Teichmüller geometry of moduli space, I: Distance minimizing rays and the Deligne-Mumford compactification

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1 Introduction

Let S be a *surface of finite type*; that is, a closed, oriented surface with a finite (possibly empty) set of points removed. In this paper we classify (globally) geodesic rays in the moduli space $\mathcal{M}(S)$ of Riemann surfaces, endowed with the Teichmüller metric, and we determine precisely how pairs of rays asymptote. We then use these results to relate two important but disparate topics in the study of $\mathcal{M}(S)$: Teichmüller geometry and the Deligne-Mumford compactification. We reconstruct the Deligne-Mumford compactification (as a metric stratified space) purely from the intrinsic metric geometry of $\mathcal{M}(S)$ endowed with the Teichmüller metric. We do this by constructing an "iterated EDM ray space" functor, which is defined on a quite general class of metric spaces. We then prove that this functor applied to $\mathcal{M}(S)$ produces the Deligne-Mumford compactification.

Rays in $\mathcal{M}(S)$. A *ray* in a metric space X is a map $r : [0, \infty) \to X$ which is locally an isometric embedding. In this paper we initiate the study of (globally) isometrically embedded rays in $\mathcal{M}(S)$. Among other things, we classify such rays, determine their asymptotics, classify almost geodesic rays, and work out the Tits angles between rays. We take as a model for our study the case of rays in locally symmetric spaces, as in the work of Borel, Ji, MacPherson and others; see [JM] for a summary.

In [JM] it is explained how the continuous spectrum of any noncompact, complete Riemannian manifold M depends only on the geometry of its ends, and in some cases (e.g. when M is locally symmetric) the generalized eigenspaces can be parametrized by a compactification constructed from asymptote classes of certain rays. The spectral theory of $\mathcal{M}(S)$

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endowed with the Teichmüller metric was initiated by McMullen [Mc], who proved positivity of the lowest eigenvalue of the Laplacian. Our compactification of $\mathcal{M}(S)$ by equivalence classes of certain rays might be viewed as a step towards further understanding its spectral theory. We remark that the Teichmüller metric is a Finsler metric.

Following [JM], we will consider two natural classes of rays.

Definition 1.1 (EDM rays). *A ray $r : [0, \infty) \to X$ in a metric space X is* eventually distance minimizing*, or* EDM*, if there exists t_0 such that for all $t \geq t_0$:*

$$d(r(t), r(t_0)) = |t - t_0|$$

Note that, if r is an EDM ray, after cutting off an initial segment of r we obtain a globally geodesic ray, i.e. an isometric embedding of $[0, \infty) \to X$.

Definition 1.2 (ADM rays). *The ray $r(t)$ is* almost distance minimizing*, or* ADM*, if there are constants $C, t_0 \geq 0$ such that for $t \geq t_0$:*

$$d(r(t), r(t_0)) \geq |t - t_0| - C$$

It is easy to check that a ray r is ADM if and only if, for every $\epsilon > 0$ there exists $t_0 \geq 0$ so that for all $t \geq t_0$:

$$d(r(t), r(t_0)) \geq |t - t_0| - \epsilon$$

As with locally symmetric manifolds, there are several ways in which a ray in $\mathcal{M}(S)$ might not be ADM: it can traverse a closed geodesic, it can be contained in a fixed compact set, or it can return to a fixed compact set at arbitrarily large times. More subtly, there are rays which leave every compact set in $\mathcal{M}(S)$ and are ADM but are not EDM; these rays "spiral" around in the "compact directions" in the cusp of $\mathcal{M}(S)$. This phenomenon does not appear in the classical case of $\mathcal{M}(T^2) = \mathbf{H}^2/\operatorname{SL}(2, \mathbf{Z})$, but it does appear in all moduli spaces of higher complexity, as we shall show.

The set of rays in $\operatorname{Teich}(S)$ through a basepoint $Y \in \operatorname{Teich}(S)$ is in bijective correspondence with the set of elements $q \in \operatorname{QD}^1(Y)$, the space of unit area holomorphic quadratic differentials q on Y (see §2 below). We now describe certain kinds of Teichmüller rays that will be important in our study.

Recall that a quadratic differential q on Y is *Strebel* if all of its vertical trajectories are closed. In this case Y decomposes into a union of flat cylinders. Each cylinder is swept out by vertical trajectories of the same length. The *height* of the cylinder is the distance across the cylinder.

We say q is *mixed Strebel* if it contains at least one cylinder of closed trajectories.

Definition 1.3 (**(Mixed) Strebel rays).** *A ray in $\mathcal{M}(S)$ is a* (mixed) Strebel ray *if it is the projection to $\mathcal{M}(S)$ of a ray in* Teich(S) *corresponding to a pair (Y, q) with q a (mixed) Strebel differential on Y.*

Our first main result is a classification of EDM rays and ADM rays in moduli space $\mathcal{M}(S)$.

Theorem 1.4 (**Classification of** EDM **rays in $\mathcal{M}(S)$).** *Let r be a ray in $\mathcal{M}(S)$. Then*

1. *r is* EDM *if and only if it is Strebel.*
2. *r is* ADM *if and only if it is mixed Strebel.*

One of the tensions arising from Theorem 1.4 is that for any $\epsilon > 0$, there exist very long local geodesics γ between points x, y in $\mathcal{M}(S)$ which are only ϵ longer than any (global) geodesic from x to y. As distance in $\mathcal{M}(S)$ is difficult to compute precisely, the question arises as to how such "fake global geodesics" γ can be distinguished from true global geodesics. This is done in §3.2. The idea is to use the input data of being non-Strebel to build by hand a map whose log-dilatation equals the length of γ, but which has nonconstant pointwise quasiconformal dilatation. By Teichmüller's uniqueness theorem, since the actual Teichmüller map from x to y has constant pointwise dilatation, this dilatation, and thus the length of the Teichmüller geodesic connecting x to y, is strictly smaller than the length of γ.

We also determine finer information about EDM rays. In Section 3.4 we determine the limiting asymptotic distance between EDM rays: it equals the Teichmüller distance of their endpoints in the "boundary moduli space" (see Theorem 3.9 below). This precise behavior of rays in $\mathcal{M}(S)$ lies in contrast to the behavior of rays in the Teichmüller space of S, which themselves may not even have limits. Theorem 3.9 is crucial for our reconstruction of the Deligne-Mumford compactification. In Section 5.3 we compute the Tits angle of any two rays, showing that only 3 possible values can occur. This result contrasts with the behavior in locally symmetric manifolds, where a continuous spectrum of Tits angles can occur.

Reconstructing the topology of Deligne-Mumford. Deligne-Mumford [DM] constructed a compactification $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ of $\mathcal{M}(S)$ whose points are represented by conformal structures on noded Riemann surfaces. They proved that $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ is a projective variety. As such, $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ as a topological space comes with a natural stratification: each stratum is a product of moduli spaces of surfaces of lower complexity. We will equip each moduli space with the Teichmüller metric, and the product of moduli spaces with the sup metric. In this way $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ has the structure of a *metric stratified space*, i.e. a stratified space with a metric on each stratum (see §4 below). We note that $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ was also constructed topologically by Bers in [Be].

In Section 4 we construct, for any geodesic metric space X, a space $\overline{X}^{ir}$ of X, called the *iterated EDM ray space* associated to X. This space comes from considering asymptote classes of EDM rays, endowing the set of these with a natural metric, and then considering asymptote classes of EDM rays on this space, etc. The space $\overline{X}^{ir}$ has the structure of a metric stratified space.

Theorem 1.5. *Let S be a surface of finite type. Then there is a strata-preserving homeomorphism $\overline{\mathcal{M}(S)}^{ir} \to \overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ which is an isometry on each stratum.*

Thus, as a metric stratified space, $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ is determined by the intrinsic geometry of $\mathcal{M}(S)$ endowed with the Teichmüller metric. The following table summarizes a kind of dictionary between purely (Teichmüller) metric properties of $\mathcal{M}(S)$ on the one hand, and purely combinatorial/analytic properties on the other. Each of the entries in the table is proved in this paper.

PURELY METRIC	ANALYTIC/COMBINATORIAL
EDM ray in $\mathcal{M}(S)$	Strebel differential
ADM ray in $\mathcal{M}(S)$	mixed Strebel differential
isolated EDM ray in $\mathcal{M}(S)$	one-cylinder Strebel differential
asymptotic EDM rays in $\mathcal{M}(S)$	modularly equivalent Strebel differentials with same endpoint
iterated EDM ray space of $\mathcal{M}(S)$	Deligne-Mumford compactification $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$
rays of rays of $\cdots$ of rays (k times)	level k stratum of $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$
Tits angle 0	pairs of combinatorially equivalent Strebel differentials
Tits angle 1	pairs of Strebel differentials with disjoint cylinders
Tits angle 2	all other pairs of Strebel differentials

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2 Teichmüller geometry and extremal length

In this section we quickly explain some basics of the Teichmüller metric and quadratic differentials. We also make some extremal length estimates which will be used later. The notation fixed here will be used throughout the paper.

Throughout this paper S will denote a surface of *finite type*, by which we mean a closed, oriented surface with a (possibly empty) finite set of points deleted. We call such deleted points *punctures*. The *Teichmüller space* $\mathrm{Teich}(S)$ is the space of equivalence classes of marked conformal structures (f, X) on S, where two markings $f_i : S \to X_i$ are equivalent if there is a conformal map $h : X_1 \to X_2$ with f_2 homotopic to $h \circ f_1$. We often drop the marking notation, remembering that a marked surface is the same as a surface where we "know the names of the curves".

The *Teichmüller metric* on $\mathrm{Teich}(S)$ is the metric defined by

$$d_{\mathrm{Teich}(S)}((X,g),(Y,h)) := \frac{1}{2}\inf\{\log K(f) : f : X \to Y \text{ is homotopic to } h \circ g^{-1}\}$$

where f is quasiconformal and

$$K(f) := \mathrm{ess} - \sup_{x \in S} K_x(f) \geq 1$$

is the *quasiconfromal dilatation* of f, where

$$K_x(f) := \frac{|f_z(x)| + |f_{\overline{z}}(x)|}{|f_z(x)| - |f_{\overline{z}}(x)|}$$

is the *pointwise quasiconformal dilatation* at x. We also use the notation $d_{\mathrm{Teich}(S)}(X, Y)$ with the markings implied. The *mapping class group* $\mathrm{Mod}(S)$ is the group of homotopy classes of orientation-preserving homeomorphisms of S. This group acts properly discontinuously and isometrically on $(\mathrm{Teich}(S), d_{\mathrm{Teich}(S)})$, and so the quotient

$$\mathcal{M}(S) = \mathrm{Teich}(S)/\operatorname{Mod}(S)$$

has the induced metric. $\mathcal{M}(S)$ is the moduli space of (unmarked) Riemann surfaces, or what is the same thing, conformal structures on S.

2.1 Quadratic differentials and Teichmüller rays

Quadratic differentials and measured foliations. Let S be a surface of finite type, and let $X \in \mathrm{Teich}(S)$. Recall that a *(holomorphic) quadratic differential* q on X is a tensor given in holomorphic local coordinates z by $q(z)dz^2$, where $q(z)$ is holomorphic. Let $\mathrm{QD}(X)$ denote the space of holomorphic quadratic differentials on X. Any $q \in \mathrm{QD}(X)$ determines a singular Euclidean metric $|q(z)||dz|^2$, with the finitely many singular points corresponding to the zeroes of q. The total area of X in this metric is finite, and is denoted by $\|q\|$, which is a norm on $\mathrm{QD}(X)$. We denote by $\mathrm{QD}^1(X)$ the set of elements $q \in \mathrm{QD}(X)$ with $\|q\| = 1$.

An element $q \in \mathrm{QD}(X)$ determines a pair of transverse measured foliations $\mathcal{F}_h(q)$ and $\mathcal{F}_v(q)$, called the *horizontal and vertical foliations* for q. The leaves of these foliations are paths $z = \gamma(t)$ such that

$$q(\gamma(t))\gamma'(t)^2 > 0$$

and

$$q(\gamma(t))\gamma'(t)^2 < 0,$$

In a neighborhood of a nonsingular point, there are *natural* coordinates $z = x + iy$ so that the leaves of $\mathcal{F}_h$ are given by $y =$ const., the leaves of $\mathcal{F}_v$ are given by $x =$ const., and the transverse measures are $|dy|$ and $|dx|$. The foliations $\mathcal{F}_h$ and $\mathcal{F}_v$ have the zero set of q as their common singular set, and at each zero of order k they have a $(k+2)$-pronged singularity, locally modelled on the singularity at the origin of $z^k dz^2$. The leaves passing through a singularity are the *singular leaves* of the measured foliation. A *saddle connection* is a leaf joining two (not necessarily distinct) singular points. The union of the saddle connections of the vertical foliation is called the *critical graph* $\Gamma(q)$ of q.

The components $X \setminus \Gamma(q)$ are of two types: cylinders swept out by vertical trajectories (i.e. leaves of $\mathcal{F}_v$) of equal length, and *minimal components* where each leaf of $\mathcal{F}_v$ is dense.

Teichmüller maps and rays. *Teichmüller's Theorem* states that, given any $X, Y \in \mathrm{Teich}(S)$, there exists a unique (up to translation in the case when S is a torus) quasiconformal map f, called the *Teichmüller map*, realizing $d_{\mathrm{Teich}(S)}(X,Y)$. The Beltrami coefficient $\mu := \frac{\bar\partial f}{\partial f}$ is of the form $\mu = k\frac{\bar q}{|q|}$ for some $q \in \mathrm{QD}^1(X)$ and some k with $0 \le k < 1$. In natural local coordinates given by q and a quadratic differential q' on Y, we have $f(x+iy) = Kx + \frac{1}{K}iy$, where $K = K(f) = \frac{1+k}{1-k}$. Thus f dilates the horizontal foliation by K and the vertical foliation by $1/K$.

