# Unique Continuation for Nonlinear Schrödinger Equation

Marco Aurelio Méndez Guaraco
Instituto Nacional de Matematica Pura e Aplicada

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# Abstract

Unique continuation results for Partial Differential Equations answers the question of what conditions two solutions of a PDE must satisfy in order to be the same. In this monograph we are going to present some results from the work of L. Escauriaza, C.E. Kenig, G. Ponce and L. Vega, about the Schrödinger Equation. Based on an Uncertainty Principle due to G.H. Hardy, the starting point will be the study of unique continuation properties for free waves.

**Keywords:** Unique continuation, dispersive equation, Schrödinger equation, Carleman estimate.

# Contents

# Chapter 1

# Motivation

Unique continuation results for Partial Differential Equations answers the question of what conditions two solutions of a PDE must satisfy in order to be the same. Commonly, solutions are required to agree on a certain subset of their domain of definition.

In this work we are going to present a different type of unique continuation result. We are going to ask for the solutions of an evolution equation, not to agree on a certain subset but to have comparable decays at certain times. The main theorem we are going to study is the following :

**Theorem 1.0.1.** *Let* $u \in C([0,1], L^2(\mathbb{R}^n))$ *be a strong solution of*

$$\partial_t u = i(\Delta u + V(x,t)u) \quad \text{in} \quad \mathbb{R}^n \times [0,1]$$

*where* $V$ *is a time-dependent real bounded potential that decays at infinity, i.e.*

$$\lim_{R\to\infty} \|V\|_{L^1([0,1]:L^2(\mathbb{R}^n \setminus B_0(R)))} = 0.$$

*If there exist positive constants* $A$ *and* $B$ *satisfying* $AB > 1/4$ *and such that*

$$\|e^{A|x|^2} u(0)\|_{L^2(\mathbb{R}^n)} \text{ and } \|e^{B|x|^2} u(1)\|_{L^2(\mathbb{R}^n)}$$

*are both finite, then* $u \equiv 0$.

This result is motivated from the study of solutions to Free Schrödinger equation (FSE)

$$\partial_t u + i\Delta u = 0.$$

In this case solutions can be constructed using the free Schrödinger group $\{e^{t\Delta} : t \in \mathbb{R}\}$

$$e^{it\Delta}u_0(x) = (e^{-i|\xi|^2 t}\widehat{u_0})^\vee = \frac{e^{i|\cdot|^2/4t}}{(4\pi it)^{n/2}} * u_0(x).$$

One has the identity

$$\begin{aligned} u(x,t) = e^{it\Delta}u_0(x) &= \int_{\mathbb{R}^n} \frac{e^{i|x-y|^2/4t}}{(4\pi it)^{n/2}} u_0(y)dy \\ &= \frac{e^{i|x|^2/4t}}{(4\pi it)^{n/2}} \int_{\mathbb{R}^n} e^{-2ix\cdot y/4t} e^{i|y|^2/4t} u_0(y)dy \\ &= \frac{e^{i|x|^2/4t}}{(2it)^{n/2}} \ \left(e^{i|\cdot|^2/4t}u_0(y)\right)^\wedge (x/2t). \end{aligned}$$

So, roughly speaking, the solution of the Free Schrödinger equation at time $t$ is a rescaled multiple of the Fourier transform of the initial condition $u_0$. More precisely,

$$(2it)^{n/2}e^{-i|x|^2/4t}u(x,t) = (e^{i|\cdot|^2/4t}u_0)^\wedge(x/2t).$$

By means of this observation we can relate Uncertainty Principles for the Fourier Transform to solutions of the Free Schrödinger equation.

In [6] G. H. Hardy's proved an uncertainty principle in terms of the asymptotic decay of the function and its transform. An $L^2$ version of the same result [7] is the following:

**Theorem 1.0.2** (Hardy Uncertainty Principle)**.** *If* $e^{A|x|^2}f(x)$ *and* $e^{4B|\xi|^2}\widehat{f}(\xi)$ *with* $1/AB > 1/4$*, then* $f \equiv 0$*.*

Applying this result to solutions of the FSE we obtain

**Theorem 1.0.3.** *If* $e^{A|x|^2}u(x,0)$ *and* $e^{B|x|^2}u(x,T)$ *are in* $L^2(\mathbb{R}^n)$ *and* $TAB \geqslant 1/4$*, then* $u \equiv 0$*.*

Before the work of L. Escauriaza, C.E. Kenig, G. Ponce and L. Vega, known proofs of this fact were based on adapted versions of the Phragmen-Lindelof Principle applied to the function and its Fourier transform. But since we are interested in generalizations of Theorem 1.0.3 to the nonlinear setting it is necessary to find a proof independent of analiticity. In this work we are going to present this generalization in detail following the work of Escauriaza, Kenig, Ponce and Vega, but before that we present a formal argument for the case of free waves. This formal atgument will sketch the path we will follow later.

# Chapter 2

# Unique Continuation for the Free Schrödinger Equation

Before proceeding to the formal argument, let us notice that it is enough to prove the theorem in the particular case in which the gaussian weights at time 0 and 1 have the same parameter.

