

THERE EXISTS AN INTERVAL EXCHANGE WITH A NON-ERGODIC GENERIC MEASURE

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1. INTRODUCTION

In the initial study of interval exchange transformations in the Soviet Union, the United States and western Europe a major motivation was basic ergodic theory. One aspect was to find bounds on the maximal number of ergodic measures. Katok [2] proved that n -IETs have at most $\lfloor \frac{n}{2} \rfloor$ ergodic measures. A little later Keane [3] proved that n -IETs have a finite number of ergodic measures and Veech [10] using different methods than Katok also proved that an n -IET has at most $\lfloor \frac{n}{2} \rfloor$ ergodic measures. Various examples of non-ergodic interval exchanges were constructed by Katok [2], Sataev [7] and Keane [4]. In a slightly different context examples were also constructed by Veech [9]. Masur [6] and Veech [11] later proved almost every interval exchange with an irreducible permutation is uniquely ergodic.

The present paper is devoted to the study of invariant measures of minimal and not uniquely ergodic interval exchange transformations.

Now let (X, d) be a σ -compact metric space, \mathcal{B} be the Borel σ algebra, μ be a Borel measure and $T : X \rightarrow X$ be μ -measure preserving.

Definition 1. *A point $x \in X$ is said to be generic for μ if for every continuous compactly supported $f : X \rightarrow \mathbb{R}$ we have $\sum_{i=0}^{N-1} f(T^i x) \rightarrow \int f d\mu$.*

Consider a σ -compact metric space. The Birkhoff ergodic theorem and the fact that continuous compactly supported functions with supremum norm have a countable dense set says that if a measure μ is ergodic then μ -almost every point is generic.

Definition 2. *A measure is called generic if it has a generic point.*

The question arises if a non-ergodic measure can be generic. In settings like the left shift on $\{0, 1\}^{\mathbb{Z}}$ it is straightforward to build generic points for non-ergodic measures. Interval exchanges have fewer, much more constrained orbits and so these techniques do not work. However, in this paper we show that there exists an IET with a generic non-ergodic measure.

Theorem 1. *There exists a minimal non uniquely ergodic IET on 6 intervals with 2 ergodic measures and which has a generic measure that is not ergodic.*

The same proof given in [2] gives the following.

Theorem 2. *A minimal IET has at most $\lfloor n/2 \rfloor$ generic measures.*

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Thus if the number of ergodic measures is maximal, then no other measure can be generic.

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2. RAUZY INDUCTION

We follow the description of interval exchange transformations introduced in [5] and also explicated in [1]. We have an alphabet \mathcal{A} on $d \geq 2$ letters. Break an interval $I = [0, |\lambda|)$ into intervals $\{I_\alpha\}_{\alpha \in \mathcal{A}}$ and rearrange in a new order by translations. Thus the interval exchange transformation is entirely defined by the following data:

- (1) The lengths of the intervals
- (2) Their orders before and after rearranging

The first are called length data, and are given by a vector $\lambda \in \mathbb{R}^d$. The second are called combinatorial data, and are given by a pair of bijections $\pi = (\pi_t, \pi_b)$ from \mathcal{A} to $(1, \dots, d)$. The bijections can be viewed as a pair of rows, the top corresponding to π_t and the bottom corresponding to π_b .

Given an interval exchange T defined by (λ, π) let $\alpha, \beta \in \mathcal{A}$ the last elements in the top and bottom. The operation of Rauzy induction is applied when $\lambda_\alpha \neq \lambda_\beta$ to give a new IET T' defined by (λ', π') where λ', π' are as follows. If $\lambda_\alpha > \lambda_\beta$ then π' keeps the top row unchanged, and it changes the bottom row by moving β to the position immediately to the right of the position occupied by α . We say α *wins* and β *loses*. For all $\gamma \neq \alpha$ define $\lambda'_\gamma = \lambda_\gamma$ and define

$$\lambda'_\alpha = \lambda_\alpha - \lambda_\beta.$$

If $\lambda_\beta > \lambda_\alpha$ then to define π' we keep the bottom row the same and the top row is changed by moving α to the position to the right of the position occupied by β . Then $\lambda'_\gamma = \lambda_\gamma$ for all $\gamma \neq \beta$ and $\lambda'_\beta = \lambda_\beta - \lambda_\alpha$. We say β *wins* and α *loses*.

In either case one has a new interval exchange T' determined by (λ', π') and defined on an interval $I' = [0, |\lambda'|)$ where

$$|\lambda'| = \sum_{\alpha \in \mathcal{A}} \lambda'_\alpha.$$

The map $T' : I' \rightarrow I'$ is the first return map to a subinterval of I obtained by cutting from I a subinterval with the same right endpoint and of length λ_ζ where ζ is the loser of the process described above.