Any $q \in \mathrm{QD}^1(X)$ determines a geodesic ray $r = r_{(X,q)}$ in $\mathrm{Teich}(S)$, called the *Teichmüller ray* based at X in the direction of q. The ray r is given by the complex structures determined by the quadratic differentials $q(t)$ obtained by multiplying the transverse measures of $\mathcal{F}_h(q)$ and $\mathcal{F}_v(q)$ by $\frac{1}{K} = e^{-t}$ and $K = e^t$, respectively, for $t > 0$. To summarize, for each $X \in \mathrm{Teich}(S)$, there is a bijective correspondence between the set of rays in $\mathrm{Teich}(S)$ based at X and the set of elements of $\mathrm{QD}^1(X)$.

Finally, we note that any ray in $\mathcal{M}(S)$ is the image of a ray in $\mathrm{Teich}(S)$ under the natural quotient map

$$\mathrm{Teich}(S) \to \mathcal{M}(S) = \mathrm{Teich}(S)/\mathrm{Mod}(S).$$

2.2 Extremal length and Kerckhoff's formula

Kerckhoff [Ke] discovered an elegant and useful way to compute Teichmüller distance in terms of extremal length, which is a conformal invariant of isotopy classes of simple closed curves. We now describe this, following [Ke].

Recall that a *conformal metric* on a Riemann surface X is a metric which is locally of the form $\rho(z)|dz|$, where ρ is a non-negative, measurable, real-valued function on X. A conformal metric determines a length function ℓ_ρ, which assigns to each (isotopy class of) simple closed curve γ the infimum $\ell_\rho(\gamma)$ of the lengths of all curves in the isotopy class, where length is measured with respect to the conformal metric. We denote the area of X in a conformal metric given by a function ρ by $\mathrm{Area}_\rho(X)$, or Area_ρ when X is understood.

By *cylinder* we will mean the surface $S^1 \times [0,1]$, endowed with a conformal metric. Recall that any cylinder C is conformally equivalent to a unique annulus of the form $\{z \in \mathbf{C} : 1 \leq |z| \leq r\}$. The number $(\log r)/2\pi$ will be called the *modulus* of C, denoted $\mathrm{mod}(C)$. A *cylinder in* X is an embedded cyclinder C in X, endowed with the conformal metric induced from the conformal metric on X. There are two equivalent definitions of extremal length, each of which is useful.

Definition 2.1 (Extremal length)**.** *Let X be a fixed Riemann surface, and let γ be an isotopy class of simple closed curves on X. The* extremal length *of γ in X, denoted by* $\mathrm{Ext}_X(\gamma)$*, or* $\mathrm{Ext}(\gamma)$ *when X is understood, is defined to be one of the following two equivalent quantities:*

Analytic definition:

$$\mathrm{Ext}(\gamma) := \sup_\rho \ell_\rho(\gamma)^2 / \mathrm{Area}_\rho$$

where the supremum is over all conformal metrics ρ on X of finite positive area.

Geometric definition:

$$\mathrm{Ext}(\gamma) := \inf\{\frac{1}{\mathrm{mod}(C)} : C \textit{ is a cylinder with core curve isotopic to } \gamma\}$$

As pointed out by Kerckhoff in [Ke], and as we will see throughout the present paper, the analytic definition is useful for finding lower bounds for $\mathrm{Ext}(\gamma)$, while the geometric definition is useful for finding upper bounds.

Theorem 2.2 (Kerckhoff [Ke], Theorem 4)**.** *Let S be any surface of finite type, and let X, Y be any two points of* $\mathrm{Teich}(S)$*. Then*

$$d_{\mathrm{Teich}(S)}(X,Y) = \frac{1}{2}\log\left[\ \sup_\gamma \frac{\mathrm{Ext}_X(\gamma)}{\mathrm{Ext}_Y(\gamma)}\ \right] \tag{1}$$

where the supremum is taken over all isotopy classes of simple closed curves γ on S.

Remark. The definition of extremal length is easily extended to measured foliations. The density of simple closed curves in the space $\mathcal{MF}(S)$ of measured foliations on S allows us to replace the right hand side of (1) by the supremum taken over all $\gamma \in \mathcal{MF}(S)$.

2.3 Extremal length estimates along Strebel rays

Let (X, q) be a Riemann surface $X \in \mathrm{Teich}(S)$ with Strebel differential $q \in \mathrm{QD}(X)$, and let $r = r_{(X,q)}$ be the corresponding Strebel ray. Our goal in this subsection is to estimate the extremal length $\mathrm{Ext}_{r(t)}(\beta)$ of an arbitrary (isotopy class of) simple closed curve β as the underlying Riemann surface moves along the ray r. The following estimates are due to Kerckhoff [Ke]. We include proofs here for completeness, and because these estimates are so essential for this paper.

The setup will be as follows. Let $C_i, 1 \leq i \leq n$ be the cylinders of the Strebel differential q, and for each i let α_i denote the homotopy class of the core curve of C_i. Let $a_i(t)$ denote the $q(t)$-length of α_i and let $b_i(t)$ denote the $q(t)$-height of C_i. Let $M_i(t) = \mathrm{mod}(C_i) = b_i(t)/a_i(t)$ be the modulus. Note that on the Riemann surface $r(t)$ we have

$$a_i(t) = e^{-t} a_i(0)$$

and the height $b_i(t)$ of the cylinder C_i satisfies

$$b_i(t) = e^{t} b_i(0).$$

Recall that the *geometric intersection number* of two isotopy classes of simple closed curves α, β, denoted $i(\alpha, \beta)$, is the miminal number of intersection points of curves α' and β' isotopic to α and β, respectively.

Lemma 2.3. *With notation as above, the following hold:*

1. $\lim_{t\to\infty} e^{2t} M_i(0) \mathit{Ext}_{r(t)}(\alpha_i) = 1$.
2. *There is a constant $c > 0$ such that if $i(\beta, \alpha_i) = 0$ for all i and β is not isotopic to any of the α_i, then for all t large enough,*
$$\mathit{Ext}_{r(t)}(\beta) \geq c.$$
3. *There is a constant $c > 0$ such that if β crosses C_i then for t large enough,*
$$\mathit{Ext}_{r(t)}(\beta) \geq c e^{2t}.$$

Proof. To prove Statement (1) we recall that the geometric definition of extremal length says that
$$\mathrm{Ext}_{r(t)}(\alpha_i) = \inf \frac{1}{\mathrm{mod}(A)},$$
where the infimum is taken over all cylinders $A \subset r(t)$ homotopic to α_i. Statement 1 is immediate in the case that $n = 1$, for then by Theorem 20.4(3) of [St], taken with

$i = 1$, the modulus of a one-cylinder Strebel differential realizes the supremum of the moduli of all cylinders homotopic to α_1, so that the reciprocal realizes the infimum of the reciprocals of the moduli in the geometric definition. In that case the limit in Statement 1 is actually an equality for each t. Thus assume $m > 1$. On $r(t)$, the cylinder C_i has modulus $e^{2t}b_i(0)/a_i(0) = e^{2t}M_i(0)$, giving the bound

$$\mathrm{Ext}_{r(t)}(\alpha_i) \leq \frac{e^{-2t}}{M_i(0)}.$$

We now give a lower bound. We can realize the surface $r(t)$ by cutting along the core curves of the cylinders, that are halfway across eachcylinder, inserting cylinders of circumference $a_i(0)$ and height $\frac{b_i(0)(e^{2t}-1)}{2}$ to each side of the cut and then regluing. Rescaling by e^t the flat metric induced by $q(t)$, gives a flat metric $\rho(t)$ of area e^{2t} for which the core curves have constant length $a_i(0)$ and height $e^{2t}b_i(0)$. Choose a constant b such that $b > a_i(0)$, and for t_0 sufficiently large, choose a fixed neighborhood $\mathrm{Nbhd}(C_i)$ of C_i on $r(t_0)$ such that

$$d_{\rho(t_0)}(C_i, \partial\,\mathrm{Nbhd}(C_i)) = b.$$

For some fixed $B > 0$ we have

$$\mathrm{area}_{\rho(t_0)}(\mathrm{Nbhd}(C_i) \setminus C_i) = B.$$

Via the construction described above, we may think of $\mathrm{Nbhd}(C_i) \setminus C_i$ as a subset of $r(t)$ for $t \geq t_0$. Define a conformal metric $\sigma_i(t)$ on $r(t)$ as follows. It is given by $\rho(t)$ on C_i. On $\mathrm{Nbhd}(C_i) \setminus C_i$ it is given by the metric $\rho(t_0)$, and on $r(t) \setminus \mathrm{Nbhd}(C_i)$ it is given by $\delta\rho(t)$ for some $\delta > 0$. With respect to the metric $\sigma_i(t)$ we then have

$$d_{\sigma_i(t)}(C_i, \partial\,\mathrm{Nbhd}(C_i)) = b$$

and

$$\mathrm{Area}_{\sigma_i(t)} \leq B + \delta e^{2t} + e^{2t}a_i(0)b_i(0).$$

Since the distance across $\mathrm{Nbhd}(C_i) \setminus C_i$ is at least $b \geq a_i(0)$, it is easy to see that

$$\ell_{\sigma_i(t)}(\alpha_i) = a_i(0).$$

Putting the estimates on lengths and areas together, it follows that given any $\epsilon > 0$, we may choose $\delta > 0$ so that for t large enough,

$$\mathrm{Ext}_{r(t)}(\alpha_i) \geq \frac{\ell^2_{\sigma_i(t)}(\alpha_i)}{A_{\sigma_i(t)}} \geq (1-\epsilon)\frac{e^{-2t}}{M_i(0)}.$$

Putting this lower bound together with the upper bound we have proved (1).

For the proof of (2), for t_0 large enough, take a fixed neighborhood N of the component of the critical graph Γ that contains β such that the distance across N is at least $\min_i a_i(0)$, the lengths of the core curves of the cylinders on the base surface $r(0)$. Again we may consider N as a subset of $r(t)$ for all $t \geq 0$. We put a conformal metric $\sigma(t)$ on $r(t)$ which is given by the flat metric defined by $q(t)$ on $r(t) \setminus N$ and the metric defined by $q(t)$ scaled by e^t on N. For some fixed $B > 0$ we have

$$\text{Area}_{\sigma(t)} \leq B.$$

Now any geodesic representative of β that enters $r(t) \setminus N$ must bound a disc with a core curve of C_i, and can be shortened to lie entirely inside N. Thus its geodesic representative in fact lies in the critical graph and so there is a b such that

$$\ell_{\sigma(t)}(\beta) \geq b.$$

The lower bound now follows from these last two inequalities and the analytic definition of extremal length.

The proof of (3) follows by using the given metric $q(t)$ in the analytic definition of extremal length. $\diamond$

3 EDM and ADM rays in moduli space

In this section we classify EDM and ADM rays in moduli space, giving a proof of Theorem 1.4. We then determine, in §3.4, when two EDM rays are asymptotic.

3.1 Strebel rays are EDM

Our goal in this subsection is to prove one direction of Theorem 1.4, namely that if (X, q) is Strebel then the ray $r_{(X,q)}$ in $\mathcal{M}(S)$ is eventually distance minimizing.

Since $\text{Mod}(S)$ acts properly by isometries on $\text{Teich}(S)$ with quotient $\mathcal{M}(S)$, the distance between points $x, y \in \mathcal{M}(S)$ are the same as minimal distances between orbits of any lift of x, y to $\text{Teich}(S)$. We warn the reader that while every ray in $\mathcal{M}(S)$ comes from the projection to $\mathcal{M}(S)$ of a ray in $\text{Teich}(S)$, the converse is not true; this is due to the fixed points of the action of $\text{Mod}(S)$ on $\text{Teich}(S)$.

Thus, to achieve our goal, we must find $t_0 \geq 0$ so that

$$d_{\text{Teich}(S)}(r(t), r(t_0)) \leq d_{\text{Teich}(S)}(\phi(r(t_0)), r(t)) \tag{2}$$

for all $t \geq t_0$ and for every $\phi \in \mathrm{Mod}(S)$. In fact we will prove for Strebel rays that the inequality in (2) is strict for $t > t_0$, as long as ϕ doesn't have a fixed point.

Remark. Note that while any two nonseparating curves on S can be taken to each other via some element of $\mathrm{Mod}(S)$, Strebel rays along cylinders with nonseparating core curves, based at the same $Y \in \mathrm{Teich}(S)$, project to different rays in $\mathcal{M}(S)$. Indeed, given any point $X \in \mathcal{M}(S)$, there are countably infinitely many Strebel rays in $\mathcal{M}(S)$ based at X, even though there are $[g/2]+1$ topological types of simple closed curves on S.

Let $\alpha_1, \dots, \alpha_p$ denote the core curves of the cylinders $\{C_i\}$ in the cylinder decomposition of (X, q). By Lemma 2.3, the extremal length of curves β with $i(\beta, \alpha_i) = 0$ for each $1 \leq i \leq p$ and not homotopic to any α_i remain bounded below by some $d > 0$. By Lemma 2.3 the extremal length of any curve β with $i(\beta, \alpha_i) > 0$ for some i tends to ∞ as $t \to \infty$. Choose t_0 big enough so that each of the following holds:

1. If $i(\beta, \alpha_i) > 0$ for some i, then $\mathrm{Ext}_{r(t)}(\beta) \geq d$ for $t \geq t_0$.
2. $e^{2t_0} > 2\max_i(\frac{M_i}{d})$, where M_i is the modulus of the cylinder C_i
3. For $t \geq t_0$, $\mathrm{Ext}_{r(t)}(\alpha_i) \leq 2e^{-2t} M_i$. (This can be done by Lemma 2.3).