## 2.1 Conformal Appell Transformation

Let $u$ be a solution of $\partial_t u = i(\Delta u + Vu + F)$ and define a new function

$$\widetilde{u} = \left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\right)^{n/2} u\left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}x, \frac{\beta}{\alpha(1-t)+\beta t}t\right) e^{\frac{(\alpha-\beta|x|^2)}{4i(\alpha(1-t)+\beta t)}}.$$

Then $\widetilde{u}$ satisfies $\partial_t \widetilde{u} = i(\Delta\widetilde{u} + \widetilde{V}\widetilde{u} + \widetilde{F})$ where

$$\widetilde{V} = \frac{\alpha\beta}{(\alpha(1-t)+\beta t)^2} V\left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}x\right)$$

and

$$\widetilde{F} = \left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\right)^{n/2+2} F\left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}x, \frac{\beta}{\alpha(1-t)+\beta t}t\right) e^{\frac{(\alpha-\beta|x|^2)}{4i(\alpha(1-t)+\beta t)}}.$$

Observe that

$$\widetilde{u}(0) = \left(\sqrt{\frac{\beta}{\alpha}}\right)^{n/2} u\left(\sqrt{\frac{\beta}{\alpha}}x, 0\right) e^{\frac{(\alpha-\beta)|x|^2}{4i\alpha}}$$
$$\widetilde{u}(1) = \left(\sqrt{\frac{\alpha}{\beta}}\right)^{n/2} u\left(\sqrt{\frac{\alpha}{\beta}}x, 1\right) e^{\frac{(\alpha-\beta)|x|^2}{4i\beta}}$$

then the weighted $L^2$ norms of $\widetilde{u}(0)$ and $\widetilde{u}(1)$ are

$$\begin{aligned}\|e^{\gamma|x|^2}\widetilde{u}(0)\|^2_{L^2(\mathbb{R}^n)} &= \left(\sqrt{\frac{\beta}{\alpha}}\right)^n \int_{\mathbb{R}^n} |e^{\gamma|x|^2} u\left(\sqrt{\frac{\beta}{\alpha}}x, 0\right)|^2 dx \\ &= \int_{\mathbb{R}^n} |e^{\gamma(\alpha/\beta)|x|^2} u(x,0)|^2 dx \\ &= \|e^{\gamma\frac{\alpha}{\beta}} u(0)\|^2_{L^2(\mathbb{R}^n)}\end{aligned}$$

So we conclude that

$$\|e^{\gamma|x|^2}\widetilde{u}(0)\|^2_{L^2(\mathbb{R}^n)} = \|e^{\gamma(\alpha/\beta)|x|^2} u(0)\|^2_{L^2(\mathbb{R}^n)}$$

and

$$\|e^{\gamma|x|^2}\widetilde{u}(1)\|^2_{L^2(\mathbb{R}^n)} = \|e^{\gamma(\beta/\alpha)|x|^2} u(1)\|^2_{L^2(\mathbb{R}^n)}$$

Let $A$ and $B$ be two positive numbers and choose $\alpha = \sqrt{A}$, $\beta = \sqrt{B}$ and $\gamma = \sqrt{AB}$. Then the equalities above become

$$\|e^{\gamma|x|^2}\widetilde{u}(0)\|^2_{L^2(\mathbb{R}^n)} = \|e^{A|x|^2} u(0)\|^2_{L^2(\mathbb{R}^n)}$$

and

$$\|e^{\gamma|x|^2}\widetilde{u}(1)\|^2_{L^2(\mathbb{R}^n)} = \|e^{B|x|^2} u(1)\|^2_{L^2(\mathbb{R}^n)}.$$

Now let us assume that these norms are finite, then the theorem asserts that if $AB \geqslant 1/4$ then the solution $u \equiv 0$, but given the calculations above this is equivalent to say that if $\gamma^2 = AB \geqslant 1/4$ then $\widetilde{u} \equiv 0$. So it suffices to prove the theorem for the case $A = B$.

## 2.2 Hardy Uncertainty Principle for the Free Schrödinger Equation

**Theorem 2.2.1.** *Let $u$ be a solution of the FSE such that*

$$e^{A|x|^2} u_0 \text{ and } e^{A|x|^2} u(1) = e^{A|x|^2} e^{it\Delta} u_0 \text{ are both in } L^2(\mathbb{R}^n).$$

*Then, if $A \geqslant 1/4$, $u$ is the zero solution.*

*Proof.* (Formal argument.)

Since we are interested in the weighted $L^2$-norms of the $u$, let us define $f$ to be $f = e^{\varphi}u$, where $\varphi$ is a function to be choosen.

Then $f$ satisfies the IVP

$$\begin{cases} \partial_t f = \partial_t \varphi \cdot f + i\{\Delta f - 2\nabla\varphi\nabla f + (|\nabla\varphi|^2 - \Delta\varphi)f\} \\ f(x_0)f = e^{\varphi}u_0. \end{cases}$$

We can rewrite this equation as $\partial_t f = (\mathcal{S} + \mathcal{A})f$ where

$$\begin{aligned} &\mathcal{S} = \partial_t\varphi - i(2\nabla\varphi\nabla + \Delta\varphi) \quad \text{is symmetric and} \\ &\mathcal{A} = i(\Delta + |\nabla\varphi|^2) \qquad \text{is skew-symmetric.} \end{aligned}$$

Having the operator decomposed in its symmetric and skew-symmetric parts we can apply the method of Carleman estimates to obtain bounds for $\|f\|_{L^2(\mathbb{R}^n)}$.

Choose $\varphi(x,t) = \mu|x + Rt(1-t)e_1|^2$ where $\mu$ and $R$ are positive constants and $e_1 = (1, 0, \ldots, 0)$. Then, if we call $H(t) = \|f\|^2_{L^2(\mathbb{R}^n)}$, Carleman estimates will give us the following bound for the second derivative of $\log(H(t))$.

