition on the area. Hence, we can skip the calculations for a given

multiplicity  $p$  and deduce directly

$$c^j(V_{\text{loop}}) \leq 2g_i \cdot \frac{4.1e^{2.05}}{\nu(\mathcal{H}^j)} \leq g \cdot \frac{8.2e^{2.05}}{\nu(\mathcal{H}^j)}. \quad (6)$$

Let  $c^j(V_{\text{sc}})$  be the Siegel–Veech constant for all saddle connections. Then

$$c^j(V_{\text{sc}}) \leq c^j(V_{\text{loop}}) + c^j(V_{\text{cyl}}(0)).$$

Combining the estimates in Equation 5 for  $A = 0$  and Equation 6 we get

$$c^j(V_{\text{sc}}) \leq \frac{64g}{\nu(\mathcal{H}^j)}.$$

The rest of the proof of Theorem 7.3 is then completely analogous to the proof of Theorem 7.1.

## 7.5 General Strata

We now turn our attention to the general case of a stratum  $\mathcal{H}(\kappa)$ . The proof follows the same steps as for the minimal stratum. We will outline here how to combine the arguments from the previous sections of the appendix with the results of [EMZ03, Vor05, Agg18b].

We start with the case of cylinders. For this, fix a connected component  $\mathcal{H}$  of a stratum  $\mathcal{H}(\kappa)$ . By applying the volume estimates from [Agg18b] to the formulae from [EMZ03], Zorich in an appendix to [Agg18b] gets the following estimates for connected strata. For cylinders where one boundary component is a saddle connection through a singularity of order  $k_1$  and the other through a distinct singularity of order  $k_2$  and where the multiplicity is 1, the Siegel–Veech constant is (up to lower order terms)  $\frac{(k_1+1)(k_2+1)}{d-2}$  for large  $g$ . If the singularities are the same on both boundary components, we have that the Siegel–Veech constant is (again, up to lower order terms)  $\frac{1}{2} \cdot \frac{(k_1+1)(k_1-1)}{d-2}$  for large  $g$ . Note that in the case of non-connected strata, these are not estimates for the Siegel–Veech constants on the stratum but we have the upper bounds  $\frac{(k_1+1)(k_2+1)}{d-2} \cdot \frac{\nu(\mathcal{H}_1(\kappa))}{\nu(\mathcal{H}_1)}$  and  $\frac{1}{2} \cdot \frac{(k_1+1)(k_1-1)}{d-2} \cdot \frac{\nu(\mathcal{H}_1(\kappa))}{\nu(\mathcal{H}_1)}$  on the Siegel–Veech constants on all of the connected components.

We now have to compare the Siegel–Veech constants for multiplicity 1 with these for higher multiplicity. In [Agg18a], Aggarwal shows that the Siegel–Veech constants for saddle connections of higher multiplicity are of lower order than the ones for saddle connections of multiplicity 1. As the terms for the saddle connection Siegel–Veech constants differ from the terms for the cylinder Siegel–Veech constants by a factor of  $d-2$ , the same combinatorial arguments hold for the cylinder Siegel–Veech constants. Hence, the Siegel–Veech constants for cylinders with restrictions on the order of the singularity but without the restriction on the multiplicity are bounded by  $\frac{5}{4} \cdot \frac{(k_1+1)(k_2+1)}{d-2} \cdot \frac{\nu(\mathcal{H}_1(\kappa))}{\nu(\mathcal{H}_1)}$  and  $\frac{5}{8} \cdot \frac{(k_1+1)(k_1-1)}{d-2} \cdot \frac{\nu(\mathcal{H}_1(\kappa))}{\nu(\mathcal{H}_1)}$ , respectively, for large values of  $g$ .