Let ϕ be any element of $\mathrm{Mod}(S)$ without a fixed point in $\mathrm{Teich}(S)$; this is the same as ϕ not having finite order. Suppose first that $\phi^{-1}(\alpha_i) = \beta \notin \{\alpha_j\}$ for some i. By Theorem 2.2 we have for $t > t_0$:

$$
\begin{aligned}
d_{\mathrm{Teich}(S)}(\phi(r(t_0)), r(t)) &\geq \tfrac{1}{2} \log \frac{\mathrm{Ext}_{\phi(r(t_0))}(\alpha_i)}{\mathrm{Ext}_{r(t)}(\alpha_i)} \\
&= \tfrac{1}{2} \log \frac{\mathrm{Ext}_{r(t_0)}(\beta)}{\mathrm{Ext}_{r(t)}(\alpha_i)} \\
&\geq \tfrac{1}{2} \log \frac{d}{2e^{-2t} M_i} \\
&> \tfrac{1}{2} \log \frac{e^{2t}}{e^{2t_0}} = t - t_0 = d_{\mathrm{Teich}(S)}(r(t), r(t_0))
\end{aligned}
$$

Thus we may assume that ϕ preserves $\{\alpha_i\}$ as a set. Consider the special case when $\phi(\alpha_i) = \alpha_i$ for each i. This assumption implies that ϕ^{-1} preserves the vertical foliation $\mathcal{F}_v(q)$ of q, as a measured foliation. Then

$$
d_{\mathrm{Teich}(S)}(\phi(r(t_0)), r(t)) \geq \frac{1}{2} \log \frac{\mathrm{Ext}_{\phi(r(t_0))}(\mathcal{F}_v(q))}{\mathrm{Ext}_{r(t)}(\mathcal{F}_v(q))} = \frac{1}{2} \log \frac{\mathrm{Ext}_{r(t_0)}(\mathcal{F}_v(q))}{\mathrm{Ext}_{r(t)}(\mathcal{F}_v(q))} = t - t_0 \qquad (3)
$$

and we are again done in this case. The leftmost inequality follows from the remark after Theorem 2.2.

We remark that the inequality (3) is strict. This is because equality of the leftmost terms occurs if and only if $\mathcal{F}_v(q)$ is the vertical foliation of the quadratic differential defining the Teichmuller map from $\phi(r(t_0))$ to $r(t)$. However, $\mathcal{F}_v(q)$ is the vertical foliation of the quadratic differential of the Teichmuller map from $r(t_0)$ to $r(t)$, and so it cannot be the former since ϕ is assumed to be nontrivial.

Finally, consider the general case of ϕ preserving $\{\alpha_i\}$ as a set. Let k be the smallest integer such that $\phi^k(\alpha_i) = \alpha_i$ for all i. If the desired result is not true there is a sequence of times $t_0 < t_1 < \ldots < t_k$ such that

$$d_{\mathrm{Teich}(S)}(r(t_{i-1}), \phi(r(t_i))) < d_{\mathrm{Teich}(S)}(r(t_{i-1}), r(t_i)).$$

Since ϕ acts as an isometry of $\mathrm{Teich}(S)$, applications of the triangle inequality give

$$d_{\mathrm{Teich}(S)}(r(t_0), \phi^k(r(t_k))) < d_{\mathrm{Teich}(S)}(r(t_0), r(t_k)).$$

But ϕ^k fixes each α_i, and we have a contradiction to the previous assertion.

3.2 Every EDM ray is Strebel

In this subsection we prove the other direction of Theorem 1.4, namely that if a ray $r_{(X,q)} : [0, \infty) \to \mathcal{M}(S)$ is EDM then (X, q) is Strebel. The idea of the proof is explained in the introduction above. Since r is EDM, we can change basepoint and assume that r is (globally) isometrically embedded. We henceforth assume this.

Recall that for each $t \geq 0$, the ray $r = r_{(X,q)}$ determines the following data: the Riemann surface $r(t) \in \mathcal{M}(S)$, the quadratic differential $q(t) \in \mathrm{QD}^1(r(t))$, and the vertical foliation $\mathcal{F}_v(q(t))$ for the quadratic differential $q(t)$. Let $\Gamma(t)$ denote the critical graph of $q(t)$, so that $\Gamma(t)$ is the union of the vertical saddle connections of $q(t)$. Note that $\Gamma(t)$ may be empty.

For any quadratic differential q on a Riemann surface X, let Σ denote the set of zeroes of q. We define the *diameter* of X (in the q-metric d_q), denoted $\mathrm{diam}(X)$, to be

$$\mathrm{diam}(X) := \sup_{x \in X} d_q(x, \Sigma).$$

Now suppose that the ray $r = r_{(X,q)}$ is not Strebel. This assumption implies that there is some subsurface $Y(t) \subseteq r(t)$ which contains some leaf of $\mathcal{F}_v(q(t))$ which is dense in $Y(t)$. We will find a contradiction.

Step 1 (Delaunay triangulations):

Proposition 3.1. *There is a triangulation $\Delta(t)$ on $r(t)$ with the following properties:*

1. *The vertices of $\Delta(t)$ lie in the zero set of $q(t)$.*

2. *The edges of $\Delta(t)$ are saddle connections of $q(t)$.*

3. *For t large enough, every edge of the vertical critical graph $\Gamma(t)$ is an edge of $\Delta(t)$.*

4. *There is a function $c(t)$ with $c(t) \to \infty$ as $t \to \infty$ so that every triangle in $\Delta(t)$ whose interior is contained in some minimal component Y, can be inscribed in a circle of radius at most $e^t/c(t)$.*

Proof. The triangulation $\Delta(t)$ will be the *Delaunay triangulation* $\Delta(t)$ constructed by Masur-Smillie in §4 of [MS]. In particular, $\Delta(t)$ automatically satisfies (1) and (2). We now claim something very special about $\Delta(t)$.

Lemma 3.2. *There is a function $c(t)$ with $\lim_{t\to\infty} c(t) = \infty$ with the following property: the shortest saddle connection $\beta(t)$ of the quadratic differential $q(t)$ on $r(t)$, whose endpoints lie in $\overline{Y} \cap \Sigma$, and whose interior lies in Y, has length at least $c(t)e^{-t}$.*

Proof. [of Lemma 3.2] Denote by $|\cdot|_t$ the length function associated to flat metric on $r(t)$ induced by $q(t)$. For an arc α, we denote by $|\alpha|_t^{\text{vert}}$ (resp. $|\alpha|_t^{\text{horiz}}$) the length of α as measured with respect to the transverse measure $|dy|$ on $\mathcal{F}_h(q(t))$ (resp. $|dx|$ on $\mathcal{F}_v(q(t))$).

We claim that there is a constant D such that $|\beta(t)|_t \leq D$. To prove the claim, consider an edge E of the Delaunay triangulation $\Delta(t)$ with $E \cap Y \neq \emptyset$. First suppose $|E|_t \leq s =: 2\sqrt{2/\pi}$. If $E \subset Y$ then take $D = s$ and we are done. If E is not contained in Y, then it crosses some edge α of $\Gamma(t)$. We remind the reader that, as we move out along $r(t)$, the horizontal lengths are expanded by e^t and the vertical length are contracted by e^{-t}. Thus we have the equation

$$|\alpha|_t = e^{-t}|\alpha|_0. \tag{4}$$

But then we can take some subsegment of E, together with a union of at most two subsegments of $\Gamma(t)$, to give a nontrivial homotopy class of arc with endpoints in $\overline{Y} \cap \Sigma$ and interior contained in Y. The geodesic representative $\beta(t)$ in this homotopy class has length bounded above by the length of E plus the length of $\Gamma(t)$, which for large enough t is less than $D = s + 1$, and we are done.

We are now reduced to the case where E has length at least s. By Proposition 5.4 of [MS], E must cross some flat cyclinder C in $r(t)$ whose height is greater than its circumference. If $C \subset Y$, then since Y has area at most 1, the circumference is at most 1, and so taking $\beta(t)$ to be the circumference, we have $|\beta(t)|_t \leq 1$. If C is not contained in Y, then C crosses the critical graph $\Gamma(t)$. Thus the height of C is bounded, as in (4). Thus the circumference is bounded as well. An argument similar to the previous paragraph then provides $\beta(t)$, and the claim is proved.

We now continue with the proof of the lemma. We have

$$|\beta(t)|_t \geq |\beta(t)|_t^{\text{horiz}} = e^t|\beta(t)|_0^{\text{horiz}}.$$

Since $|\beta(t)|_t$ is bounded, we must have $|\beta(t)|_0^{\text{horiz}} \to 0$ as $t \to \infty$. Since Y is assumed to be minimal, there are no vertical saddle connections in Y, and so $|\beta(t)|_0^{\text{horiz}} > 0$. Because the set of holonomy vectors of saddle connections is a discrete subset of $\mathbf{R}^2$ for a fixed flat structure (see, e.g, [HS]), this forces $|\beta(t)|_0^{\text{vert}} \to \infty$ as $t \to \infty$. Now

$$|\beta(t)|_t \geq |\beta(t)|_t^{\text{vert}} = e^{-t}|\beta(t)|_0^{\text{vert}}.$$

Thus the desired inequality holds with $c(t) = |\beta(t)|_0^{\text{vert}}$. $\diamond$

We continue with the proof of Proposition 3.1.

For t sufficiently large, by Lemma 3.2 the segments of $\Gamma(t)$ are the shortest saddle connections on $q(t)$. Now since these segments are all vertical, given such a segment α, the midpoint p of α has the property that the two endpoints of α realize the distance from p to Σ. Thus, by construction of the Delaunay triangulation (see §4 of [MS]), the entire segment α lies in $\Delta(t)$. This proves (3).

We now prove (4). By (3), no edge of the triangulation crosses $\Gamma(t)$, so any 2-cell that intersects a minimal component Y is contained in that minimal component. By Theorem 4.4 of [MS], every point in Y is contained in a unique Delaunay cell isometric to a polygon inscribed in a circle of radius $\leq \text{diam}(Y)$. It therefore remains to bound $\text{diam}(Y)$.

If $\text{diam}(Y) > 2s$, then there is a cylinder C whose height is at least s. As we have seen above, such a cylinder must be contained in Y, as it cannot cross $\Gamma(t)$ for sufficiently large t. But Lemma 3.2 gives that the circumference of C is at least $c(t)e^{-t}$, and since $r(t)$ has unit area, the height of C is at most $e^t/c(t)$, and there the diameter is at most $(e^t/2c(t))+1$ (the second term coming from a bound on the length of the circumference of C). $\diamond$

We now return to the proof that EDM rays are Strebel. Recall we are arguing by contradiction, so that we are assuming the quadratic differential q_0 defining the say is not Strebel. Thus there is at least one minimal component in the complement of the critical graph of q_0. Let $C_1, \ldots, C_r$ be the (possibly empty) collection of vertical cylinders of q_0.

Recall now that for t large enough, by Proposition 3.1, the critical graph of $q(t)$ are edges of the Delaunay triangulation $\Delta(t)$ of $q(t)$. Consider $\Delta(t)$ restricted to the complement of the cylinders C_i.

Proposition 3.3. *Let (r_0, q_0) be given with (possibly empty) cylinder data. There exist finitely many triangulations $T_1, \ldots, T_m$ of the complement of the set of cylinders of (r_0, q_0), with the following property: for any combinatorial type of triangulation Δ that appears as the Delaunay triangulation $\Delta(t_n)$ of $(r(t_n), q(t_n))$ for a sequence $t_n \to \infty$, there exists some T_i combinatorially equivalent to Δ on the complement of the cylinders.*

Proof. For any such Δ, choose $t_1 \geq 0$ to be the smallest time for which $\Delta(t_1)$ appears in its combinatorial equivalence class. Now let T_1 be the pullback of $\Delta(t_1)$ by the Teichmuller map $f : r(0) \to r(t_1)$. We remark that T_1 is not necessarily Delaunay with respect to the flat structure given by $q(0)$.

We now do this for each new combinatorial class that appears along $r(t)$. There are only finitely many such T_i since there are only finitely many combinatorial types of triangulations with a fixed number of vertices and edges. ⋄

Step 2 (Building the fake Teichmüller map): Given $r(t)$, we will build a very efficient map ψ from some $r(0)$ to $r(t)$. We first need the following lemma about Euclidean triangles.

Lemma 3.4 (Euclidean triangle lemma)**.** *Fix a triangle T_0 in the Euclidean plane. Then there is a constant b, depending only on T_0, with the following property: for any other Euclidean triangle T whose shortest side has length at least ϵ, such that each side has length at most R, and which can be inscribed in a circle of radius R, there is an affine map from T_0 to T which has quasiconformal dilatation at most bR/ϵ.*

Proof. Let $p_i, i = 1, 2, 3$ be the vertices of T on the circle arranged in counterclockwise order and assume $\overline{p_1 p_2}$ is the shortest side with length $a_1 \geq \epsilon$, $\overline{p_1, p_3}$ is the longest side with length $a_3 \leq 2R$. Let p_0 the center of the circle. Let θ the angle at p_3 of the T. We claim that

$$\sin(\theta) = \frac{a_1}{2R} \geq \frac{\epsilon}{2R}.$$

The first case is that the segment $\overline{p_1 p_3}$ separates p_0 from p_2. Let ψ_1 be the angle at p_0 of the isoceles triangle with vertices at p_0, p_1, p_2. Let ψ_2 the angle at p_0 of the isoceles triangle with vertices p_0, p_2, p_3. Let ψ the angle at p_3 of the isoceles triangle with vertices p_0, p_1, p_3. Since this triangle is isoceles, we have

$$2\psi = (\pi - (\psi_1 + \psi_2)).$$

Since the triangle with vertices at p_0, p_2, p_3 is isoceles, we have

$$2(\psi + \theta) = \pi - \psi_2.$$

Subtracting we get

$$\theta = \psi_1/2,$$

proving the claim. A similar analysis holds if $\overline{p_1p_3}$ does not separate p_0 from p_2.

Similarly we have θ', the angle of T at p_1, is given by

$$\sin(\theta') = \frac{a_2}{2R} \geq \sin(\theta),$$

where a_2 is length of the side $\overline{p_2p_3}$. Now let h be the height of the triangle T with vertex p_2 and opposite side length a_3. It divides T into a pair of triangles T_1, T_2 with bases x_1, x_2 along $\overline{p_3p_1}$ and angles θ, θ'. Since $\theta' \geq \theta$ we have

$$x_2/h \leq x_1/h = \cot(\theta) \leq 1/\sin(\theta) \leq 2R/\epsilon.$$

Thus if we double the triangles along the hypotenuse we find their moduli are bounded by $2R/\epsilon$ and so the affine map to a standard isoceles right triangle has dilatation bounded in terms of R/ϵ. ⋄

Proposition 3.5. *For t sufficiently large, there exists a map $\psi : r(0) \to r(t)$ which is at most an e^{2t}-quasiconformal map and which is not the Teichmüller map.*

Proof. For any t sufficiently large, choose T_i such that the $(r(0), q(0))$ triangulation T_i described in Proposition 3.3 is combinatorially equivalent to the Delaunay triangulation $\Delta(t)$ on $r(t)$, say via a homeomorphism $h : r(0) \to r(t)$.