Let Δ be the standard simplex in \mathbb{R}^d and let \mathcal{P} be the set of permutations on n letters. We can normalize so that all IET are defined on the unit interval. Let

$$\mathcal{R} : \Delta \times \mathcal{P} \rightarrow \Delta \times \mathcal{P}$$

then denote Rauzy induction.

There is a corresponding visitation matrix $M = M(T)$. Let $\{e_\gamma\}_{\gamma \in \mathcal{A}}$ be the standard basis. If α is the winner and β the loser then $M(e_\gamma) = e_\gamma$ for $\gamma \neq \alpha$ and $M(e_\alpha) = e_\alpha + e_\beta$. We can view M as simply arising from the identity matrix by adding the α column to the β column. We can projectivize the matrix M and consider it as $M : \Delta \rightarrow \Delta$.

When the interval exchange T is understood, and we perform Rauzy induction n times then define $M(1) = M(T)$ and inductively

$$M(n) = M(n-1)M(\mathcal{R}^{n-1}T).$$

That is, the matrix $M(n)$ comes from multiplying $M(n-1)$ on the right by the matrix of Rauzy induction applied to the IET after we have done Rauzy $n-1$ times. Let $C_\sigma(n)$ the column of $M(n)$ corresponding to the letter $\sigma \in \mathcal{A}$.

Definition 3. *We say an IET satisfies the Keane condition if the orbit of every point of discontinuity is infinite and misses every other such point. It is a fact that such an IET is minimal.*

If an interval exchange has all powers of Rauzy induction defined on it then it satisfies the Keane condition (see [8, Corollary 5.4] for a proof in a survey).

Theorem 3. ([10, Section 1 and Proposition 3.22] *Let T be an IET satisfying the Keane condition. The space of invariant measures of T is $\cap_{n=1}^\infty M(n)\Delta$.*

We now specialize to IET on 6 intervals. We will take paths that return infinitely often to the arrangement

$$\begin{pmatrix} A & B & C & D & E & F \\ F & E & D & C & B & A \end{pmatrix}.$$

Our paths will be of two types which we call the left and right paths. Our paths will depend on a sequence of parameters a_k . Inductively we assume that for the k^{th} time having just traveled on the left hand side we have taken a total of n_k steps of Rauzy induction and returned to the above arrangement. Our matrix is $M(n_k)$.

Remark 1. In what follows we suppress the matrix that the columns are being taken from. So $C_A(M(n_k + r_k))$ is denoted $C_A(n_k + r_k)$. Given a vector \bar{v} let $|\bar{v}|$ denote the sum of the absolute values of its entries (that is its ℓ^1 norm).

Inductively assume the columns satisfy

$$(A) \quad \min\{|C_A(n_k)|, |C_B(n_k)|\} > a_k \max\{|C_E(n_k)|, |C_F(n_k)|\}$$

and

$$(B) \quad \min\{|C_E(n_k)|, |C_F(n_k)|\} > \frac{a_k-1}{k^2} \max\{|C_C(n_k)|, |C_D(n_k)|\}.$$

Now we go along the loop on the right side. In the first step of Rauzy induction F beats A . The second step F beats B . In the third step C beats F , and then C beats E . Now for a total of r_k consecutive steps C loses to D . The number r_k is chosen so that it is smallest odd number satisfying

$$(C) \quad |C_C(n_k + 4 + r_k)| > k|C_F(n_k + 4 + r_k)|$$

Now C will beat D . Now F beats C and then F beats D . Next E beats F consecutively s_k times for a total of $s_k + r_k + 7$ steps until the first time that

$$(D) \quad |C_F(n_k + r_k + s_k + 7)| > a_{k+1} \cdot \max\{|C_A(n_k + 7 + r_k + s_k)|, |C_B(n_k + 7 + r_k + s_k)|\}.$$

Then F beats E and we return to

$$\begin{pmatrix} A & B & C & D & E & F \\ F & E & D & C & B & A \end{pmatrix}$$

for the $k + 1^{st}$ time. The number of steps we have taken is $r_k + s_k + 8$ so

$$n_{k+1} = n_k + r_k + s_k + 8.$$

Notice when we return to $\begin{pmatrix} A & B & C & D & E & F \\ F & E & D & C & B & A \end{pmatrix}$ we have by (A)

$$\min\{|C_E(n_{k+1})|, |C_F(n_{k+1})|\} > a_{k+1} \max\{|C_A(n_{k+1})|, |C_B(n_{k+1})|\}$$

and for a_k sufficiently large compared to k ,

$$\min\{|C_A(n_{k+1})|, |C_B(n_{k+1})|\} > \frac{a_k}{k^2} \max\{|C_C(n_{k+1})|, |C_D(n_{k+1})|\}$$

so (B) holds.