**Claim 1.**

$$[\log(H(t))]'' \geqslant -\frac{R^2}{8\mu}.$$

*In particular the function $e^{-\frac{R^2 t(1-t)}{16\mu}}H(t)$ is logarithmically convex.*

Let us now finish the proof of the theorem assuming the claim.

Call $F(t) = e^{-\frac{R^2 t(1-t)}{16\mu}}H(t)$, this function is logarithmically convex, in particular

$$e^{-\frac{R^2}{64\mu}}H(1/2) = F(1/2) \leqslant F(0)^{1/2}F(1)^{1/2} = H(0)^{1/2}H(1)^{1/2}.$$

By the definition of $H$ and passing the exponential to the right hand side we obtain

$$\int_{\mathbb{R}^n} |u(x,1/2)|^2 e^{2\mu|x+(R/4)e_1|^2}dx \leqslant \|e^{\mu|x|^2}u_0\|_{L^2(\mathbb{R}^n)}\|e^{\mu|x|^2}u(1)\|_{L^2(\mathbb{R}^n)}.$$

Now observe that if $|x| \leqslant \epsilon R/4$ then $|x + \frac{R}{4}e_1| \geqslant \frac{R}{4}(1-\epsilon)$, so that if we integrate on the sphere $B(0, \epsilon R/4)$ we obtain

$$e^{2\mu\frac{R^2}{16}(1-\epsilon)^2}\int_{B(0,\epsilon R/4)}|u(x,1/2)|^2dx \leqslant \int_{\mathbb{R}^n}|u(x,1/2)|^2e^{2\mu|x+(R/4)e_1|^2}dx.$$

From this and the preeceding formula we finally got

$$\int_{B(0,\epsilon R/4)}|u(x,1/2)|^2 \leqslant \epsilon^{\frac{R^2}{64\mu}(1-8\mu^2(1-\epsilon))^2}C$$

where C is a finite constant depending only on $\mu$ and the solution, which are fixed quantities.

Then if $\mu \geqslant 1/\sqrt{8}$ the parameter on the exponential will be negative and taking $R \to \infty$ in the above inequality we obtain $\|u(1/2)\|_2 = 0$ and therefore $u = 0$ by the theory of existence and uniqueness. This concludes the proof of theorem. □

Now let us proceed to prove the Claim.

*Proof of Claim 1.* Remember that the function $u$ satisfies the FSE $\partial_t u = i\Delta u$, and we want to study the behavior of the weighted norms $\|e^{\varphi}u\|_{L^2(\mathbb{R}^n)}$, with $\varphi$ being a real function depending on time. The function we defined before $f \equiv e^{\varphi}u$ satisfies the equation

$$\partial_t f = \partial_t\varphi \cdot f + i\{\Delta f - 2\nabla\varphi\nabla f + (|\nabla\varphi|^2 - \Delta\varphi)f\}.$$

Let us diferenciate the function $H = \langle f, f\rangle$ to see what we obtain. Rembember that the equation for $f$ can be written as $\partial_t f = \mathcal{S}f + \mathcal{A}f$ where $\mathcal{S} = \partial_t\varphi - i(2\nabla\varphi\nabla + \Delta\varphi)$ is a symmetric operator and $\mathcal{A} = i(\Delta + |\nabla\varphi|^2)$ is skew-symmetric.

Then

$$\begin{aligned} H' &= \langle\partial_t f, f\rangle + \langle f, \partial_t f\rangle \\ &= 2\,\mathrm{Re}\langle\partial_t f, f\rangle \\ &= 2\,\mathrm{Re}\langle\mathcal{S}f + \mathcal{A}f, f\rangle \\ &= 2\langle\mathcal{S}f, f\rangle \end{aligned}$$

Then we concluded that $H' = 2D$ where $D = \langle\mathcal{S}f, f\rangle$. Then the second derivative of $\log H$ will be

$$\big(\log H\big)'' = \left(\frac{H'}{H}\right)' = \left(\frac{2D}{H}\right)' = \frac{2D'}{H} - \frac{4D^2}{H^2},$$

where

$$
\begin{aligned}
D' &= \langle \mathcal{S}f, f\rangle' \\
&= \langle (\partial_t \mathcal{S})f, f\rangle + \langle \mathcal{S}\partial_t f, f\rangle + \langle \mathcal{S}f, \partial_t f\rangle \\
&= \langle (\partial_t \mathcal{S})f, f\rangle + \langle \mathcal{S}(\mathcal{S}f + \mathcal{A}f), f\rangle + \langle \mathcal{S}f, (\mathcal{S}f + \mathcal{A}f)\rangle \\
&= \langle (\partial_t \mathcal{S} + [\mathcal{S}; \mathcal{A}])f, f\rangle + 2\langle \mathcal{S}f, f\rangle.
\end{aligned}
$$

Substituting this in the equation for $(\log H)''$ we obtain

$$
\big(\log H\big)'' = \frac{2\langle (\partial_t \mathcal{S} + [\mathcal{S}; \mathcal{A}])f, f\rangle}{\langle f, f\rangle} + \frac{4\langle \mathcal{S}f, \mathcal{S}f\rangle}{\langle f, f\rangle} - \frac{4\langle \mathcal{S}f, f\rangle^2}{\langle f, f\rangle^2}.
$$

We want a lower bound for this quantity, and observe that the last two terms together are already greater than zero by the Cauchy-Schwarz inequality, i.e.

$$
\frac{4\langle Sf, Sf\rangle}{\langle f, f\rangle} - \frac{4\langle \mathcal{S}f, f\rangle^2}{\langle f, f\rangle^2} \geqslant 0.
$$