Let a be the length of the shortest vertical saddle connection of $(r(0), q(0))$. We now build the map $\psi : r(0) \to r(t)$. On each vertical cylinder, ψ will be the linear map of least quasiconformal dilatation, which is e^{2t}. Notice this is the map that agrees with the Teichmuller map $f : r(0) \to r(t)$ on the cylinder. Extend the map to the obvious linear map on the critical graph $\Gamma(0)$. We are left with having to define ψ on the nonempty collection of complementary minimal components of $r(0) \setminus \Gamma(0)$.

The homeomorphism h gives a bijective mapping between the set of triangles of T_i and those of $\Delta(t)$. For each triangle P of T_i, each of P and $h(P)$ has a given Euclidean structure. Define ψ to be the unique affine map wihich identifies edges in the same combinatorial way as h does.

Let a be the length of the shortest edge in the critical graph Γ. By property (4) of Proposition 3.1 applied to $\Delta(t)$, we can apply Lemma 3.4 with $\epsilon \geq ae^{-t}$ and $R = e^t/c(t)$ to conclude that on the union of the interiors of the triangles which are not in any vertical cylinder, the pointwise quasiconformal dilatation is at most

$$\frac{e^t b}{ac(t)e^{-t}} = \frac{be^{2t}}{ac(t)}.$$

Note that since there are only finitely many T_i, the constant b is universal.

Since $c(t) \to \infty$, this number can be taken to be smaller than e^{2t}. Note that with quasiconformal maps we only need to check dilatation on a set of full measure, since quasiconformal dilatation is an L^∞ norm.

Thus the (global) dilatation $K(\psi)$, as a supremum of the dilatation over all points on the surface, equals e^{2t}, but ψ is not the Teichmüller map since the dilatation is not constant. Namely, it is strictly smaller than e^{2t} for any point in the minimal component. ⋄

Step 3 (The trick): Since ψ is not the Teichmüller map, there is a Teichmüller map $\Phi : r(0) \to r(t)$ in the same homotopy class of ψ, with dilatation strictly smaller than that of ψ, which is e^{2t}. Hence the distance in moduli space from $r(0)$ to $r(t)$ is strictly less than $\frac{1}{2}\log e^{2t} = t$, and we are done.

3.3 ADM rays

Our goal in this subsection is to prove the following.

Theorem 3.6. *A Teichmuller geodesic $r(t)$ determined by (X_0, q_0) is ADM if and only if it is mixed Strebel.*

Proof. Suppose (X_0, q_0) is mixed Strebel. Let C be a cylinder with modulus M in the homotopy class of some β. Let

$$b := \inf\{\mathrm{Ext}_{X_0}(\alpha) : \alpha \text{ is a simple closed curve}\} > 0$$

On $r(t)$ the image of C has modulus $e^{2t}M$. By the geometric definition of extremal length, the extremal length of β on $r(t)$ is at most e^{-2t}/M. By Kerckhoff's distance formula (Theorem 2.2 above), for any $\phi \in \mathrm{Mod}(S)$,

$$d_{\mathrm{Teich}(S)}(\phi(r(0)), r(t)) \geq \frac{1}{2}\log\frac{Mb}{e^{-2t}} = t + \frac{1}{2}\log M + \frac{1}{2}\log b.$$

We have thus proved with $C = -\frac{1}{2}\log M - \frac{1}{2}\log b$ that mixed Strebel implies ADM.

Now assume that $r(t)$ is ADM. We need to show that (X_0, q_0) is mixed Strebel. We argue by contradiction: assume that q_0 has no vertical cylinder.

Since $r(t)$ is ADM, it cannot return to any compact set in $\mathcal{M}(S)$ for arbitrarily large times. Therefore, for sufficiently large t, there is a nonempty maximal collection $\beta_1(t), \ldots, \beta_n(t)$ of simple closed curves whose hyperbolic length is less than some fixed ϵ, the Margulis constant.

We have

$$|\beta_j(t)| \geq e^{-t}|\beta_j(t)|_0^{\mathrm{vert}} \geq ce^{-t}|\beta_j(t)|_0,$$

for some fixed $c > 0$. By Theorems 4.5 and 4.6 of [Mi2], since by assumption (X_0, q_0) has no vertical cylinder, we have that for some fixed $\delta > 0, \delta' > 0$:

$$\mathrm{Ext}_{r(t)}(\beta_j(t)) \geq \frac{\delta}{-\log|\beta_j(t)|_t} \geq \frac{\delta}{t - \log(|\beta_j(t)|_0)} \geq \frac{\delta'}{t}, \tag{5}$$

for t sufficiently large.

By a theorem of Maskit (see [Mas]), the ratio of the hyperbolic length of $\beta_j(t)$ to its extremal length tends to 1 as $t \to \infty$, so we can assume that the hyperbolic lengths of β_j satisfy the same lower bounds.

Now fix a collection of uniformly bounded length curves $\gamma_1, \ldots, \gamma_n$ on X_0 combinatorially equivalent to the collection of $\beta_i(t)$, which means that the is an element $\phi(t)$ of the mapping class group taking the $\beta_i(t)$ to the γ_i. Since the curves in any complementary component Y of the β_i have length bounded below, we can further choose ϕ on Y so that for any curve of $\phi(Y)$ the extremal lengths on $\phi(r(t))$ and X_0 have bounded ratio.

By moving a bounded Teichmuller distance we can shorten the γ_i so that they have fixed length ϵ. We can now apply the Minsky product theorem (see [Mi1]) to find constants C_1, C_2 such that that

$$d_{\mathrm{Teich}(S)}(X_0, \phi(r(t))) \leq \max_j\{\frac{1}{2}\log\frac{C_1}{\mathrm{Ext}_{r(t)}(\beta_j(t))}\} + C_2$$

which by (5) is at most

$$\log t - \log\delta' + \log C_1 + C_2$$

for t sufficiently large. Thus $r(t)$ is not almost length minimizing. $\diamond$

3.4 Asymptote classes of EDM rays

We say that two rays r, r' are *asymptotic* if there is a choice of basepoints $r(0), r'(0)$ so that $\lim_{t\to\infty} d(r(t), r'(t)) \to 0$. In this section we determine the asymptote classes of EDM rays. We will then use these rays in Section 4.3 to compactify $\mathcal{M}(S)$.

Definition 3.7 (Endpoint of a ray)**.** *Let (X, q) be a Strebel differential with maximal cylinders $C_1, \ldots, C_p$, determining a ray $r : [0, \infty) \to \mathrm{Teich}(S)$. Cut each C_i along a circle and glue into each side of the cut an infinite cylinder. The resulting surface with punctures $\hat{X}$ is the* endpoint *of r, denoted $r(\infty)$. It carries a quadratic differential $q(\infty)$ with double poles at the punctures, with equal residues, such that the vertical trajectories are closed leaves isotopic to the punctures.*

The surface $\hat{X}$ can be considered as an element of the product of Teichmuller spaces of its connected components. We denote this moduli space, or product of moduli spaces, which we endow with the sup *metric, by $\mathrm{Teich}(\hat{X})$.*

We note that $\hat{X}$ and $q(\infty)$ do not depend on where C_i is cut. The following definition is due to Kerckhoff [Ke].

Definition 3.8 (Modularly equivalent differentials)**.** *Suppose that* $(X, q), (X', q')$ *are Strebel differentials with maximal cylinders* $C_1, C_2, \ldots, C_p$ *and* $C'_1, \ldots, C'_r$ *respectively. We say that these differentials are* modularly equivalent *if each of the following holds:*

1. $p = r$.
2. *After reindexing, up to the action of there is an element* $\mathrm{Mod}(S)$, *for each* i, $\phi(C_i)$ *is homotopic to* C'_i.
3. *There exists* $\lambda > 0$ *so that* $Mod(C_i) = \lambda Mod(C'_i)$ *for each* i.

Suppose a pair of rays r, r' are modularly equivalent. Since the moduli change by a fixed factor along rays, we can choose basepoints $r(0), r'(0)$ so that the cylinders have the same moduli at the basepoints, and define

$$d(r, r') = \lim_{t \to \infty} d_{\mathcal{M}(S)}(r(t), r'(t))$$

if the limit exists.

Theorem 3.9. *With the notation as above, suppose that* r *and* r' *are modularly equivalent. Then* $d(r, r')$ *exists and* $d(r, r') = d_{\mathcal{M}(\hat{X})}(r(\infty), r'(\infty))$.

Assuming Theorem 3.9 for the moment, we have the following.

Corollary 3.10. *Two rays* r, r' *are asymptotic if and only if they are modularly equivalent and they have the same endpoints* $r(\infty) = r'(\infty)$.

This corollary was proven by Kerckhoff [Ke] in the case of a maximal collection of cylinders.

Proof. [of Corollary 3.10] The "if" direction follows immediately from Theorem 3.9. For the "only if" direction, we first note that the hypothesis implies that for each n sufficiently large, there is a sequence of $(1 + o(1))$-quasiconformal maps $f_n : r(n) \to r'(n)$. Since uniformly quasiconformal maps form a normal family (see, e.g., [Hu], Theorem 4.4.1) and $r(n), r'(n)$ converge to $r(\infty), r'(\infty)$, there is a subsequence of $\{f_n\}$ which converges to a conformal map $f_\infty : r(\infty) \to r'(\infty)$, so that $r(\infty) = r'(\infty)$. Modular equivalence of r and r' follows immediately from (1) of Lemma 2.3 and Kerckhoff's distance formula (Theorem 2.2). ⋄

We now begin the proof of Theorem 3.9.

Proof. We first note that, exactly as in the proof of Corollary 3.10, we have

$$d_{\mathcal{M}(\hat{X})}(r(\infty), r'(\infty)) \leq \liminf_{t\to\infty} d_{\mathcal{M}(S)}(r(t), r'(t)).$$

To prove the opposite inequality we first need the following lemma.

Lemma 3.11. *Suppose $\epsilon > 0$ is given. Let C_1, C_2 be Euclidean cylinders with heights R_1, R_2 and circumference 1. Now in coordinates (x, y) in the upper half-space model $\mathbf{H}^2$ of the hyperbolic plane, given any $n \in \mathbf{Z}$ we let $z_1 = (0, R_1)$ and $z_2 = (n, R_2)$ be points in $\mathbf{H}^2$. Let*

$$d_0 := d_{\mathbf{H}^2}(z_1, z_2).$$

Let p_1, q_1 marked points on the boundary of C_1 assumed to be at $(0, 0)$ and $(0, R_1)$. Let p_2, q_2 marked points on the boundary of C_2 at $(0, 0)$ and (α, R_2) in polar coordinates (θ, h) on C_2. Let $f(\theta)$ be a real analytic function defined from the base $h = 0$ of C_1 to the base of C_2 such that $f(0) = 0$ and

$$\sup_{\theta} |f'(\theta) - 1| \leq \epsilon.$$

Let γ_1 be the vertical line in C_1 joining p_1 to q_1. Let β be the Euclidean geodesic joining $(0, 0)$ to (α, R_2) in C_2. Let γ_2 be the local geodesic in the relative homotopy class of β twisted n times about the core curve of C_2. Then for R_1, R_2 large enough, there is a $(1 + O(\epsilon))e^{2d_0}$-quasiconformal map $F : C_1 \to C_2$ such that

- *$F(\theta, 0) = (f(\theta), 0)$.*
- *$F(q_1) = q_2$.*
- *$F(\gamma_1)$ is homotopic to γ_2 relative to the boundary of C_1.*

Proof. Define $F = (F_1, F_2)$ by

$$F(\theta, h) = ((1 - \frac{h}{R_1})f(\theta) + \frac{h(\theta + \alpha + n)}{R_1}, \frac{hR_2}{R_1});$$

the first coordinate taken modulo 1. We have $F(\theta, 0) = (f(\theta), 0)$ and $F(0, R_1) = (\alpha, R_2)$ and $F(\gamma_1) = \gamma_2$. We compute

$$\partial F_1/\partial\theta = \frac{h}{R_1}(1 - f'(\theta)) + f'(\theta)$$

$$\partial F_2/\partial h = \frac{R_2}{R_1}$$

$$\partial F_1/\partial h = \frac{1}{R_1}(\theta + \alpha + n - f(\theta))$$

and

$$\partial F_2/\partial\theta = 0.$$

So we have

$$|\partial F_1/\partial\theta - 1| < 2\epsilon$$

and for R_1 sufficiently large we have

$$|\partial F_1/\partial h - n/R_1| \leq \epsilon.$$

Thus

$$|\mathrm{Jac}(F) - \begin{pmatrix} 1 & n/R_1 \\ 0 & R_2/R_1 \end{pmatrix}| = O(\epsilon),$$

where Jac stands for Jacobian. Note that the above linear map is the Teichmuller map taking the marked torus spanned by $\{(1,0),(0,R_1)\}$ to the marked torus spanned by $\{(1,0),(n,R_2)\}$. These tori correspond to the given points in $\mathbf{H}^2$ and therefore the dilatation of the linear map is precisely e^{2d_0}, as claimed. $\diamond$

We also need the following lemma.