The diagrams below describe the paths on the right side and left side of the Rauzy diagram. The permutation $\begin{pmatrix} A & B & C & D & E & F \\ F & E & D & C & B & A \end{pmatrix}$ is a common point.

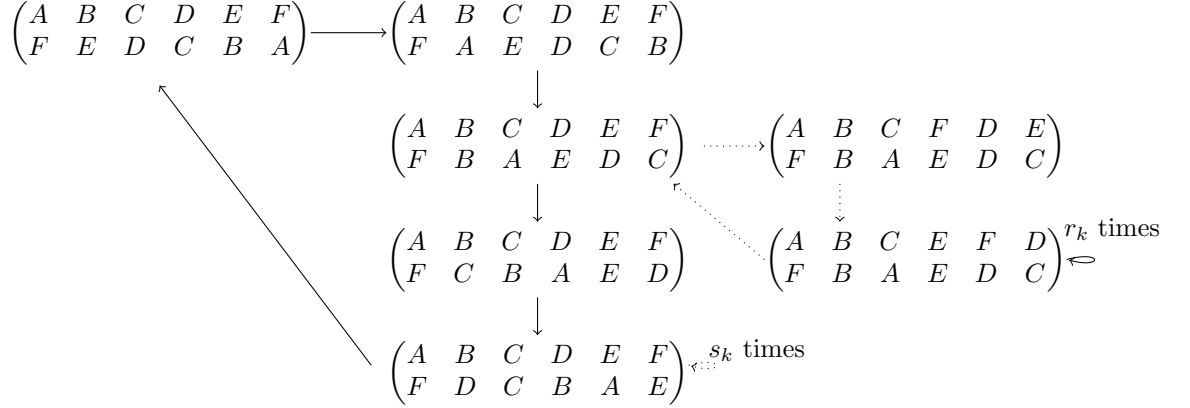


FIGURE 1. The path we take on the right hand side of the Rauzy diagram

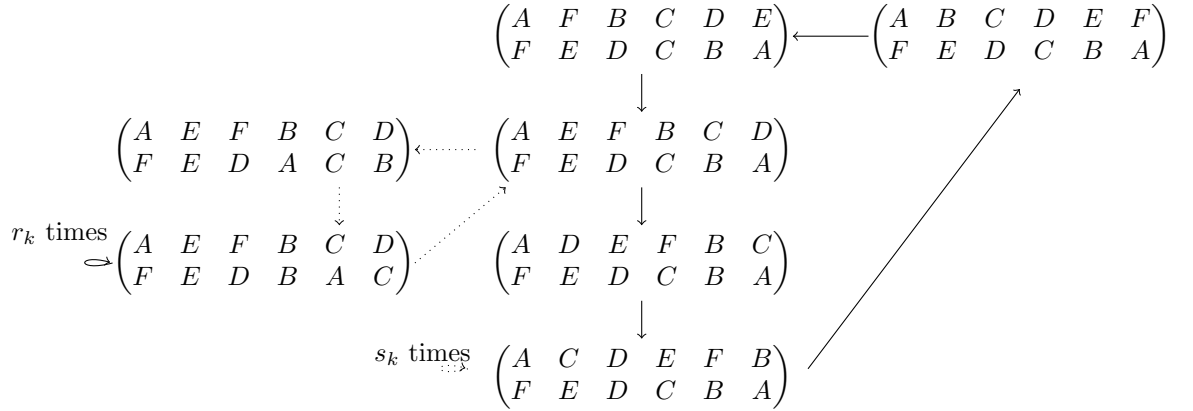


FIGURE 2. The path we take on the left hand side of the Rauzy diagram

On the left hand side A beats F and then beats E ; then D beats A and then B ; then C beats D repeatedly r_{k+1} times until the first time that

$$C_D > (k+1)C_A.$$

Then D beats C . Then A beats D and then C and then B beats A repeatedly s_{k+1} times until the first time that

$$C_A(n_{k+1}+r_{k+1}+7+s_{k+1}) > a_{k+2} \min\{(C_E(n_{k+1}+7+s_{k+1}+m_{k+1}), C_F(n_{k+1}+7+r_{k+1}+s_{k+1}))\}.$$

Finally A beats B and we return.

Our main Theorem is then the following.

Theorem 4. *For choices of a_k there is a minimal 6 interval exchange transformation T with the property that the resulting simplices $\Delta_k = M(k)(\Delta)$ converge to a line segment as $k \rightarrow \infty$. The endpoints are ergodic measures for T . One endpoint is the common projective limit of the column vectors C_A, C_B ; the other endpoint is the common projective limit of the column vectors C_E, C_F . Along the sequence n_k the column vectors corresponding to C_C, C_D converge projectively to a common interior point of the segment and this limit point is a generic measure.*

2.1. Outline of proof. The proof is in two main steps. In the next subsection we show that $\cap_{n=1}^{\infty} M(n)\Delta$ converges to a line segment. This is given by Proposition 2 with preliminary Proposition 1. This shows that there are two ergodic measures. We also show that the sequence of vectors given by the C^{th} and D^{th} columns of the matrices $M(n)$ converge to a single point away from the endpoints in $\cap_{n=1}^{\infty} M(n)\Delta$. This gives the invariant measure ν we are interested in. In the section after that we show that ν is generic by finding a generic point x .