Thus we can restrict ourselves to obtain lower bounds for $\langle (\partial_t \mathcal{S} + [\mathcal{S}; \mathcal{A}])f, f\rangle$ and for that we must write explicitly the operator $\partial_t \mathcal{S} + [\mathcal{S}; \mathcal{A}]$

The derivatives of $\varphi = \mu|x + Rt(1-t)e_1|^2$ are

$$
\begin{aligned}
\partial_t \varphi &= 2\mu R(1-2t)(x_1 + Rt(1-t)) \\
\partial_t^2 \varphi &= 2\mu R^2(1-2t)^2 - 4\mu R(x_1 + Rt(1-t)) \\
\nabla(\partial_t \varphi) &= 2\mu R(1-2t)e_1 \\
\Delta(\partial_t \varphi) &= 0 \\
\partial_{x_1} \varphi &= 2\mu(x_1 + Rt(1-t)) \\
\partial_{x_j} \varphi &= 2\mu x_j \qquad \text{(for every } j \geqslant 2) \\
\nabla \varphi &= 2\mu(x + Rt(1-t)e_1) \\
|\nabla \varphi|^2 &= 4\mu^2(|x|^2 + 2Rt(1-t)x_1 + (Rt(1-t))^2) \\
\partial_{x_j}^2 \varphi &= 2\mu \qquad \text{(for every } j \geqslant 1) \\
\Delta \varphi &= 2n\mu \\
D^2 \varphi &= 2\mu I.
\end{aligned}
$$

Replacing these expressions in the formula

$$(\partial_t \mathcal{S} + [\mathcal{S};\mathcal{A}])f = (\partial_t^2\varphi)f - 2i\nabla(\partial_t\varphi)\nabla f - i\Delta\partial_t\varphi\cdot f + [\partial_t\varphi - 2i\nabla\varphi\nabla - i\Delta\varphi; i(\Delta + |\nabla\varphi|^2)]f$$

we obtain:

$$\begin{aligned}
\langle(\partial_t \mathcal{S} + [\mathcal{S};\mathcal{A}])f, f\rangle =& \int_{\mathbb{R}^n} \partial_t^2\varphi|f|^2 - 4i\int_{\mathbb{R}^n} \nabla(\partial_t\varphi)\nabla f\cdot\bar{f} \\
&+ 4\int_{\mathbb{R}^n} \nabla\bar{f}D^2\varphi\nabla f + 4\int_{\mathbb{R}^n} \nabla\varphi D^2\varphi\nabla\varphi|f|^2 \\
=& 2\mu R^2(1-2t)^2\int_{\mathbb{R}^n}|f|^2 - 4\mu R\int_{\mathbb{R}^n} x_1|f|^2 \\
&- 4\mu R^2 t(1-t)\int_{\mathbb{R}^n}|f|^2 - 8\mu Ri(1-2t)\int_{\mathbb{R}^n} f_{x_1}\bar{f} \\
&+ 8\mu\int_{\mathbb{R}^n}|\nabla f|^2 + 32\mu^3\int_{\mathbb{R}^n}|x|^2|f|^2 \\
&+ 64\mu^3 Rt(1-t)\int_{\mathbb{R}^n} x_1|f|^2 + 32\mu^3(Rt(1-t))^2\int_{\mathbb{R}^n}|f|^2
\end{aligned}$$

Recall that we want to find a lower bound for this expression. There are some "good" terms which are already positive, you can use them, along with the inequality $2ab < a^2 + b^2$ to bound the "bad" ones. On this process you will end up adding a term of the form

$$c\frac{R^2}{\mu}\int|f|^2$$

which will complete the proof.

□

# Chapter 3

# Proof Of The Main Theorem

We want to prove

**Theorem 3.0.2.** *Let* $u \in C([0,1] : L^2(\mathbb{R}^n \times [0,1]))$ *be a (strong) solution of*

$$\partial_t u = i(\Delta u + V(x,t)u)$$

*where* $V$ *is a real potential and*

$$V \in L^\infty(\mathbb{R}^n \times [0,1]) \text{ and } \lim_{R\to\infty} \|V\|_{L^1([0,1]:L^\infty(\mathbb{R}^n - B_R(0)))} = 0.$$

*If* $e^{A|x|^2}u(0)$ *and* $e^{B|x|^2}u(1)$ *are both in* $L^2(\mathbb{R}^n)$ *with* $AB > 1/4$ *then* $u \equiv 0$.

But before proving Theorem 3.0.2, in the following section we will present three technical lemmas that we are going to need in the demostration. We decided to present them first since the proofs of these lemmas enclose some of the steps we did on the formal argument in Chapter 2.

## 3.1 Technical Lemmas

The first Lemma we are going to present deals with the exponential decay of a solution between two given times. Since in this work we are concerned with gaussian (and not exponential) decays, after the proof of the lemma we are going to present a trick for passing from exponential to gaussian estimates.