Lemma 3.12. *Let $g : \hat{X} \to \hat{X}'$ a Teichmuller map with dilatation K_0. Given $\epsilon > 0$, there is a $(K_0 + \epsilon)$-quasiconformal map $f : \hat{X} \to \hat{X}'$ which is conformal in a neighborhood of of the punctures.*

Proof. Let μ be the dilatation of g. For any small neighborhood $0 < |z| < |t|$ of the punctures, let μ_t be the Beltrami differential which is 0 in $0 < |z| < |t|$ and μ in the complement. For some surface $\hat{X}_t$, there is a K_0-quasiconformal map $f_t : \hat{X} \to \hat{X}_t$ with dilatation μ_t; in particular f_t is conformal in $0 < |z| < |t|$. As $t \to 0$, $\mu_t \to \mu$ and therefore $\lim_{t\to 0} f_t = g$ and so $\lim_{t\to 0} \hat{X}_t = \hat{X}'$. Choose a nonempty open set $\mathcal{V}$ on $\hat{X}$. We can find a collection of Beltrami differentials supported in $\mathcal{V}$ that form a basis for the tangent space to Teich at $\hat{X}'$. This implies that for t small enough we can find a $(1+\epsilon)$-quasiconformal map $h_t : \hat{X}_t \to \hat{X}'$ which is conformal in a neighborhood of the punctures. Our desired map is $f = h_t \circ f_t$. $\diamond$

Now we begin the proof of the bound

$$\limsup_{t\to\infty} d_{\mathcal{M}(S)}(r(t), r'(t)) \leq d_{\mathcal{M}(\hat{X})}(r(\infty), r'(\infty)).$$

Let $p_i, q_i, i = 1, \ldots, p$ be the paired punctures on $r(\infty)$, and let z_i be the coordinate at p_i so that for some $a_i > 0$,

$$q(\infty) = \frac{a_i^2}{z_i^2} dz_i^2,$$

we have a similar coordinate in a neighborhood of q_i. Let ζ_i the corresponding coordinate for $q'(\infty)$ on $r'(\infty)$ in a neighborhood of p'_i so that

$$q'(\infty) = \frac{b_i^2}{\zeta_i^2} d\zeta_i^2.$$

Circles in these coordinates are vertical leaves for $q(\infty)$ and $q'(\infty)$ and have lengths $2\pi a_i$ and $2\pi b_i$ respectively. For some $\delta_j(t)$ we recover the surfaces along $r(t)$ by removing punctured discs of radius $\delta_i^{1/2}(t)$ around p_i and q_i and glueing the resulting surfaces along their boundary. We have

$$\lim_{t \to 0} \delta_j(t) = 0.$$

We have a similar picture for r' with corresponding $\delta_i'^{1/2}(t)$. The assumption that r, r' are modularly equivalent means that for each δ_i there is δ'_i, such that the resulting cylinders A_i, A'_i on $r(t), r'(t)$ have the same modulus. For convenience we drop the subscript i.

Let $K = e^{d_{\mathcal{M}(\hat{X})}(r(\infty), r'(\infty))}$. Given ϵ, let $F_2 : r'(\infty) \to r(\infty)$ be the $(K+\epsilon)$-quasiconformal map given by Lemma 3.12 that is conformal in a neighorhood of all of the punctures. We may take a fixed κ' so that F_2 is conformal inside the circle of radius κ' inside each punctured disc. This means that we can take ζ as a conformal coordinate in a neighborhood of the puncture on $r(\infty)$ and so the map F_2 is the identity on the circle $|\zeta| = \kappa'$ in these coordinates.

Consider the annulus $B' \subset A'$ defined by

$$B' = \{\zeta : |\delta'^{1/2}| < |\zeta| < \kappa'\}.$$

Consider also the annulus $B \subset r(\infty)$ which in the z plane is bounded by the circle of radius $|\delta^{1/2}|$ and the curve ω which is the image under F_2 of the circle of radius κ'. In the ζ coordinates on $r(\infty)$, B is bounded by the circle $|\zeta| = \kappa'$ and an analytic curve γ which is the image under the holomorphic change of coordinate map $\zeta = \zeta(z)$ of the circle of radius $|\delta^{1/2}|$.

Since κ' is fixed, we have

$$\lim_{\delta' \to 0} \frac{\text{Mod}(B')}{\text{Mod}(A')} = 1$$

and since ω is fixed,

$$\lim_{\delta \to 0} \frac{\text{Mod}(B)}{\text{Mod}(A)} = 1.$$

Since $\text{Mod}(A) = \text{Mod}(A')$ we therefore have

$$\lim_{\delta \to 0} \frac{\text{Mod}(B')}{\text{Mod}(B)} \to 1. \tag{6}$$

For small enough δ we wish to find a $(1 + O(\epsilon))$ quasiconformal map F_1 from B' to B such that

- for $\zeta = \delta'^{1/2}e^{i\theta}, z = F_1(\zeta) = \delta^{1/2}e^{i\theta}$
- for $|\zeta| = \kappa'$, $F_1(\zeta) = \zeta$.

In other words, the desired F_1 is the identity on the circle of radius κ' and takes the circle of radius $\delta'^{1/2}$ in the ζ coordinates to the circle of radius $\delta^{1/2}$ in the z coordinates. We also find a corresponding map F_1 for neighborhoods of the punctures q_i, q_i'. We then will glue these maps F_1 along the circle of radius $\delta'^{1/2}$ together to give a $1+O(\epsilon)$ quasiconformal map, again denoted F_1, on the glued annulus to the annulus found by gluing along the circle of radius $\delta^{1/2}$ in the z coordinates. We then glue F_1 to F_2 along the circles of radius κ' to give a $(K + O(\epsilon))$-quasiconformal map from $r'(t)$ to $r(t)$.

We now find the map F_1. By (6) for all sufficiently small δ,

$$|\frac{\mathrm{Mod}(B')}{\mathrm{Mod}(B)} - 1| \leq \epsilon/2$$

Find a conformal map $h_\delta(z)$ from B to a round annulus

$$B_1 = \{w : \delta''^{1/2} < |w| < \kappa'\}$$

with the normalization that $h_\delta(\kappa') = \kappa'$. The composition

$$\zeta = \delta'^{1/2}e^{i\theta} \to z = \delta^{1/2}e^{i\theta} \to h_\delta(z)$$

is a map $w = f_\delta(\zeta)$ from the circle of radius $\delta'^{1/2}$ in the ζ plane to the circle of radius $\delta''^{1/2}$ in the w plane. Similarly we have a map $w = g_\delta(\zeta)$ from the circle of radius κ' in the ζ-plane to the circle of radius κ' in the w-plane. These two maps can be thought of as boundary maps of B' to B_1.

We wish to show that, as $\delta \to 0$, we have $|f_\delta'(\zeta) - (\delta''/\delta')^{1/2}| \to 0$ and $|g_\delta'(\zeta) - 1| \to 0$. In that case after mapping the annuli B_1, B' to flat cylinders with base 0, circumference 1 and heights R_1, R_2 respectively, by a logarithm map, the induced maps on the top and bottom of the cylinders have derivatives almost constantly 1. Since the ratio of moduli has limit 1, we then can apply Lemma 3.11 with $R_1/R_2 \to 1$ and $n = 0$.

We now show the desired above limits hold. Considering B as an annulus in the ζ coordinates, with outer boundary the fixed circle $|\zeta| = \kappa'$, as $\delta \to 0$, the conformal maps h_δ converge to a conformal self map of the punctured disc $0 < |\zeta| < \kappa'$. It extends to a conformal map taking 0 to 0. The only such conformal maps are rotations. But by our normalization of the h_δ's to fix a point, that map must be the identity. Thus as $\delta \to 0$, the maps h_δ converge uniformly to the identity, and therefore g_δ' converges uniformly to 1 on the circle of radius κ'.

By replacing z with $z/\delta^{1/2}$, and w with $w/\delta''^{1/2}$ we also can consider h_δ as a map from the annulus B in the z plane with inner boundary the unit circle, to B_1, another annulus with inner boundary the unit circle. As $\delta \to 0$, h_δ converges to a conformal map of the exterior of the unit disc to the exterior of the unit disc, taking ∞ to ∞. The limiting conformal map is therefore again the identity. Thus the map h_δ from the circle of radius $\delta^{1/2}$ to the circle of radius $\delta''^{1/2}$ in the w plane has derivative approaching $(\delta''/\delta)^{1/2}$ as $\delta \to 0$. Since the map from the circle of radius $\delta'^{1/2}$ in the ζ plane to the circle of radius $\delta^{1/2}$ in the z plane has derivative $(\delta/\delta')^{1/2}$, applying the chain rule the composition f_δ has derivative converging to $(\delta''/\delta')^{1/2}$ as $\delta \to 0$. We are now in a position to apply Lemma 3.11. This completes the proof. $\diamond$

4 The iterated EDM ray space and the Deligne-Mumford compactification

In this section we introduce a functor $X \mapsto \overline{X}^{ir}$ defined on a certain collection of metric spaces X. The space $\overline{X}^{ir}$ will be constructed via certain equivalence classes of EDM rays, and will have the structure of a metric stratified space (see below). We will then prove that this functor applied to $\mathcal{M}(S)$ produces the Deligne-Mumford compactification $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$; that is, we will find a stratification-preserving homeomorphism from $\overline{\mathcal{M}(S)}^{ir}$ to the Delgine-Mumford compactification $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ which is an isometry on each stratum.

4.1 The iterated EDM ray space

Before defining $\overline{X}^{ir}$, we will have to deal with a technical issue. The boundary pieces of $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ are naturally products of smaller moduli spaces. We will need to canonically pick out the factors in such products by studying uniqueness of product decompositions. Unfortunately, the fact that $\mathcal{M}(S)$ has orbifold points slightly complicates matters, as we will now see.

A metric space Y is said to have the *unique local geodesic property* if for every $y \in Y$ there is a neighborhood U of y with the property that any two points in U can be connected by a unique geodesic in U. It is well-known that Teich(S) has the unique local geodesic property. It follows easily from the proper discontinuity of the action of Mod(S) on Teich(S) that $\mathcal{M}(S)$ has this property in the complement of its orbifold locus. However, for points $s \in \mathcal{M}(S)$ in the orbifold locus, this is not true: every neighborhood of s in $\mathcal{M}(S)$ has some pair of points x, y so that the number $n(x,y)$ of (globally length minimizing) geodesics from x to y is greater than 1. Since there is a uniform bound (of $84(g-1)$) of the order of any group stabilizing any point of Teich(S), it follows that there is a uniform upper bound for $n(x,y)$ for any $x, y \in \mathcal{M}(S)$.

Theorem 4.1 (Uniqueness of product decomposition)**.** *For each $1 \leq i \leq m$, let Y_i be a connected metric space, not equal to a point, with the following property:*

1. *The complement of the set of points $S_i \subset Y_i$ without the unique local geodesic property is open and dense in Y_i, and*
2. *there exists $N_i \geq 1$ so that for all $x, y \in Y_i$, the number $n_i(x,y)$ of (globally length-minimizing) geodesics in Y_i from x to y is at most N_i.*

Let $Z = Y_1 \times Y_2 \ldots \times Y_n$, endowed with the sup *metric. Then given any other way of writing $Z = X_1 \times \cdots X_m$ with the* sup *metric, it must be that $m = n$ and, after perhaps permuting factors, $X_i = Y_i$ for all i.*

As the proof of Theorem 4.1 is independent of the rest of this paper, we leave it for the Appendix (Section 6) below. One key ingredient is a recent theorem of Malone [Mal].

As discussed above, $Y_i = \mathcal{M}(S)$ satisfies the hypotheses of Theorem 4.1. In this case the set S_i is precisely the orbifold locus of $\mathcal{M}(S)$.

Now consider a metric space (X,d) with $X = X_1 \times \ldots X_m$ (possibly with $m = 1$). Assume that (X,d) satisfies the hypotheses of Theorem 4.1. We will consider rays in each factor.

Definition 4.2 (Isolated rays)**.** *We say that a ray r is* isolated *if the following two properties hold*

1. *There is a factor X_j such that $r \subset X_j$ and r is an EDM ray in X_j.*
2. *For every $p \in X_j$, the set of asymptote classes of EDM rays $[r'] \subset X_j$ which are a bounded distance from r, and which have some representative passing through p, is countable.*

We will now define a space $\overline{X}^{ir}$ inductively, building it inductively, stratum by stratum. The level k stratum will be denoted $D_k(X)$.

Henceforth every metric space (Y,d) that appears as a factor in a product will be assumed to have the following three properties:

Standing Assumption I (Limits exist): For any two isolated EDM rays r_1, r_2 in Y that are a bounded distance apart, there are initial points $r_1(0), r_2(0)$ such that $\lim_{t\to\infty} d(r_1(t), r_2(t))$ exists and is a minimum among all choices of basepoints.

Standing Assumption II (Asymptotes are uniformly asymptotic): For any $\epsilon > 0$, any asymptote class of isolated EDM rays $[r]$, any representative r of $[r]$, and any choice of basepoint $r(0)$, there is a $T = T(r, r(0), \epsilon)$ such that for any such asymptotic pairs r, r' the rays $r([T, \infty))$ and $r'([T', \infty))$ are within Hausdorff distance ϵ of each other.

Standing Assumption III (Almost locally unique geodesics): Y satisfies the hypotheses (and hence the conclusions) of Theorem 4.1

If a metric space X contains isolated rays, we consider the set $\mathrm{Asy}(X)$ of all asymptote classes of isolated EDM rays $[r]$ in X. With Standing Assumption I in hand, we can endow $\mathrm{Asy}(X)$ with a distance function via $d_{\mathrm{asy}}([r_1], [r_2]) = \lim_{t\to\infty} d(r_1(t), r_2(t))$ for choice of basepoints that minimizes this limit. It is easy to check that this defines a metric.

Let $(D_0(X), d_0) := (X, d)$.

Step 1 (Inductive step): Suppose we are given the metric space $D_k(X)$, written as a product of factors $X_1 \times \ldots \times X_m$ with the metric $d_k(\cdot, \cdot)$, where d_k is the sup of the metrics d^j of the factors. Remove each factor that is a point. If none of the factors X_j contains isolated EDM rays, define $D_m(X) = \emptyset$ for all $m > k$ and stop the inductive process. If some factor X_j contains isolated rays then we set

$$D^j_{k+1}(X) = X_1 \times \ldots \times X_{j-1} \times \mathrm{Asy}(X_j) \times X_{j+1} \times \ldots \times X_m.$$

We can endow $D^j_{k+1}(X)$ with a distance function d^j_{k+1} as the sup metric on the factors. From Standing Assumption III, we have that if $\mathrm{Asy}(X_j)$ is a product, then it can be written uniquely as a product. Thus, given the product representation of $D_k(X)$, we have a unique product representation of $D^j_{k+1}(X)$.