2.2. Convergence of column vectors. The first Lemma is simple Euclidean geometry. Our angles will be bounded away from $\frac{\pi}{2}$ and so $\sin x$ will be comparable with x .

Lemma 1. *Let $v, w \in \mathbb{R}^n$, and let $\theta(v, w)$ denote the angle between v and w . If $\theta(v+w, w)$ denotes the angle between $v+w$ and w , we have*

$$|\sin \theta(v+w, w)| = \frac{\|v\|}{\|v+w\|} |\sin \theta(v, w)|.$$

We will now simplify notation for the column vectors, using v for a vector and eliminating some of the times of Rauzy induction.

We assume that we enter the right side after an even number of returns. We define $v_F(2k+1), v_E(2k+1)$ to be the column vectors $C_F(n_{2k}+3), C_E(n_{2k}+4)$ after F, E lose to C on the right side. We choose $v_F(2k+2)$ to be the column vector $C_F(n_{2k}+r_{2k}+m_{2k}+7)$ after E is added to F on the right side and $v_E(2k+2)$ the column vector $C_E(n_{2k}+r_k+8) = C_E(n_{2k+1})$ after C_F is added to C_E .

Similarly let $v_A(2k)$ be the column vector $C_A(n_{2k-1}+3)$ on the left side just after A has lost to D and $v_A(2k+1)$ is the column vector $C_A(n_{2k}) = C_A(n_{2k-1}+r_{2k-1}+m_{2k-1}+7)$ just after B has finished beating A . Similarly the column vectors $v_B(2k)$ and $v_B(2k+1)$ are defined in terms of C_B .

We define $v_C^R(2k+1)$ to be the column vector $C_C(n_{2k}+r_{2k}+4)$ when C has finished losing to D on the right side and $v_C^R(2k+2)$ the column vector for C after C loses to F . This ensures that $v_C^R(2k+2)$ lies on the line joining $v_C^R(2k+1)$ and $v_F(2k+1)$ which will be needed later. Similarly, $z_C^L(2k)$ is the column vector for

C on the left side after D has beaten C , and $z_C^L(2k+1)$ is the column vector after A has beaten C . We have similar definitions for $v_D^L(2k), v_D^L(2k+1)$ in terms of the column vectors C_D .

The following Proposition gives the properties these vectors satisfy. They will be used to prove the desired convergence results.

Proposition 1. *With a choice of a_k there exist constants $\delta > 0$, $\alpha < 1$ such that for $i, j \in \{1, 2\}$, and all k*

- (1) *for $\sigma, \sigma' \in \{E, F\}$, $\theta(v_\sigma(k), v_{\sigma'}(k+1)) < \alpha^k$ and similarly for A, B .*
- (2) *for $\sigma \in \{E, F\}$ and $\sigma' \in \{A, B\}$, $\theta(v_\sigma(k), v_{\sigma'}(k)) \geq \delta$.*
- (3) *$\theta(v_C^R(2k+2), v_F(2k+1)) = (1 - \frac{1}{k} + O(\frac{1}{k^2}))\theta(v_C^R(2k+1), v_F(2k+1))$, with the same formula for v_D^R .*
- (4) *$\theta(v_D^L(2k+1), v_A(2k)) = (1 - \frac{1}{k} + O(\frac{1}{k^2}))\theta(v_D^L(2k), v_A(2k))$, with the same formula for v_C^L .*
- (5) *$\theta(v_C^R(k), v_D^R(k)) \leq \alpha^k$ for all k . The same holds on the left.*
- (6) *$\theta(v_C^L(2k), v_C^R(2k)) \leq \alpha^k$.*
- (7) *$\theta(v_D^R(2k+1), v_D^L(2k+1)) \leq \alpha^k$.*

Moreover $v_C^R(2k+2)$ lies on the line connecting $v_C^R(2k+1)$ and $v_F(2k+1)$ and $v_C^L(2k+1)$ lies on the line connecting $v_C^L(2k)$ and $v_A(2k)$ with the same for v_D .