**Lemma 3.1.1.** *There exists $\epsilon_0 > 0$ such that if $V : \mathbb{R}^n \times [0,1] \longrightarrow \mathbb{C}$ satisfies*

$$\|V\|_{L^1_t L^\infty_x} \leqslant \epsilon_0,$$

*and $u \in C([0,1] : L^2_x(\mathbb{R}^n))$ is a (strong) solution of the IVP*

$$\begin{cases} i\partial_t u + \Delta u = Vu + H & (x,t) \in \mathbb{R}^n \times [0,1], \\ u(x,0) = u_0(x), \end{cases}$$

*with $H \in L^1_t([0,1] : L^2_x(\mathbb{R}^n))$, and for some $\beta \in \mathbb{R}$,*

$$u_0, u_1 \equiv u(\cdot,1) \in L^2(e^{2\beta x_1}dx), \quad H \in L^1_t([0,1] : xL^2(e^{2\beta x_1}dx)),$$

*then*

$$\sup_{0\leqslant t\leqslant 1} \|u(\cdot,t)\|_{L^2(e^{2\beta x_1}dx)} \leqslant c(\|u_0\|_{L^2(e^{2\beta x_1}dx)} + \|u_1\|_{L^2(e^{2\beta x_1}dx)} + \|H\|_{L^1_t L^2_x(e^{2\beta x_1}dx)}),$$

*with $c$ independent of $\beta$.*

Roughly speaking what this lemma is saying is that if at two times the quantity $\|e^{\varphi}u\|_{L^2(\mathbb{R}^n)}$ is finite, then it is bounded for every time in the middle. We repeat the proof on [2].

*Proof.* Define $\varphi_n \in C^\infty(\mathbb{R})$ such that $0 \leqslant \varphi_n \leqslant 1$ and

$$\varphi_n(s) = \begin{cases} 1 & \text{if } s \leqslant n \\ 0 & \text{if } s \geqslant 10n \end{cases}$$

with $|\varphi_n^{(j)}(s)| \leqslant c_j/n^j$. Let

$$\theta_n(s) = \beta \int_0^s \varphi_n(l)dl$$

then $\theta$ is nondecreasing with

$$\theta_n(s) = \begin{cases} \beta s & \text{if } s < n \\ c_n\beta & \text{if } s > 10 \end{cases},$$

and

$$\theta_n'(s) = \beta\varphi_n^2(s) \leqslant \beta, \quad |\theta_n^(j)(s)| \leqslant \frac{\beta c_j}{n^{j-1}}.$$

Finally let $\phi_n(s) = \exp(2\theta_n(s))$, then $\phi_n(s) \leqslant \exp(2\beta s)$ and $\phi_n(s)$ converges to $\exp(2\beta s)$ when $t \to \infty$.

We define a truncation of $u(x,t)$ by

$$w_n(x,t) = e^{\mu}\phi_n(x_1)u(x,t)$$

where $\mu = -4i\beta^2\phi_n^4(x_1)t$.

This funtion satisfies the equation

$$i\partial_t w_n + \Delta w_n = Vw_n + \tilde{H}_n + hw_n + a^2(x_1)\partial_{x_1}w_n + itb(x_1)\partial_{x_1}w_n$$

where

$$\begin{aligned}
\tilde{H}_n(x,t) =& e^{\mu}\phi_n(x_1)H(x,t),\\
h(x,t) =& (i16\beta^2\varphi_n^3\varphi_n' t)^2 + i48\beta^2\varphi_n^2(\varphi_n')^2 t + i16\beta^2\varphi_n^3\varphi_n'' t\\
&+ 4\beta\varphi_n\varphi_n' + i64\beta^2\varphi_n^3\varphi_n' t,\\
a^2(x) =& 4\beta\varphi_n^2(x_1),\\
b(x) =& -32\beta^2\varphi_n^3(x_1)\varphi_n'(x_1).
\end{aligned}$$

Note that $a$ and $b$ are real and

$$\|\partial_{x_1}^j h\|_{L^\infty(\mathbb{R}\times[0,1])} \leqslant \frac{c_j}{n^{j+1}}, \qquad a^2(x_1) \geqslant 0,$$
$$\|\partial_{x_1}^j a^2(x_1)\|_{L^\infty(\mathbb{R})} \leqslant \frac{c_j}{n^j}, \qquad \|\partial_{x_1}^j b(x_1)\|_{L^\infty(\mathbb{R})} \leqslant \frac{c_j}{n^j}.$$

Now, to perform energy estimates for $w_n$, take $\epsilon \in (0,1]$ and define the multiplier operators

$$\widehat{P_\epsilon f}(\xi) = \eta(\epsilon\xi)\hat{f}(\xi) \;\text{ and }\; \widehat{P_\pm f}(\xi) = \chi_\pm(\xi_1)\hat{f}(\xi)$$

where $\eta \in C_0^\infty(\mathbb{R}^n)$ is such that $0 \leqslant \eta \leqslant 1$, $\eta(x) = 1$ for $|x| \leqslant 1/2$, $\eta(x) = 0$ for $|x| \geqslant 1$ and

$$\chi_\pm(\xi_1) = \begin{cases} 1 \quad \text{if} \quad \xi_1 > 0 \ (\xi < 0) \\ 0 \quad \text{if} \quad \xi_1 < 0 \ (\xi > 0). \end{cases}$$

Now we derive the equations for $P_\epsilon P_\pm w_n$. For $P_\epsilon P_\pm w_n$

$$\begin{aligned}
i\partial_t P_\epsilon P_+ + \Delta P_\epsilon P_+ w_n =& P_\epsilon P_+(Vw_n) + P_\epsilon P_+(\tilde{H}_n) + P_\epsilon P_+(hw_n)\\
&+ P_\epsilon P_+(a^2(x_1)\partial_{x_1}w_n) + P_\epsilon P_+(ib(x_1)\partial_{x_1}w_n)
\end{aligned}$$