Note also that if two points in $D^j_{k+1}(X)$ have an infinite distance from each other, then they are in different components of $D^j_{k+1}(X)$. We then set

$$D_{k+1}(X) = \sqcup_{j=1}^m D^j_{k+1}(X)$$

with metric d_{k+1} which is the corresponding metric d^j_{k+1} on each term in the disjoint union.

Step 2 (Topology): We will inductively define a topology on the disjoint union $Y := \cup_{j=0}^{\infty} D_j(X)$, as follows.

Using Standing Assumption II, for every $[r_0] \in \mathrm{Asy}(X_j)$ and every $\epsilon > 0$ we can define an ϵ-neighborhood $V_\epsilon([r_0])$ of $[r_0]$ in $\mathrm{Asy}(X_j) \cup X_j$. Consider the set of equivalence classes of isolated rays $[r] \in \mathrm{Asy}(X_j)$ such that $d^j([r], [r_0]) < \epsilon$ and set $V^j_\epsilon([r_0])$ to be the union of the set of such rays and the following set. For each such ray $[r]$ and each $r \in [r]$ include in $V^j_\epsilon([r_0])$ the set $\{r(t) : t \geq T(r, r(0), \epsilon)\}$.

We are now ready to define the topology.

Definition 4.3. *Let $j \geq 0$. Suppose $\vec{x}(n)$ is a sequence in $D_k(X)$ and $(\vec{x}, [r]) \in D^j_{k+1}(X)$. We say $\vec{x}(n) \to (\vec{x}, [r])$ if there exists $t_n \to \infty$ such that*

1. *for $i \neq j$, $\lim_{n\to\infty} d^i(x_i(n), x_i) = 0$*
2. *$\lim_{n\to\infty} d^j(x_j(n), r(t_n)) = 0$ for some representative r of $[r]$.*

Now suppose inductively for each k, m, and for each sequence $\vec{x}(n) \in D_k(X)$, and $y \in D_{k+m}(X)$ we have defined what it means for $\vec{x}(n)$ to converge to y.

Definition 4.4. *Suppose $\vec{x}(n) \in D_k(X)$ and $z \in D_{k+m+1}(X)$. We say $\vec{x}(n) \to z$ if there exists j, points $(\vec{x'}(n), [r_n]) \in D^j_{k+1}(X)$, a sequence $\epsilon_n \to 0$, representatives r_n and times t_n such that*

1. *$\lim_{n\to\infty} d^i(x_i(n), x'_i(n)) = 0$ for $i \neq j$.*
2. *$\lim_{n\to\infty} d^j(x_j(n), r_n(t_n)) = 0$.*
3. *$r_n(t_n) \in V_{\epsilon_n}([r_n])$.*
4. *$\lim_{n\to\infty}(\vec{x'}(n), [r_n]) = z$.*

The first condition just says that one has convergence in the factors where one is not considering isolated rays. Notice the last condition inductively makes sense since $(\vec{x'}(n), [r_n]) \in D_{k+1}(X)$ and $z \in D_{k+m+1}(X)$ and $k + m + 1 - (k + 1) = m$.

We thus obtain a topological space which is stratified by $\{D_k(X)\}$, and in fact each stratum is a metric space (by Standing Assumption I). Note that X is open and dense in Y. We are actually interested in a somewhat simpler space, obtained as a certain quotient of Y, as follows.

Step 3 (Identifications): The space Y provides a natural "boundary" for X, although the construction may give multiple copies of the same boundary component. To remedy this, we will identify points that "should" be distance zero from each other. In some sense this is like Cauchy's scheme for completing metric spaces.

We make no identifications of points in $D_0(X)$. Now suppose inductively we have made identifications of points in $D_j(X)$ for all $j \leq k$ and $P, Q \in D_{k+1}(X)$.

Definition 4.5. *We say $P \sim Q$ if there exist sequences x_n, y_n in the same component of $D_{k-1}(X)$ such that*

1. *$\lim_{n\to\infty} x_n = P$ and $\lim_{n\to\infty} y_n = Q$.*
2. *$\lim_{n\to\infty} d_{k-1}(x_n, y_n) = 0$.*

This is clearly an equivalence relation. We denote the quotient space of Y by this equivalence relation by $\overline{X}^{ir}$, and call it the *iterated EDM ray space* associated to X. This is evidently a functor from metric spaces (whose D_j's satisfy the standing assumptions) and isometries to metric spaces and isometries. If Y turns out to be a compactification of X, then since we only identified certain points in $Y \setminus X$, it follows that $\overline{X}^{ir}$ is also a compactification of X.

Example 4.6. *For X the upper quadrant in $\mathbf{R}^2 = \mathbf{R}^+ \times \mathbf{R}^+$ with the* sup *metric, D_1 has two components, each of which is an infinite ray. A point in one component corresponds to a vertical ray, with the distance function equal to the distance function between vertical rays, i.e. the difference of their x coordinates. The points in the other component correspond to horizontal rays, with the distance being the difference of their y coordinates. Since D_1 is a disjoint union of two rays, D_2 consists of two points. The sequence (n, n) converges to each of the two points in D_2, and so these points are identified. Thus in this case $\overline{X}^{ir}$ is a closed square.*

4.2 Metric stratified spaces

We would like to keep track of structures finer than topological type. To do so we will need the following standard concept.

Definition 4.7. *A* stratification *of a second countable, locally compact Hausdorff space X is a locally finite partition $\mathcal{S}_X$ into open sets S satisfying:*

1. *Each element $S \in \mathcal{S}_X$, called a* stratum*, is a connected topological space in the induced topology.*

2. *For any two strata $S_1, S_2 \in \mathcal{S}_X$, if $\overline{S_1} \cap S_2 \neq \emptyset$ then $\overline{S_1} \supset S_2$.*

A space X with a stratification, with each stratum endowed with the structure of a metric space, is called a metric stratified space*.*

Inclusion $\overline{S_1} \supset S_2$ defines a partial ordering $S_1 > S_2$ on the elements of $\mathcal{S}_X$. The *depth*, or *level* of a stratum T is the maximal n so that there is a chain

$$S_0 > \cdots > S_n = T$$

with $S_i \in \mathcal{S}_X$. Note that since $\mathcal{S}_X$ is locally finite, any such chain is finite, although *a priori* one might have strata of infinite depth.

Example 4.8. *The iterated EDM ray space $\overline{X}^{ir}$ of §4.1 has a natural stratification, where the level k strata are the components of $D_k(X)$.*

4.3 The Deligne-Mumford compactification

Deligne-Mumford [DM] constructed a compactification $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ of $\mathcal{M}(S)$, called the *Deligne-Mumford compactification*, which they proved is a projective variety. As such, $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ is endowed with the structure of a stratified space. Bers [Be] also gave a construction of $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ as a stratified space. Points of the level k strata of $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ are given by conformal structures on k-noded Riemann surfaces; the set of strata are parametrized by the set of combinatorial types of collections of nodes (see [Be, DM]).

The topology on $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ is as follows. On each stratum the topology is just that of the corresponding moduli space. Points X_n converge to some Y in a lower level stratum if for every neighborhood N of the union of nodes in Y, there is a conformal map $(Y \setminus N) \to X_n$ for n sufficiently large. We endow each stratum of $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ with the corresponding Teichmüller metric, thus giving $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ the structure of a metric stratified space.

Our goal in this section is to reconstruct $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ as a metric stratified space (but not as a projective variety) as the iterated EDM ray space $\overline{\mathcal{M}(S)}^{ir}$ associated to $\mathcal{M}(S)$. We therefore begin by applying the construction from the previous subsection to $\mathcal{M}(S)$, endowed with the Teichmüller metric. .

We characterize the isolated rays in $\mathcal{M}(S)$, and identify the metric they give on the stratum $D_1(\mathcal{M}(S))$.

Proposition 4.9. *Let S be a surface of finite type. Then a ray in $\mathcal{M}(S)$ is an isolated EDM ray if and only if it is a one-cylinder Strebel ray. Let r and r' be one-cylinder Strebel rays. Suppose the cylinders of r and r' both have core curves of the same topological type as a fixed simple closed curve γ. Then $d_1(r, r')$ in $D_1(\mathcal{M}(S))$ exists, and coincides with the Teichmüller distance between $r(\infty)$ and $r'(\infty)$ in the boundary moduli space $\mathcal{M}(S \setminus \gamma)$.*

We remark that if the cylinder defining the Strebel ray is given by a separating curve, then S' is disconnected, and so $\mathcal{M}(S \setminus \gamma)$ is itself a product of smaller moduli spaces.

Proof. By Theorem 1.4, a ray in $\mathcal{M}(S)$ is EDM if and only if it is Strebel. By Theorem 21.7 of [St], on each Riemann surface there is a unique one-cylinder Strebel differential in each homotopy class of simple closed curve. There are only countably many such homotopy classes. Moreover, given a collection of more than one distinct homotopy class of disjoint curves, the set of Strebel differentials with cylinders in those homotopy classes is uncountable (again, by Theorem 21.7 of [St]). Moreover by Theorem 2 of [Ma1], any two Strebel differentials with homotopic cylinders are a bounded distance apart. However (again by Theorem 21.7 of [St]) they are not modularly equivalent and so these classes are not isolated. It follows easily from Lemma 2.3 that each of these is an unbounded distance from a ray defined by a one-cylinder Strebel differential. These facts together imply that the isolated rays coincide with the one-cylinder Strebel rays.

The fact that the set of asymptote classes of one-cylinder Strebel rays on any moduli space is homeomorphic to the moduli spaces of one smaller complexity, and that the distance between one cylinder Strebel rays of the same type exists and is equal to the Teichmuller distance on the corresponding one complexity smaller moduli space, is the content of Theorem 3.9. The fact that isolated EDM rays determined by combinatorially inequivalent curves are not bounded distance apart follows from Lemma 2.3. ⋄

With the setup above, we can now prove the main result of this section: that $\overline{\mathcal{M}(S)}^{ir}$ and $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ are isomorphic as metric stratified spaces.

Theorem 4.10. *The iterated EDM ray space* $\overline{\mathcal{M}(S)}^{ir}$ *associated to* $\mathcal{M}(S)$ *is homeomorphic to the Deligne-Mumford compactification* $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ *via a stratification-preserving homeomorphism which is an isometry on each stratum.*

Proof. First recall that the set of level k strata of $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ is parametrized by the set of combinatorial types of k-tuples of simple closed curves on S, representing the curves that are pinched to nodes. Each level k stratum corresponding to a k-tuple $\{\alpha_1, \ldots, \alpha_k\}$ is a product of the moduli spaces of the punctured surfaces consisting of the components of $S \setminus \{\alpha_1, \ldots, \alpha_k\}$. Further, we have endowed each stratum with the Teichmüller metric of the corresponding moduli space or, in the case of disconnected surfaces, with the sup metric on the product of moduli spaces.

Step 1 (Defining a surjective map): We first define a map

$$\psi : \cup_{k=0}^{\infty} D_k(\mathcal{M}(S)) \to \overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$$

inductively, as follows. On $D_0(\mathcal{M}(S))$ we simply let ψ be the identity map. Each factor that was a point that was removed is sent to the moduli space of a three times punctured sphere which is itself a point. By Proposition 4.9, the isolated EDM rays in $D_0(\mathcal{M}(S))$ are precisely the one-cylinder Strebel rays. The equivalence classes of one-cylinder Strebel differentials correspond precisely to the topological types of simple closed curves on S. By Corollary 3.10, the asymptote classes of one-cylinder Strebel rays r correspond to the possible endpoints $r(\infty)$. By Strebel's existence theorem (Theorem 23.5 of [St])), every possible endpoint can occur, so that $D_1(\mathcal{M}(S))$ consists of all possible surfaces obtainable by pinching a single simple closed curve on S. Thus $D_1(\mathcal{M}(S))$ is the disjoint union of moduli spaces, one for each topological type of simple closed curve. By Theorem 3.9, the metric d_1 on D_1 coincides with the corresponding Teichmüller metric on each component of $D_1(\mathcal{M}(S))$. We define ψ on each component of $D_1(\mathcal{M}(S))$. If the component is not a product we map an asymptote class $[r]$ of rays to the corresponding endpoint $r(\infty)$. If the component is a product, then for each factor we define ψ by fixing the coordinates of the other factors and map an asymptote class of rays in the factor to its endpoint. By the above, on each component, this map is an isometry onto the component of $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$ corresponding to the appropriate combinatorial type of simple closed curve.

Suppose now inductively that we have proven that each component of $D_k(\mathcal{M}(S))$ is isometric via a map ψ to a (products of) moduli spaces, and the map is onto the collection of moduli spaces, one for each combinatorial type of k-tuple of simple closed curves. Fix any component of $D_k(\mathcal{M}(S))$, corresponding to a k-tuple $\{\alpha_1, \ldots, \alpha_k\}$, and let $\mathcal{M}(S')$ be the corresponding (products of) moduli spaces $\mathcal{M}(S_1)\times\ldots\times\mathcal{M}(S_p)$, where $S' = S\backslash\{\alpha_1, \ldots, \alpha_k\}$. For each factor in this product we find the asymptote classes of isolated EDM rays, again given by the one cylinder Strebel differentials. We thus obtain components of $D_{k+1}(\mathcal{M}(S))$, and these components correspond to the possible combinatorial types of $(k+1)$-tuples obtainable from $\alpha_1, \ldots, \alpha_k$ by adding a single simple closed curve. We again define ψ on each component by sending each asymptote class $[r]$ to $r(\infty)$, and if the component is a product, defining it to be the identity on the other coordinates. As above, we see that ψ is an isometry when restricted to any of the fixed components just obtained. By Strebel's theorem again, the map is onto all $(k+1)^{\mathrm{st}}$ strata in $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$.

We have therefore inductively defined a map

$$\psi : \bigcup_k D_k(\mathcal{M}(S)) \to \overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$$

which we have shown to be onto (by Strebel's existence theorem), and which is an isometry when restricted to any fixed component of any fixed $D_k(\mathcal{M}(S))$.