Recall that  $V_{\text{cyl}} := V_{\text{cyl}}(0)$  is the set of holonomy vectors for all saddle connections bounding a cylinder. To obtain a bound on the Siegel–Veech constant  $c(V_{\text{cyl}})$  for  $\mathcal{H}$ , we

have to sum over all possible ordered pairs  $(k_1, k_2)$ :

$$\begin{aligned}
c(V_{\text{cyl}}) &\leq \frac{5}{4} \cdot \frac{1}{d-2} \cdot \frac{\nu(\mathcal{H}_1(\kappa))}{\nu(\mathcal{H}_1)} \cdot \sum_{k_i \geq 2} \frac{1}{2} (k_i + 1)(k_i - 1) + \sum_{i \neq j} (k_i + 1)(k_j + 1) \\
&\leq \frac{5}{4} \cdot \frac{1}{d-2} \cdot \frac{\nu(\mathcal{H}_1(\kappa))}{\nu(\mathcal{H}_1)} \cdot \sum_{i,j} (k_i + 1)(k_j + 1) = \frac{1}{d-2} \left( \sum_i k_i + 1 \right)^2 \\
&\leq \frac{5}{4} \cdot \frac{1}{d-2} (4g-4)^2 \cdot \frac{\nu(\mathcal{H}_1(\kappa))}{\nu(\mathcal{H}_1)}
\end{aligned}$$

In particular, note that  $d-2$  is larger than  $2g-2$  and hence we have

$$\begin{aligned}
c(V_{\text{cyl}}) &\leq \frac{5}{4} \cdot \frac{1}{d-2} (4g-4)^2 \cdot \frac{\nu(\mathcal{H}_1(\kappa))}{\nu(\mathcal{H}_1)} \\
&\leq \frac{5}{4} \cdot \frac{1}{2g-2} (4g-4)^2 \cdot \frac{\nu(\mathcal{H}_1(\kappa))}{\nu(\mathcal{H}_1)} \\
&\leq \frac{5}{2} (4g-4) \cdot \frac{\nu(\mathcal{H}_1(\kappa))}{\nu(\mathcal{H}_1)} \leq 10g \cdot \frac{\nu(\mathcal{H}_1(\kappa))}{\nu(\mathcal{H}_1)}.
\end{aligned}$$

As in the proof of Theorem 7.1, we let  $f$  be the characteristic function of the ball of radius  $\delta$  and  $\hat{f}: \mathcal{H}^j \rightarrow \mathbb{R}$  be the associated function defined in Equation 1. Again,  $\hat{f}(X) = N_{\text{cyl}}(X, \delta)$  outside of a measure zero set. This is because, otherwise the holonomy vectors associated to two not homologous saddle connections are parallel and this is a measure zero property. As every stratum has at most three connected components, we can again use the calculation

$$\begin{aligned}
\frac{\int_{\mathcal{H}_1(\kappa)} N_{\text{cyl}}(X, \delta) d\nu(X)}{\nu(\mathcal{H}_1(\kappa))} &\leq \frac{1}{\nu(\mathcal{H}_1(\kappa))} \cdot 3 \cdot \int_{\mathcal{H}} \hat{f} d\nu \\
&\leq \frac{1}{\nu(\mathcal{H}_1(\kappa))} \cdot 3 \cdot 10g \cdot \frac{\nu(\mathcal{H}_1(\kappa))}{\nu(\mathcal{H}_1)} \cdot \pi \delta^2 \cdot \nu(\mathcal{H}_1) \\
&\leq 30\pi g \delta^2.
\end{aligned}$$

Including the requirement on the area of the cylinder gives a factor of  $(1-A)^{d-2}$  in the very first calculation of Siegel–Veech constants for a given configuration (compare the proof for the minimal stratum or [Vor05]). Hence, this factor carries through the full proof and appears in the end as claimed.

For the case of saddle connections, a comparison of Corollary 1 and 3 with Corollary 4 and 5 in the appendix of [Agg18b] shows that the estimates of the corresponding Siegel–Veech constants are larger by a factor of  $d-2$  than in the case of cylinders. The inequality  $3g \leq d-2$  then implies Theorem 7.3.

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