Now we multiply this equation by $\overline{P_\epsilon P_+ w_n}$, its complex conjugate by $-P_\epsilon P_+ w_n$ and adding the results we obtain

$$\begin{aligned}
& i\partial_t |P_\epsilon P_+ w_n|^2 + \Delta P_\epsilon P_+ w_n \cdot \overline{P_\epsilon P_+ w_n} - \overline{\Delta P_\epsilon P_+ w_n} \cdot P_\epsilon P_+ w_n \\
&\quad = P_\epsilon P_+(V w_n)\overline{P_\epsilon P_+ w_n} - \overline{P_\epsilon P_+(V w_n)} P_\epsilon P_+ w_n \\
&\quad + P_\epsilon P_+(\tilde{H}_n)\overline{P_\epsilon P_+ w_n} - \overline{P_\epsilon P_+(\tilde{H}_n)} P_\epsilon P_+ w_n \\
&\quad + P_\epsilon P_+(h w_n)\overline{P_\epsilon P_+ w_n} - \overline{P_\epsilon P_+(h w_n)} P_\epsilon P_+ w_n \\
&\quad + P_\epsilon P_+(a^2 \partial_{x_1} w_n)\overline{P_\epsilon P_+ w_n} - \overline{P_\epsilon P_+(a^2 \partial_{x_1} w_n)} P_\epsilon P_+ w_n \\
&\quad + i\big(P_\epsilon P_+(b\partial_{x_1} w_n)\overline{P_\epsilon P_+ w_n} + \overline{P_\epsilon P_+(b\partial_{x_1} w_n)} P_\epsilon P_+ w_n\big).
\end{aligned}$$

Taking the imaginary part of this equation

$$\begin{aligned}
& \partial_t |P_\epsilon P_+ w_n|^2 + 2\,\mathrm{Im}\left(\Delta P_\epsilon P_+ w_n \cdot \overline{P_\epsilon P_+ w_n}\right) \\
&\quad = 2\,\mathrm{Im}\left(P_\epsilon P_+(V w_n)\overline{P_\epsilon P_+ w_n}\right) + 2\,\mathrm{Im}\left(P_\epsilon P_+(\tilde{H}_n)\overline{P_\epsilon P_+ w_n}\right) \\
&\quad + 2\,\mathrm{Im}\left(P_\epsilon P_+(h w_n)\overline{P_\epsilon P_+ w_n}\right) + 2\,\mathrm{Im}\left(P_\epsilon P_+(a^2 \partial_{x_1} w_n)\overline{P_\epsilon P_+ w_n}\right) \\
&\quad + 2\,\mathrm{Re}\left(P_\epsilon P_+(b\partial_{x_1} w_n)\overline{P_\epsilon P_+ w_n}\right).
\end{aligned} \tag{3.1}$$

By hypotesis we can integrate in $x$ for almost every $t$, and simple calculations leads to the following results

$$\mathrm{Im} \int_{\mathbb{R}^n} \Delta P_\epsilon P_+ w_n \cdot \overline{P_\epsilon P_+ w_n} dx = 0,$$

$$\left| \mathrm{Im} \int_{\mathbb{R}^n} P_\epsilon P_+(V w_n)\overline{P_\epsilon P_+ w_n} dx \right| \leqslant c\|V\|_{L^\infty} \|w_n\|_{L^2}^2,$$

$$\left| \mathrm{Im} \int_{\mathbb{R}^n} P_\epsilon P_+(\tilde{H}_n)\overline{P_\epsilon P_+ w_n} dx \right| \leqslant c\|\tilde{H}_n\|_{L^2} \|w_n\|_{L^2}^2,$$

and

$$\left| \mathrm{Im} \int_{\mathbb{R}^n} P_\epsilon P_+(h w_n)\overline{P_\epsilon P_+ w_n} dx \right| \leqslant c\|h\|_{L^\infty} \|w_n\|_{L^2}^2 \leqslant \frac{c}{n} \|w_n\|_{L^2}^2,$$

where the constant $c$ is independent of $\epsilon$ and $n$.

It remains to bound the last two terms in 3.1. For that we recall Calderon's commutator estimates [8]:

$$\|[P_\pm; a]\partial_{x_1} f\|_{L^2} \leqslant c\|\partial_{x_1} a\|_{L^\infty} \|f\|_{L^2}$$

and

$$\|\partial_{x_1}[P_\pm; a] f\|_{L^2} \leqslant c\|\partial_{x_1} a\|_{L^\infty} \|f\|_{L^2}$$

also we have [9]:

$$\|[P_\epsilon; a]\partial_{x_1} f\|_{L^2} \leqslant \frac{c}{n}\|f\|_{L^2}$$

and

$$\|\partial_{x_1}[P_\epsilon; a] f\|_{L^2} \leqslant \frac{c}{n}\|f\|_{L^2}.$$

Observe that these estimates also hold with $b$ instead of $a$, and that the constat $c$ is independent $\epsilon$.

Now we apply this estimates of the commutators to obtain

**Claim 1.**

$$\int_{\mathbb{R}^n} P_\epsilon P_+(b\partial_{x_1} w_n)\overline{P_\epsilon P_+ w_n} dx =$$
$$= -\overline{\int_{\mathbb{R}^n} P_\epsilon P_+(b\partial_{x_1} w_n)\overline{P_\epsilon P_+(w_n)} dx} + O\Big(\frac{\|w_n\|_{L^2}}{n}\Big)$$

and

**Claim 2.**

$$\begin{aligned}
&\mathrm{Im}\int_{\mathbb{R}^n} P_\epsilon P_+(a^2\partial_{x_1} w_n)\overline{P_\epsilon P_+ w_n} dx \\
&= \mathrm{Im}\int_{\mathbb{R}^n} \partial_{x_1} P_\epsilon P_+(a(x_1) w_n)\overline{P_\epsilon P_+(a(x_1) w_n)} dx + O\Big(\frac{\|w_n\|_{L^2}}{n}\Big) \\
&\leqslant O\Big(\frac{\|w_n\|_{L^2}}{n}\Big).
\end{aligned}$$

We will present the proof of Claim 1. The proof of Claim 2 is analogue.