Step 2 (The standing assumptions hold): Standing Assumption I holds by the fact discussed above, that if two EDM rays are defined by pinching the same combinatorial type of curve then the rays have an asymptotic distance apart, and by the fact that if the topological types are different then the rays are not bounded distance apart. The latter follows from Lemma 2.3

Now we show Standing Assumption II holds. Let $[r]$ be an asymptotic class of isolated EDM ray on any moduli space with r any representative. As we have seen, on the surface $r(\infty)$ there is a quadratic differential $q(\infty)$ with double poles at the paired punctures, such that the vertical trajectories are all closed curves of equal length isotopic to the punctures. Since $q(\infty)$ is the unique (up to scalar multiple) quadratic differential with this property, any two representatives determine the same $q(\infty)$. Since the Strebel differentials along r can be reconstructed by cutting out punctured discs on $r(\infty)$ and gluing along the boundary circles of $q(\infty)$, the ray r is determined by a single twist parameter; namely, how the circles are glued to each other. Thus the Strebel differentials on any two rays differ by only a twist about the core curve, and the amount of twisting is bounded by the length of the curve. For any two points $r_1(t_1)$ and $r_2(t_2)$ along two such rays, if the moduli of the cylinders M_1, M_2 are equal and large, then $d(r(t_1), r_2(t_2))$ is small; there is a $O(1+1/M_1)$-quasiconformal map of the cylinders that realizes the twisting. Standing Assumption II follows.

Standing Assumption III holds since , as disscussed before Theorem 4.1, the hypotheses of that theorem are satisfied by a product of Teichmüller spaces.

Step 3 (ψ is continuous): Suppose $x_n \in D_k(\mathcal{M}(S))$ converges to $z \in D_{k+m}(\mathcal{M}(S))$ as in Definitions 4.3 or 4.4. The proof of continuity of ψ is by induction on m. Assume $m = 1$. If the component of D_k containing x_n is a product, then by definition all of the coordinates but one of $\psi(x_n)$ in the product coincide with the corresponding coordinates of x_n. By assumption, these converge to the corresponding coordinates of $\psi(z)$. Thus we can assume that the component of D_k is not a (nontrivial) product. Then $\psi(z)$ is the Riemann surface $r(\infty)$, where r is an EDM ray in $D_k(\mathcal{M}(S))$, and $d_k(x_n, r(t_n)) \to 0$ for a sequence $t_n \to \infty$. The fact that $r(\infty)$ is the endpoint of r says that $\psi(r(t_n)) \to \psi(z)$ as $t_n \to \infty$ in the topology of $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$. The fact that $d_k(x_n, r(t_n)) \to 0$ says there is a sequence of $(1+o(1))$-quasiconformal maps of $\psi(x_n)$ to $\psi(r(t_n))$. These converge to a conformal map of a limit $\psi(r(t_n))$ to $\psi(z)$. Thus any such limit must in fact coincide with $\psi(z)$.

Now suppose the continuity of ψ has been proved for all $p \leq m$ and $m = p+1$. Again it suffices to assume that D_k is not a product. Let y_n a sequence in $D_{k+1}(\mathcal{M}(S))$ such that $y_n \to z$ as in Definition 4.4. There is a sequence of isolated rays r_n in D_k defined by one-cylinder Strebel differentials with core curve some γ such that $y_n = r_n(\infty)$. By the induction hypothesis $\psi(y_n) \to \psi(z)$. Now assumption (2) in the definition of the topology implies that

$$\mathrm{Ext}_{r_n(t_n)}(\gamma) \to 0,$$

for otherwise there would be rays in the same asymptote class whose distance from $r_n(t_n)$ does not tend to 0. Consider the $p+1$ nodes of $\psi(z)$ corresponding to pinching $p+1$ curves. Without loss of generality we can assume the last p of them are pinched along $\psi(y_n)$. Form small neighborhoods of the corresponding paired punctures on $\psi(z)$. By definition of the topology, since $\psi(y_n) \to \psi(z)$, there is a conformal map of the complement of the last p pair of neighborhoods to $\psi(y_n)$ for n large. For each such n, there is a conformal map of the complement of the first pair of neighborhoods to $r_n(t_n)$ for t_n sufficiently large. This shows that $\psi(r_n(t_n)) \to \psi(z)$. By assumption, there is a sequence of $(1+o(1))$-quasiconformal maps from $\psi(x_n)$ to $\psi(r_n(t_n))$, and therefore $\psi(x_n) \to \psi(z)$ as well. This shows that ψ is continuous.

Step 4 (Factoring ψ): Now the map ψ itself is not injective, since one can have two combinatorially distinct j-tuples of curves which become combinatorially equivalent when one additional curve is added. For example, if S is closed of genus 2, then in $D_2(\mathcal{M}(S))$ the component corresponding to pinching a separating and nonseparating curve is counted twice. However we show now that the final identification Step 4 precisely identifies, by definition, such tuples. Namely we show that the map ψ factors through a map

$$\Psi : \overline{\mathcal{M}(S)}^{ir} \to \overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}.$$

Suppose $z, z' \in D_{k+1}(\mathcal{M}(S))$ and $z \sim z'$. We have to show $\psi(z) = \psi(z')$. By definition there are sequences $x_n, x'_n \in D_{k-1}(\mathcal{M}(S))$ that satisfy $d_{k-1}(x_n, x'_n) \to 0$; $x_n \to z$, $x'_n \to z'$. By the continuity of ψ we have $\psi(x_n) \to \psi(z)$ and $\psi(x'_n) \to \psi(z')$. Since $d_{k-1}(x_n, x'_n) \to 0$, there is a sequence of $(1 + o(1))$-quasiconformal maps from $\psi(x_n)$ to $\psi(x'_n)$. Therefore we also have $\psi(x'_n) \to \psi(z)$ and so $\psi(z) = \psi(z')$. We have shown that there is a well-defined map $\Psi : \overline{\mathcal{M}(S)}^{ir} \to \overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$.

Step 5 (Ψ is injective): We must prove that if $\Psi(z) = \Psi(z')$, then z has been identified with z'. We can assume z, z' are in different components of $D_{k+1}(\mathcal{M}(S))$. Again we can assume the components are not products; hence they are endpoints of rays r, r' in different components E, E' of $D_k(\mathcal{M}(S))$. Let $x_n \in D_{k-1}(\mathcal{M}(S))$ such that $x_n \to z$. We wish to show $x_n \to z'$ as well, for then z is identified with z'. We have $X_n := \Psi(x_n) \to Z := \Psi(z)$.

4.4 (s, t) coordinate system

Before continuing the proof we need to describe a coordinate system about Z which allows us to represent any surface near Z in the coordiante system. This coordinate system is due to [EM] (see also [Ma2] and [W]).

We may lift so that Z is in the augmented Teichmuller space. We will find a neighborhood $\mathcal{V}$ of Z whose intersection with Teich(S) will not be locally compact. We can separate the nodes of Z into pairs of punctures, denoted p_i, q_i. Choose conformal neighborhoods $V_i = \{z_i : 0 < |z_i| < 1\}$ and $W_i = \{w_i : 0 < |w_i| < 1\}$ of p_i and q_i. Also choose points P and Q on the pairs of circles of radius 1. The discs may be taken to be mutually disjoint. For each component Z_l of Z choose a nonempty open set $\mathcal{W}$ disjoint from $\cup_i(V_i \cup W_i)$. Let n_l denote the complex dimension of Teich(Z_l). There exist Beltrami differentials $\nu_1, \ldots, \nu_{n_l}$ supported in $\mathcal{W}$ whose equivalence classes form a basis for the tangent space to T_{Z_l} at Z_l. This implies that for any Y_l sufficiently close to Z_l, there is a n_l-tuple $s(Y) = (s_1, \ldots s_{n_l})$ of complex numbers close to 0 and a quasiconformal map $f : Z_l \to Y_l$ such that the dilatation $\mu(f)$ of f satisfies

$$\mu(f) = \sum_{i=1}^{n_l} s_i \nu_i.$$

We do this for each component of Z. The result is a parametrization of surfaces in a neighborhood of $Z \in \mathcal{V}$ that lie in the bordification, by $s \mapsto Z(s)$ for a neighborhood of 0 in $\mathcal{C}^N$, for some N.

Since the map f (on each component) is conformal in $U_i \cup V_i$, the coordinates z_i, w_i are local holomorphic coordinates in neighborhoods V_i, W_i of the punctures on each $Z(s)$. Now choose a p-tuple $t = (t_1, \ldots, t_p)$ complex numbers in a small neighborhood of the origin. For each surface Z_s, and for each $1 \leq i \leq p$, remove the disc of radius $|t_i|$ from each of V_i and W_i, and then glue z_i to t_i/w_i. We note that in this notation $Z(s, 0) = Z(s)$; so if all $t_i = 0$, then there are no disks to remove.

To define the neighborhod in Teich(S) we need to choose markings on $Z(s, t)$ by choosing a homotopy class of arcs joining P and Q crossing the glued annulus. Thus we have a marking of the surface $Z(s, t)$ consisting of the marking of $Z = Z(0, 0)$, the curves along which we glued, and for each such curve, a transverse arc crossing the annulus. Note that markings differ by Dehn twists about the glued curve, and since these are arbitrary the resulting neighborhood is not locally compact.

We continue the proof that ψ is injective. We can lift to Teichmuller space and find the coordinate system (s, t) around Z. Since Z lies in a moduli space of two fewer dimensions than X_n, there are two plumbing coodinates t_1, t_2 such that the coordinates $t_1(n), t_2(n)$ of X_n are both nonzero.

We can assume that points of E' have coordinate $t_1 = 0$, and the t_2 coordinate tends to 0 along the ray $r'(u)$ as $u \to \infty$. We can assume that points of E have $t_2 = 0$. The s coordinate of X_n approaches 0. For each n, we can find a time u_n such that the modulus of the cylinder on $\Psi(r'(u_n))$ coincides with the modulus of the corresponding annulus on X_n. For each such $r'(u_n)$ there is a ray $r'_n \subset D_{k-1}(\mathcal{M}(S))$ such that $r'(u_n) = r'_n(\infty)$. We can choose a time l_n so that the corresponding cylinder on $r'_n(l_n)$ has the same modulus as the corresponding annulus on X_n. Now, just as in the proof of Theorem 3.9, as $n \to \infty$ there is a sequence of $(1 + o(1))$-quasiconformal maps from X_n to $\Psi(r'(l_n))$, and by the definition of the topology on the union of the $D_j(\mathcal{M}(S))$, we have that $x_n \to z'$.

Step 6 (Ψ^{-1} is continuous): Suppose then that $X_n \in \mathcal{M}(S')$ converges to Z in $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$. Again we can form an (s, t) coordinate neighborhood system about Z such that, after re-indexing, the t coordinates of X_n are given by $(t_1(n), \ldots, t_k(n)) \neq 0$. Here k is the number of curves of X_n that we pinch to get Z. The proof is by induction on k and resembles the proof that Ψ is injective. Suppose $k = 1$. Let r be the Strebel ray with endpoint $r(\infty) = Z$, so by definition, $\Psi([r]) = Z$. For each n, we can find a time u_n such that the modulus of the cylinder on $r(u_n)$ is the same as the modulus about the pinching curve on X_n found by the plumbing construction. Now again just as in the proof of Theorem 3.9, for any ϵ, for n large enough, we can find a $(1 + \epsilon)$-quasiconformal map from X_n to $r(u_n)$. Then by definition, $X_n \to [r] = \Psi^{-1}(Z)$ in the topology of $\overline{\mathcal{M}(S)}^{ir}$.

Now for the induction step. Suppose we have proven the desired limit for $k-1$, where Z is found by pinching along k curves. We have $Z = \Psi([r_0])$ for some ray r_0. Let Y_n have the same (s,t) coordinates as X_n except that we require $t_1 = 0$. This means that we find Z from Y_n by pinching $k-1$ curves. Let q_n be the Strebel differential on Y_n with double poles at the punctures corresponding to $t_1 = 0$, and let r_n be the corresponding Strebel ray with endpoint $r_n(\infty) = Y_n$. By definition, $\Psi([r_n]) = Y_n$. Now $Y_n \to Z$ in $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$, and by the induction hypothesis on the continuity of the map Ψ^{-1}, we see that $[r_n] \to [r_0]$. Just as above we may choose u_n so that the modulus of the cylinder on $r_n(u_n)$ is the same as the modulus of the annulus corresponding to the t_1 coordinate in the plumbing construction. By definition of the topology of $\overline{\mathcal{M}(S)}^{ir}$ it is again enough to prove that $d_{\mathcal{M}(S)}(X_n, r_n(t_n)) \to 0$. But this again follows just as in the proof of Theorem 3.9: there is a conformal map $X_n \to \Psi(r(u_n))$ in the complement of annuli with large but equal moduli; then for any ϵ, for n large enough, we can find a $(1+\epsilon)$-quasiconformal map from X_n to $\Psi(r(u_n))$. This completes the proof. $\diamond$

5 Further geometric properties

5.1 A strange example

In this subsection we indicate some of the difficulties of the Teichmüller geometry of $\mathcal{M}(S)$ by exhibiting two sequences of EDM rays r_n, r'_n, with the following properties: there exists a constant $D > 0$ and sequences of times $t_n, t'_n \to \infty$ such that $d_{\mathcal{M}(S)}(r_n(t_n), r'_n(t'_n)) \leq D$, each sequence r_n, r'_n converges to an EDM ray r_∞, r'_∞ uniformly on compact intervals of time, and yet r_∞ does not stay within a bounded distance of r'_∞. This example violates Assumption 9.11 of [JM], so that the Ji-MacPherson compactification method cannot be applied to $\mathcal{M}(S)$. This partially explains why we took a different approach.

We construct a sequence of rays r_n as follows. Let r_0 be a Strebel ray corresponding to a maximal collection of curves $\beta_1, \ldots, \beta_{3g-3+n}$ whose cylinders have equal moduli. Note that $r_0(\infty)$ is the unique maximally noded Riemann surface within its combinatorial equivalence class. Let α be a curve distinct from the β_i and therefore it has positive intersection with some β_j. Let T_α denote the Dehn twist about α. Let r_n be the Strebel ray through $r_0(0)$ corresponding to the Strebel differential whose set of core curves is $\{T^n_\alpha(\beta_i)\}$ and whose cylinders have equal moduli. This is possible by a theorem of Strebel ([St], Theorem 21.7). Note that $r_n(\infty) = r_0(\infty)$ for each n, since the collection $\{T^n_\alpha(\beta_i)\}$ is combinatorially equivalent to $\{\beta_i\}$. Since the rays are modularly equivalent they are asymptotic (Corollary 3.10 above), so we can choose times $t_n, t'_n \to \infty$ such that $d_{\mathcal{M}(S)}(r_n(t_n), r_0(t'_n))$ is uniformly bounded.