Proof. The vectors $v_E(2k+1)$ (resp. $v_F(2k+1)$) arises from $v_E(2k)$ (resp. $v_F(2k)$) from E (resp. F) losing first to A on the left side and then to C back on the right side. Recall this means that the A column and then the C column are added to the E (resp. F column). Then by (A) and (B) and Lemma 1,

$$\theta(v_E(2k+1), v_E(2k)) \leq \frac{1}{a_{k+1} + \frac{a_{k+1}}{k^2}},$$

and similarly for v_F . Similarly we have

$$\theta(v_A(2k+2), v_A(2k+1)) \leq \frac{1}{a_{k+2} + \frac{a_{k+2}}{(k+1)^2}}$$

and the same for v_B .

On the right side the C_E column is added to the C_F column many times and then C_F is added to C_E . By (A) and (D) and Lemma 1, this gives

$$\theta(v_F(2k+2), v_E(2k+2)) \leq \frac{1}{a_k a_{k+1}} \theta(v_F(2k+1), v_E(2k+1)).$$

The same argument shows that

$$|\theta(v_F(2k+2), v_F(2k+1)) - \theta(v_F(2k+1), v_E(2k+1))| \leq \frac{1}{a_k a_{k+1}}.$$

We have similar inequalities for the v_A, v_B . If we let a_k be exponentially small, putting these together we see (1) is satisfied.

We also note that on the right side F beats A, B only when smaller by the factor of a_k and similarly on the left side A beats E, F only when smaller by a factor of a_{k+1} which gives

$$\theta(v_E(k+1), v_A(k+1)) \geq (1 - (\frac{1}{a_k} + \frac{1}{a_{k+1}}))\theta(v_E(k), v_A(k)).$$

Thus since initially the angle between the column for E and A is positive we can choose a_k so (2) is satisfied.

We prove (3),(4). When C or D loses to F , C_C moves towards C_F so

$$\theta(v_C^R(2k+2), v_C^R(2k+1)) \leq \frac{1}{k} \theta(v_C^R(2k+1), v_F(2k)).$$

The same holds for $v_C^L(2k), v_C^L(2k+1)$ and also for the corresponding v_D . These imply (3),(4).

When we have finished on the right we have $v_C^R(2k), v_D^R(2k)$. When we move to the left we change from $v_C^R(2k), v_D^R(2k)$ to $v_C^L(2k), v_D^L(2k)$. In doing so we add the C column to the D column r_{2k} times and then the D column is added back to the C column. Since r_k grows exponentially this give (5) for j even and left. Moving from left to right similarly gives (5) when j is odd and we are on the right. In addition in increasing the index on both sides means adding the same column (either C_A or C_F) to each so they remain exponentially close. This proves (5). Similarly the fact that C beats D repeatedly on the left means $v_D^L(2k)$ moves close to $v_C^L(2k)$ and then the fact that D is added to C means that $v_C^L(2k)$ is close to $v_C^R(2k)$. This is (6). We have the corresponding computation on the right giving (7). \square

The following is the basic Proposition that will allow us to find limiting invariant measures.

Proposition 2. *Consider the sequence of vectors $v_\sigma(k) \in \mathbb{R}^6$ that satisfy the conclusions of Proposition 1, projectivized to lie in the simplex*

$$\Delta = \{x_1, \dots, x_6 : \sum_{i=1}^6 x_i = 1\}.$$

Then

- $v_E(k), v_F(k)$ have limit v_0 and $v_A(k), v_B(k)$ have a distinct limit v_1 .
- $\lim_{k \rightarrow \infty} v_C^L(k) = \lim_{k \rightarrow \infty} v_C^R(k) = \lim_{k \rightarrow \infty} v_D^L(k) = \lim_{k \rightarrow \infty} v_D^R(k) = v'$ exists in $\mathbb{P}(\mathbb{R}^6)$
- v' is on the line between v_0 and v_1 in $\mathbb{P}(\mathbb{R}^6)$
- $v' \neq v_0, v_1$.

Proof. We can replace angles with Euclidean distance. The fact that the limits v_0, v_1 of $v_{E,F}(k), v_{A,B}(k)$ exist follows immediately from (1). In fact, they converge exponentially fast. The fact that the limits are distinct comes from (2).

Let \bar{u} be the direction of the line connecting v_0 and v_1 . Let y denote a coordinate perpendicular to \bar{u} . We claim that the y coordinates of $v_C^R(k), v_D^R(k), v_C^L(k), v_D^L(k)$ go to 0. Without loss of generality consider $v_C^R(k), v_C^L(k)$. The fact that $v_C^R(2k+1)$ and $v_C^R(2k+2)$ lie on a line through $v_F(2k+1)$ whose y coordinate goes to 0 exponentially fast together with (3) implies that the y coordinate of $v_C^R(2k+2)$ must decrease by a factor of approximately $1 - \frac{1}{k} + O(\frac{1}{k^2})$ from the coordinate of $v_C^R(2k+1)$. The same is true for the change from even to odd indices of v_C^L , also by (3). But then (6) says that $d(v_C^R(2k), v_C^L(2k)) \rightarrow 0$ exponentially fast and together with the fact that

$$\prod_{k=2}^{\infty} (1 - \frac{1}{k}) = 0$$

implies the y coordinates of $v_C^{R,L}(k)$ must converge to 0. The same is true of $v_D^{R,L}(k)$ by (3) and (7).