*Proof of Claim 1.* By the commutator estimates we have that

$$P_+(b\partial_{x_1} w_n) = bP_+(\partial_{x_1} w_n) + O\Big(\frac{\|w_n\|_{L^2}}{n}\Big),$$

and

$$P_\epsilon P_+(b\partial_{x_1} w_n) = bP_\epsilon P_+(\partial_{x_1} w_n) + O\Big(\frac{\|w_n\|_{L^2}}{n}\Big).$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^n} P_\epsilon P_+(b\partial_{x_1} w_n)\overline{P_\epsilon P_+ w_n} dx &= \int_{\mathbb{R}^n} bP_\epsilon P_+(\partial_{x_1} w_n)\overline{P_\epsilon P_+ w_n} dx + O\Big(\frac{\|w_n\|_{L^2}}{n}\Big) \\
&= -\int_{\mathbb{R}^n} P_\epsilon P_+ w_n \overline{\partial_{x_1}(bP_\epsilon P_+ w_n)} dx + O\Big(\frac{\|w_n\|_{L^2}}{n}\Big) \\
&= -\int_{\mathbb{R}^n} P_\epsilon P_+ w_n \overline{P_\epsilon P_+ b\partial_{x_1} w_n} dx + O\Big(\frac{\|w_n\|_{L^2}}{n}\Big).
\end{aligned}$$

□

In particular

$$2\,\mathrm{Re}\int_{\mathbb{R}^n} P_\epsilon P_+(b\partial_{x_1}w_n)\overline{P_\epsilon P_+ w_n}dx = O\bigg(\frac{\|w_n\|_{L^2}}{n}\bigg).$$

Thus, substituting the quantities on the claims, and integrating the equation for $\partial_t|P_\epsilon P_+ w_n|^2$ we obtain

$$\partial_t\int |P_\epsilon P_+ w_n|^2dx \leqslant 2c\|V\|_{L^\infty}\|w_n\|_{L^2}^2 + 2c\|\tilde{H}\|_{L^2}\|w_n\|_{L^2} + \frac{c}{n}\|w_n\|_{L^2}^2$$

with $c$ independent of $\epsilon$ and $n$.

Similarly, for $P_-$ we have

$$\partial_t\int |P_\epsilon P_- w_n|^2dx \geqslant -2c\|V\|_{L^\infty}\|w_n\|_{L^2}^2 - 2c\|\tilde{H}\|_{L^2}\|w_n\|_{L^2} - \frac{c}{n}\|w_n\|_{L^2}^2.$$

Now note that since for each fixed $n$ we have

$$\sup_{0\leqslant t\leqslant 1}\|w_n\|_{L^2} < \infty,$$

so there is a $t_n$ such that

$$\|w_n(\cdot,t_n)\|_{L^2}^2 \geqslant \frac{1}{2}\sup_{0\leqslant t\leqslant 1}\|w_n\|_{L^2}^2.$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\sup_{0\leqslant t\leqslant 1}\|w_n\|_{L^2}^2\\
&\qquad\leqslant\|w_n(\cdot,t_n)\|_{L^2}^2\\
&\qquad=\|P_+w_n(\cdot,t_n)\|_{L^2}^2 + \|P_-w_n(\cdot,t_n)\|_{L^2}^2\\
&\qquad=\lim_{\epsilon\to 0}\left(\|P_\epsilon P_+ w_n(\cdot,t_n)\|_{L^2}^2 + \|P_\epsilon P_- w_n(\cdot,t_n)\|_{L^2}^2\right)\\
&\qquad=\lim_{\epsilon\to 0}\bigg(\int_0^{t_n}\partial_s\|P_\epsilon P_+ w_n(\cdot,s)\|_{L^2}^2ds + \|P_\epsilon P_+ w_n(\cdot,0)\|_{L^2}^2\\
&\qquad\quad-\int_{t_n}^1\partial_s\|P_\epsilon P_- w_n(\cdot,s)\|_{L^2}^2ds + \|P_\epsilon P_- w_n(\cdot,1)\|_{L^2}^2\bigg)\\
&\qquad\leqslant 2c\int_0^1\|V\|_{L^\infty}ds\cdot\sup_{0\leqslant t\leqslant 1}\|w_n(\cdot,t)\|_{L^2}^2\\
&\qquad\quad+2c\int_0^1\|H\|_{L^2(e^{2\beta x_1})}ds\cdot\sup_{0\leqslant t\leqslant 1}\|w_n\|_{L^2}^2\\
&\qquad\quad+\frac{c}{n}\sup_{0\leqslant t\leqslant 1}\|w_n\|_{L^2}^2 + \|u(0)\|_{L^2(e^{2\beta x_1})}^2 + \|u(1)\|_{L^2(e^{2\beta x_1})}^2.
\end{aligned}$$

Now taking $n$ large enough such that $c/n < 1/4$ and choosing $\epsilon_0$ such that

$$2c \int_0^1 \|V(\cdot, s)\|_{L^\infty(\mathbb{R}^n)} ds < \frac{1}{8},$$

we obtain from the last inequality

$$\begin{aligned} \frac{1}{16} \sup_{0 \leqslant t \leqslant 1} \|w_n(\cdot, t)\|_{L^2}^2 \leqslant & 32c^2 \left( \int_0^1 \|H\|_{L^2(e^{2\beta x_1})} ds \right)^2 \\ & + \|u(0)\|_{L^2(e^{2\beta x_1})}^2 + \|u(1)\|_{L^2(e^{2\beta x_1})}^2. \end{aligned}$$

Letting $n \to \infty$ we obtain the result.