On the other hand the rays r_n converge uniformly on compact sets in time to a ray r_∞, where r_∞ corresponds to the unique one cylinder Strebel differential with core curve α. Taking $r'_n = r_0$ so that $r'_\infty = r_0$ for all n, we have $d(r_\infty, r'_\infty) = \infty$ by Lemma 2.3.

5.2 The set of asymptote classes of all EDM rays

In this subsection we give a parametrization of the set of asymptote classes of all (not necessarily isolated) EDM rays. As we will see, this space is naturally a closed simplex bundle B over $\overline{\mathcal{M}(\mathrm{S})}^{\mathrm{DM}}$. Let S be a surface of genus g with n punctures. The fiber over a point $\hat{X} \in \mathcal{M}_{g',n'}$, where $(g', n') \neq (g, n)$, consists of projective classes $(b_1, \dots, b_p)$ of vectors. Let Σ be the collection of all asymptotic classes of EDM rays on $\mathcal{M}_{g,n}$. We define a map

$$\Phi : \Sigma \to \mathcal{B}\,.$$

Let $[r]$ be an equivalence class of rays. Let r any representative with cylinders $C_1, \dots, C_p$ with moduli $\mathrm{mod}(C_1), \dots, \mathrm{mod}(C_p))$. By Corollary 3.10 the projective class of the vector of moduli is independent of the choice of representative and the endpoint $r(\infty)$ is independent of the representative. Define $\Phi([r])$ to be the point whose base is $r(\infty)$ and whose fiber is the projective vector $(\mathrm{mod}(C_1), \dots, \mathrm{mod}(C_p))$

Theorem 5.1. *The map Φ is a homeomorphism onto the open simplex subbundle $\mathcal{B}_0$ where no coordinate is* 0.

Proof. The map Φ is clearly injective. To show surjectivity let $\hat{X} \in \mathcal{M}_{g',n'}$ any point; $v = (M_1, \dots M_j)$ a projective vector. Pick a representative vector v and let $(\hat{X}, \hat{q})$ be the (unique) quadratic differential on $\hat{X}$ such that

- $(\hat{X}, \hat{q})$ has double poles at the punctures,
- the vertical trajectories are closed loops isotopic to the punctures
- the lengths of the vertical trajectories are $1/M_i$ for each paired puncture.

This is possible by Theorem 23.5 of [St]. Remove a punctured disc around each paired puncture so that the remaining cylinder has height 1/2. Glue together along the circles. The corresponding cylinders C_i have height 1. The moduli of the cylinders are therefore M_i. We may choose the representative v so that the area of the resulting (X, q) is 1. This gives a corresponding geodesic ray $r(t)$. We have that $\hat{X} = r(\infty)$, so that $\Phi([r]) = (\hat{X}, M_1, \dots, M_j)$

The quadratic differential $(\hat{X}, \hat{q})$ depends continously on $\hat{X}$ and the vector v, which implies that the ray $[r]$ depends continuously on these parameters so that the map Φ^{-1} is continuous. The map Φ is continuous because the endpoints and moduli depend continously on the quadratic differentials defining the ray. $\diamond$

5.3 Tits geometry of the space of EDM rays

In this section we compute some invariants for pairs of EDM rays. These invariants are fundamental in the study of nonpositively curved manifolds (see, e.g., [Eb], Chapter 3).

Definition 5.2. *Let $r(t), r'(t)$ a pair of EDM rays in a metric space (X, d). We define the* pre-Tits distance *$\ell(r, r')$ between r and r' to be*

$$\ell(r, r') := \lim_{t\to\infty} \frac{d(r(t), r'(t))}{t}$$

if the limit exists.

For simply-connected, nonpositively curved manifolds X, the *Tits distance* on the visual boundary ∂X is equal to the path metric induced by ℓ ([Eb], Prop. 3.4.2). The quantity ℓ is related to the *angle metric* $\angle(r, r')$ on ∂X via

$$\ell(r, r') = 2\sin(\frac{1}{2}\angle(r, r'))$$

(see [Eb], Prop. 3.2.2).

Our goal now is to compute ℓ for pairs of EDM rays in $\mathcal{M}(S)$.

Theorem 5.3. *Let r, r' be EDM rays defined by Strebel differentials (X, q) and (X', q') with core curves $\{\gamma_i\}$ and $\{\gamma'_j\}$. The Tits angle between r and r' is 0 if there is an element ϕ of the mapping class group sending $\{\gamma_i\}_{i=1}^{p}$ to $\{\gamma'_j\}_{i=i}^{p'}$. The angle is 1 if the above does not hold but there is an element ϕ of the mapping class group such that $i(\phi(\gamma_i), \gamma'_j) = 0$ for all γ_i, γ'_j. The angle is 2 otherwise.*

This discretization of Tits angles lies in contrast to what happens for higher rank locally symmetric spaces $\Gamma\backslash G/K$, where one has a continuous values of the Tits angles coming from almost isometrically embedded Weyl chambers.

Proof. The first case is if the collection of curves $\{\gamma_i\}$ is combinatorially equivalent to the collection of curves $\{\gamma'_j\}$. That is, there is an element ϕ of the mapping class group sending one collection to the other. Then the corresponding geodesics stay bounded distance apart by [Ma1]. Thus the Tits angle is 0.

Thus assume the collections are not combinatorially equivalent. Assume further that any collection of curves combinatorially equivalent to $\{\gamma_i\}$ must intersect some γ'_j. By reindexing we can assume

$$i(\gamma_1, \gamma'_1) > 0.$$

Now by Lemma 2.3

$$e^{2t}\,\mathrm{Ext}_{r(t)}(\gamma_1)\to c_1,$$

for some $c_1>0$. Since γ_1 crosses C_1',

$$\mathrm{Ext}_{r'(t)}(\gamma_1)\geq c_2e^{2t},$$

for some $c_2>0$. By Theorem 2.2

$$d_{\mathcal{M}(S)}(r(t),r'(t))\geq 1/2\log(c_1c_2e^{4t})$$

and so

$$\liminf_{t\to\infty}\frac{d_{\mathcal{M}(S)}(r(t),r'(t))}{t}\geq 2.$$

On the other hand by the triangle inequality

$$\limsup_{t\to\infty}\frac{d_{\mathcal{M}(S)}(r(t),r'(t))}{t}\leq 2,$$

and we are done in this case.

The remaining case is that there is some ϕ so that $i(\phi(\gamma_i),\gamma_j')=0$ for all i,j. There are several possibilities with similar analyses. Assume for example that after reindexing and applying an element of $\mathrm{Mod}(S)$ that $\gamma_1\neq\gamma_j'$ for all j. Now since

$$i(\gamma_1,\gamma_j')=0$$

for all j', by Lemma 2.3 we have $\mathrm{Ext}_{r'(t)}(\gamma_1)$ bounded below, and so by Theorem 2.2

$$\liminf_{t\to\infty}\frac{d_{\mathcal{M}(S)}(r(t),r'(t))}{t}\geq 1.$$

We need to show the opposite inequality. That is, we need to show

$$\sup_\beta\frac{Ext_{r(t)}(\beta)}{Ext_{r'(t)}(\beta)}\leq c(t)e^{2t}, \tag{7}$$

where

$$\frac{\log c(t)}{t}\to 0.$$

We will use results of Minsky [Mi1] to compare extremal lengths of any β along $r(t)$ and $r'(t)$. We will say that two functions f,g are *comparable*, denoted $f\asymp g$, if f and g differ by fixed multiplicative constants (which in our case will depend only on the genus of S).

Fix some $\epsilon>0$, smaller than the Margulis constant for S. For sufficiently large t_0, and for each cylinder C_i along $r(t)$, find a pair of curves γ_i^1,γ_i^2 with the following properties:

1. γ_i^1,γ_i^2 are isotopic to γ_i.

2. Each has fixed hyperbolic length ϵ.

3. γ_i^1 and γ_i^2 bound a cylinder $\hat{C}_i \subset C_i$ such that $\frac{\mathrm{mod}(\hat{C}_i)}{\mathrm{mod}(C_i)} \to 1$ as $t \to \infty$.

Note that

$$\mathrm{mod}(C_i) = c_i e^{2t}$$

for some fixed c_i. Let $M_i(t) = \mathrm{mod}(\hat{C}_i)$. The curves $\gamma_i^j; j = 1, 2$ define the thick-thin decomposition of $r(t)$. The components Ω_j of the complement of the cylinders $\hat{C}_i$ are thick. According to [Mi1], for any β we have

$$\mathrm{Ext}_{r(t)}(\beta) \asymp \max_{i,j}(\mathrm{Ext}_{\hat{C}_i}(\beta), \mathrm{Ext}_{\Omega_j}(\beta)), \tag{8}$$

which is the maximum of the contribution to the extremal length of β from its intersections with the $\hat{C}_i$ and the Ω_j. These quantities are given below.

For the first, the hyperbolic geodesic representative of β crosses each $\hat{C}_i$ a total of n_i times, twisting t_i times. The contribution to extremal length $\mathrm{Ext}_{\hat{C}_i}(\beta)$ from its intersection with $\hat{C}_i$ is given by

$$\mathrm{Ext}_{\hat{C}_i}(\beta) = n_i^2(M_i(t) + t_i^2/M_i(t)). \tag{9}$$

By [Mi1] the contribution to extremal length $\mathrm{Ext}_{\Omega_j}(\beta)$ of β from Ω_j is comparable to $\ell^2(\beta \cap \Omega_j)$, where $\ell(\cdot)$ is length in the hyperbolic metric. This quantity can be computed as follows. Let $\Gamma_j = \Gamma \cap \Omega_j$, the component of the critical graph contained in Ω_j. Choose generators $\omega_1, \ldots, \omega_n$ for $\pi_1(\Gamma_j)$, where $n = n(j)$. Since Ω_j is thick, we have

$$\ell^2(\beta \cap \Omega_j) \asymp (\max_i i(\beta, \omega_i))^2. \tag{10}$$

and so

$$\mathrm{Ext}_{\Omega_j}(\beta) \asymp (\max_i i(\beta, \omega_i))^2 \tag{11}$$

Similar estimates hold for the extremal length of β on $r'(t)$. Now assume β crosses C_1. By assumption, the core curve γ_1 of C_1 lies in a thick component Ω'_j of $r'(t)$. By (11), the contribution to the extremal length of β in the thick part of Ω'_j from the n_i crossings of β with γ_1 with t_i twists, is comparable to $n_i^2 t_i^2$. The contribution to extremal length of intersections with curves whose homotopy classes lie in both critical graphs are comparable, by (11). Comparing the estimate $n_i^2 t_i^2$ to (9) we see that for some $c > 0$,

$$\frac{\mathrm{Ext}_{r(t)}(\beta)}{\mathrm{Ext}_{r'(t)}(\beta)} \leq c\frac{n_i^2(M_i(t) + t_i^2/M_i(t)}{n_i^2 t_i^2} \leq cM_i(t) \leq cc_i e^{2t}.$$

The same estimates hold if β crosses a collection of $\hat{C}'_i$ while the γ'_i lie in thick components Ω_j. Thus we see that (7) holds. $\diamond$

6 Appendix: Proof of Theorem 4.1

Before we begin the proof of Theorem 4.1 we will need some definitions and lemmas. By a *geodesic* in a metric space we will mean a globally length-minimizing geodesic. Suppose $Y = Y_1 \times \ldots \times Y_m$ is a product of metric spaces, given the sup metric. A pair of points $p = (p_1, \ldots, p_m)$ and $q = (q_1, \ldots, q_m)$ in Y is called a *diagonal pair* if $d_{Y_i}(p_i, q_i) = d_{Y_j}(p_j, q_j)$ for $1 \leq i, j \leq m$. If one of the points is understood, we call the other a *diagonal point*.

The following lemma follows directly from the definition of the sup metric on Y.

Lemma 6.1 (Characterizing diagonal pairs)**.** *Let Y be as above, and suppose $m \geq 2$. If p, q is a diagonal pair, then any geodesic between p and q is of the form $(r_1(t), \ldots, r_m(t))$, where each $r_i(t)$ is a geodesic segment in Y_i, and the $r_i(t)$ have the same parametrizations. Thus if there is a unique geodesic from p_i to q_i for each $1 \leq i \leq m$, then there is a unique geodesic from p to q. If p, q is not a diagonal pair, then there are infinitely many geodesics in Y from p to q.*

Now suppose that we are in the situation of the hypotheses of Theorem 4.1. By the previous paragraph, Lemma 6.1, and the definition of the sets S_i, we have the following.

Lemma 6.2. *The set of points*

$$(S_1 \times Y_2 \cdots \times Y_m) \cup (Y_1 \times S_2 \times \cdots \times Y_m) \cup (Y_1 \times \cdots \times Y_{m-1} \times S_m)$$

in Y is precisely the set of points $z = (z_1, \ldots, z_m) \in Y$ with the following property: there exists an integer $N > 1$ such that for every neighborhood U of z, there exists a pair of points $x, y \in U$ such that the number of geodesics in Y from x to y is greater than one and at most N. In fact we can take $N = N_1 \cdot N_2 \cdots N_m$.

Note that the given set in Lemma 6.2 is just $(Y_1 \setminus S_1) \times \cdots \times (Y_m \setminus S_m)$. The characterization of the points in this set given by Lemma 6.2 is purely metric, and is therefore clearly preserved by any isometry of Y. It follows that any isometry of Y preserves this set. But the metric space $(Y_1 \setminus S_1) \times \cdots \times (Y_m \setminus S_m)$ is a product of geodesic metric spaces, none of which is a point, and each of which has the locally unique geodesics property. Malone proved that any such product decompositions (in the sup metric) is unique. As each $(Y_i \setminus S_i)$ is open and dense in Y_i, any isometry of $Y_i \setminus S_i$ has a unique extension to Y_i. Theorem 4.1 follows.

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