It follows from the above conditions that these $v_C^R(k), v_C^L(k), v_D^R(k), v_D^L(k)$ have a single limit v' on the axis. Indeed since the \bar{u} coordinate of $v_C^R(2k+2)$ decreases by $\frac{1}{2k+1}$ with small error from that of $v_C^R(2k+1)$ and then increases by $\frac{1}{2k+2}$ with small error from even to odd implies that the distance in the \bar{u} coordinate between $v_C^R(2k)$ and $v_C^R(2k+2)$ is $O(\frac{1}{k^2})$ so that these coordinates form a convergent Cauchy sequence. The fact that the distance in the \bar{u} coordinate between $v_C^R(2k)$ and $v_C^R(2k+1)$ goes to 0 means the entire sequence converges. The fact that (5) holds implies that the $v_D^{R,L}(k)$ have the same limit.

We claim that $v' \notin \{v_0, v_1\}$. Suppose by contradiction that $v' = v_0$. (The argument for v_1 is similar) Assume without loss of generality $|v_1 - v_0| = 1$. For k even

$$d(v_D^L(k+1), v_D^L(k)) = (1 + o(1)) \frac{1}{k} (|1 - v_D^L(k)|).$$

That is, since $v_D^L(k+1)$ is assumed near 0, we move towards v_1 an amount about $\frac{1}{k}$. When we move towards 0 on the next move we move not as far;

$$d(v_D^R(k+2), v_D^R(k+1)) = (1 + o(1)) \frac{1}{k+1} v_D^R(k+1).$$

Then again on the next move we move further away. On odd moves we will move further away from 0 than we move closer to 0 on even moves as soon as say the \bar{u} coordinate is at most some positive ϵ . This says we do not converge to 0. By a symmetric argument we see that no subsequence converges to q . \square

3. GETTING A GENERIC POINT

Lemma 2. *Let μ be a T -invariant measure with intervals I_j . A point x is generic for μ if and only if $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{I_j}(T^k(x)) = \mu(I_j)$ for all j .*

Proof. The only if direction holds by definition. For the other direction, the point x defines an invariant measure ν such that

$$\nu(I_j) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{I_j}(T^k(x)).$$

We conclude that $\nu(I_j) = \mu(I_j)$. However the measures of initial intervals determine the invariant measure. This is because if T is minimal then $\{T^i I_\sigma\}_{i \in \mathbb{Z}, \sigma \in \{A, \dots, F\}}$ generates the Borel σ -algebra. \square

Given $x \in [0, 1)$, for each j let

$$v_j(x) = \left(\sum_{i=0}^{j-1} \chi_A(T^i x), \dots, \sum_{i=0}^{j-1} \chi_F(T^i x) \right)$$

the vector of visits to the original intervals. If T^i is continuous on an interval U for $0 \leq i < j$ then v_j is constant on U and in an abuse of notation let $v_j(U)$ be this vector.

Recall n_k is the number so that after n_k steps of Rauzy induction we have hit

$$\begin{pmatrix} A & B & C & D & E & F \\ F & E & D & C & B & A \end{pmatrix}.$$

Let J_k be the interval so that after n_k steps of Rauzy induction the induced map, denoted S_k , is defined on J_k . Let A_k, \dots, F_k be the six subintervals of the IET S_k . For σ_k one of these intervals and $x \in \sigma_k$

$$m_k(x) = \min\{i > 0 : T^i x \in J_k\}$$

is constant on σ_k and is $|C_{\sigma_k}|$.

Let $m_k(\sigma_k)$ be this number. In other words in this notation

$$v_{m_k(\sigma_k)}(\sigma_k) = C_{\sigma_k}.$$

Also let

$$\mathcal{O}_k(\sigma_k) = \cup_{i=0}^{m_k(\sigma_k)-1} T^i(\sigma_k).$$

This is the orbit of σ_k under T before σ_k returns to J_k .

Now suppose we have the induced map S_k on J_k and we are about to go to the left side of Rauzy induction. Recall r_{k+1} is the number of times on the left side that C beats D , and so for $0 \leq i \leq r_{k+1}$,

$$S_k^i(C_k) \cap C_k \neq \emptyset.$$

If on the other hand we are about to go to the right side then for $0 \leq i \leq r_{k+1}$

$$S_k^i(D_k) \cap D_k \neq \emptyset.$$

For simplicity let us assume that we are about to go to the left hand side. The arguments are symmetric if we are about to go to the right hand side. Recall r_{k+1} is odd. Let

$$u_k := \lfloor \frac{r_{k+1}}{2} \rfloor.$$

Let

$$\hat{I}_k = S_k^{u_k}(C_k) \cap S_k^{-u_k}(C_k) \cap C_k \subset J_k.$$

Lemma 3. \hat{I}_k consists of two intervals of $\mathcal{O}_{k+1}(C_{k+1})$ and one interval of $\mathcal{O}_{k+1}(D_{k+1})$

Proof. On the left side D_k beats A_k and B_k . Then C_k as the bottom letter beats D_k , the top letter r_{k+1} times before losing to it. Set

$$D'_k = D_k \setminus (S_k(A_k) \cup S_k(B_k)).$$

For each i , after C_k has beaten D'_k i times, we have $S_k^{-i}(D'_k) \subset C_k$. Thus

$$(1) \quad C_k = S_k^{-1}(D'_k) \cup S_k^{-2}(D'_k) \cup S_k^{-3}(D'_k) \dots \\ \cup S_k^{-r_{k+1}}(D'_k) \cup (S_k^{-r_{k+1}-1}(D'_k) \cap C_k),$$

with this disjoint union moving right to left along C_k . The last term in the union which is at the left end of C_k is an image of C_{k+1} . Moreover D'_k is a disjoint union of an image of C_{k+1} and an image of D_{k+1} . In addition S_k acts by translation by the length of D'_k along C_k . Then for $i < r_{k+1}$, $S_k^i(C_k) \cap C_k$ is C_k with the i copies of $C_{k+1} \cup D_{k+1}$ furthest on the left of C_k deleted. Similarly, $S_k^{-i}(C_k) \cap C_k$ deletes the i copies of $C_{k+1} \cup D_{k+1}$ furthest on the right of C_k . It follows then from the definition of \hat{I}_k that it is a single union of an image of $C_{k+1} \cup D_{k+1}$ and the last term in the union, which is an image of C_{k+1} . \square

Remark 2. If the top letter σ beats the bottom letter τ then $S_k(\tau) \subset \sigma$. Similarly if the bottom letter τ beats the top letter σ then $S^{-1}(\sigma) \subset \tau$. With this observation, the proof of the previous lemma on the right hand side would change the negative powers of S_k to positive powers and the positive powers of S_k to negative powers.

Now let $I_k = \hat{I}_k \cap \mathcal{O}_{k+1}(D_{k+1})$. We conclude from the previous lemma that

Lemma 4. $S_k^i I_k \subset C_k$ for $-u_k < i < u_k$.

Let

$$\hat{m}_k = \min\{i \geq 0 : T^i I_k \subset J_{k+1}\}.$$

Then $v_{\hat{m}_k}(I_k)$ consists of u_k iterates of the entire orbit $\mathcal{O}_k(C_k)$, followed by the entire orbit of $\mathcal{O}(D_k)$, and then followed by the entire orbit of $\mathcal{O}(A_k)$. Now the fact that

$$u_k m_k(C_k) = (1 + o(1)) \frac{k}{2} m_k(A_k)$$

implies

$$m_k(A_k) = o(\hat{m}_k).$$

Let \mathcal{I}_{k-1} be an interval that has been inductively defined so that it contains a unique interval of $\mathcal{O}_k(C_k)$. Let \mathcal{I}_k be defined as $\mathcal{O}_k(I_k) \cap \mathcal{I}_{k-1}$. Observe that it contains a unique interval of $\mathcal{O}_{k+1}(D_{k+1})$ and therefore the number

$$j_k = \min\{j : T^j \mathcal{I}_k \subset J_{k+1}\}$$

is well-defined and T^{j_k} is continuous on \mathcal{I}_k .

Lemma 5. $\theta(v_{m_k(C_k)}(C_k), v_{j_k}(\mathcal{I}_k)) = o(1)$ and $\theta(v_{m_k(D_k)}(D_k), v_{j_k}(\mathcal{I}_k)) = o(1)$.

In fact these angles approach 0 exponentially fast.

Proof. This is similar to the proof of Proposition 1 because $v_{j_k}(\mathcal{I}_k)$ is made up of blocks of C_k or D_k and a smaller column of $\mathcal{O}(A_k)$. \square

We conclude the proof of the Main Theorem. We let

$$x = \cap_{k=1}^{\infty} \mathcal{I}_k.$$

We will show that x is generic for ν . To do that we show that the angle its visitation vector makes with the column C_k goes to 0 as $k \rightarrow \infty$. This is shown in the next Proposition.

Proposition 3. For $j \in [j_{k-1}, j_k]$ we have that

$$\lim_{k \rightarrow \infty} \theta(v_j(\mathcal{I}_k), v_{m_k(C_k)}(C_k)) = 0.$$

Proof. By Lemma 5, the angle $v_{j_{k-1}}(\mathcal{I}_k) = v_{j_{k-1}}(\mathcal{I}_{k-1})$ makes with $v_{m_{k-1}(D_{k-1})}(D_{k-1})$ goes to 0 with k and by Proposition 1 the angle of the latter with $v_{m_k(C_k)}(C_k)$ also goes to 0. Thus the conclusion of the Proposition holds for $j = j_{k-1}$. Moreover

$$T^{j_{k-1}}(\mathcal{I}_k) = S_k(I_k) \subset C_k.$$

We wish to increase the set of times in the orbit of \mathcal{I}_k . Again recall that we are assuming that we have just finished the right hand side. Since

$$j_k = j_{k-1} + (u_k - 1)m_k(C_k) + m_k(A_k),$$

we first consider $m_k(C_k)$. Now $v_{m_k(C_k)}(C_k)$ consists of r_k copies of the vector of visits of the entire orbit $\mathcal{O}_{k-1}(D_{k-1})$ to the original intervals followed by a single copy of the vector of visits of $\mathcal{O}_{k-1}(C_{k-1})$ to the original intervals followed by a

single copy of $\mathcal{O}_{k-1}(F_{k-1})$. Given that $m_{k-1}(D_{k-1}) = o(j_{k-1})$, Proposition 3 holds for any i in the initial interval $[j_{k-1}, j_{k-1} + m_{k-1}(D_{k-1})]$. Then by Proposition 1 and the fact that the Proposition 3 holds for j_{k-1} , we have that

$$\theta(v_{j_{k-1}}(\mathcal{I}), v_{m_{k-1}(D_{k-1})}(D_{k-1})) = o(1).$$

This implies that for any

$$i \in [j_{k-1}, j_{k-1} + r_k m_{k-1}(D_{k-1})],$$

if one considers the orbit of \mathcal{I}_k at time i , then

$$\lim_{k \rightarrow \infty} \theta(v_i(\mathcal{I}_k), v_{m_k(C_k)}(C_k)) = 0.$$

Since $m_{k-1}(F_{k-1}) = o(r_k m_{k-1}(D_{k-1}))$ we have that if $i \in [j_{k-1}, j_{k-1} + m_k(C_k)]$ then

$$\lim_{k \rightarrow \infty} \theta(v_i(\mathcal{I}_k), v_{m_k(C_k)}(C_k)) = 0.$$

Iterating this shows that if

$$i \in [j_{k-1}, j_{k-1} + (u_k - 1)m_k(C_k)]$$

we have that

$$\lim_{k \rightarrow \infty} \theta(v_i(\mathcal{I}_k), v_{m_k(C_k)}(C_k)) = 0.$$

Finally since $m_k(A_k) = o(u_k m_k(C_k))$ and

$$j_k = j_{k-1} + (u_k - 1)m_k(C_k) + m_k(A_k),$$

this finishes the proof of the Proposition and the proof of the main theorem is complete. \square

We finish the paper with the proof of Theorem 2. This is the same proof given in [2] where it was shown that there are at most $\lfloor \frac{n}{2} \rfloor$ ergodic measures. We include it for completeness.

Proof of Theorem 2. We suspend the interval exchange to a translation surface (X, ω) of genus $g = \lfloor \frac{n}{2} \rfloor$. The interval I becomes horizontal. Let x be a generic point for an invariant measure μ . Let $n_k \rightarrow \infty$ a sequence such that for each n_k and $0 < i < n_k$ there is no point of $T^i(x)$ in the interval between x and $T^{n_k}(x)$. This means that if we take the vertical line through x to $T^{n_k}(x)$ we can join x to $T^{n_k}(x)$ by a segment $J_{n_k} \subset I$ and the resulting closed curve γ_{n_k} is simple. Since x is generic the homology class $\frac{\gamma_{n_k}}{|\gamma_{n_k}|}$ converges in $H_1(X, \mathbb{R})$ to what is called an asymptotic cycle, denoted $[\gamma(\mu)]$.

It is easy to see that if μ and ν are different invariant measures they define different measures to intervals and so the corresponding asymptotic homology classes $[\gamma(\mu)] \neq [\gamma(\nu)]$. Moreover since the measures are nonatomic, for intervals J_{n_k} with $|J_{n_k}| \rightarrow 0$, $\mu(J_{n_k}) \rightarrow 0$ which implies that the asymptotic cycles have 0 intersection number that is,

$$\langle [\gamma(\mu)], [\gamma(\nu)] \rangle = 0.$$

This says they form a Lagrangian subspace of $H_1(X, \mathbb{R})$. Since the dimension of homology is $2g$, where g is the genus of X , a Lagrangian subspace has dimension at most g . \square

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