□

Now that we have proved the lemma, let us see how we can use it to obtain estimates for the gaussian weights.

We proved that there is $C > 0$ such that

$$\int_{R^n} e^{2\beta x_1} |u(x, t)|^2 \leqslant C \left( \int_{R^n} e^{2\beta x_1} |u(x, 0)|^2 dx + \int_{R^n} e^{2\beta x_1} |u(x, 1)|^2 dx \right).$$

Since it doesn't makes any difference to put $x_1$ or any of the $x_i$, with $i = 1, \ldots, n$, we can put

$$\int_{\mathbb{R}^n} e^{2\lambda \cdot x} |u(x, t)|^2 \leqslant C \left( \int_{\mathbb{R}^n} e^{2\lambda \cdot x} |u(x, 0)|^2 dx + \int_{\mathbb{R}^n} e^{2\lambda \cdot x} |u(x, 1)|^2 dx \right),$$

where $\lambda \in \mathbb{R}^n$. Then multiplying the left hand side of the inequality by $e^{-|\lambda|^2}$, integrating in $\lambda$ and applying Fubini, we obtain:

$$\int_{\mathbb{R}^n} e^{-|\lambda|^2} \int_{\mathbb{R}^n} e^{2\lambda \cdot x} |u(x, t)|^2 dx d\lambda = \int_{\mathbb{R}^n} |u(x, t)|^2 \int_{\mathbb{R}^n} e^{-|\lambda|^2 + 2\lambda \cdot x} d\lambda dx = \int_{\mathbb{R}^n} |u(x, t)|^2 e^{|x|^2}.$$

Where we used the formula $\int_{\mathbb{R}^n} e^{-|\lambda|^2 + 2\lambda \cdot x} = C_n e^{|x|^2}$, where $C_n$ is a positive constant.

Doing the same on the right hand side we obtain

$$\|u(x, t) e^{|x|^2}\|_{L^2(\mathbb{R}^n)}^2 \leqslant C \left( \|u(x, 0) e^{|x|^2}\|_{L^2(\mathbb{R}^n)}^2 + \|u(x, 1) e^{|x|^2}\|_{L^2(\mathbb{R}^n)}^2 \right).$$

Actually you can put any positive constant multiplying the $|x|^2$ inside the exponential.

The second lemma we will use is the following:

**Lemma 3.1.2.** *For every positive constants* $\epsilon, \mu, R$ *and a function* $g \in C_0(\mathbb{R}^{n+1})$ *we have the estimate*

$$R\sqrt{\frac{\epsilon}{8\mu}}\|e^{\mu|x+Rt(1-t)e_1|^2-\frac{(1+\epsilon)R^2t(1-t)}{16\mu}}g\|_{L^2(\mathbb{R}^{n+1})}$$
$$\leqslant \|e^{\mu|x+Rt(1-t)e_1|^2-\frac{(1+\epsilon)R^2t(1-t)}{16\mu}}(\partial_t - i\Delta)g\|_{L^2(\mathbb{R}^{n+1})}.$$

*Proof.* Set $\varphi(x,t) = \mu|x+Rt(1-t)e_1|^2 - \frac{(1+\epsilon)R^2t(1-t)}{16\mu}$.

We will obtain the result by means of a typical Carleman estimate for the operator $e^{\varphi}(\partial_t - i\Delta)e^{-\varphi}$ applied to the function $f = e^{\varphi}g$. For this we need to split the operator in its symmetric and skew-symmetric parts.

Observe that

$$\begin{aligned} e^{\varphi}(\partial_t - i\Delta)e^{-\varphi}f =& f\Big[-2\mu x_1 R(1-2t) - 2\mu R^2 t(1-t)(1-2t) + \frac{(1+\epsilon)R^2}{16\mu}(1-2t)\Big] \\ &+ \partial_t f - if\Big[-2n\mu + 4\mu^2|x+Rt(1-t)e_1|^2\Big] \\ &+ 4i\mu(x+Rt(1-t)e_1)\cdot\nabla f - i\Delta f. \end{aligned}$$

Then if we call

$$\mathcal{A} = i\Delta + 4i\mu^2|x+Rt(1-t)e_1|^2$$

and

$$\mathcal{S} = -(4i\mu(x+Rt(1-t)e_1)\cdot\nabla + 2\mu n i) + 2\mu R(1-2t)(x_1 + Rt(1-t)) - \frac{(1+\epsilon)R^2}{16\mu}(1-2t)$$

, the operator will just be $\partial_t - \mathcal{A} - \mathcal{S}$, where $\partial_t - \mathcal{A}$ and $\mathcal{S}$ are the skew-symmetric and symmetric parts, respectively.

In a general setting, in a situation like this we can do the following computation:

$$\begin{aligned} \|\partial_t f - \mathcal{A}f - \mathcal{S}f\|^2_{L^2(\mathbb{R}^{n+1})} &= \|\partial_t f - \mathcal{A}f\|^2_{L^2(\mathbb{R}^{n+1})} + \|\mathcal{S}f\|^2_{L^2(\mathbb{R}^{n+1})} - 2\,\mathrm{Re}\int \mathcal{S}f\overline{(\partial_t f - \mathcal{A}f)} \\ &\geqslant -\int \mathcal{S}f\overline{(\partial_t f - \mathcal{A}f)} - \int \overline{\mathcal{S}f}(\partial_t f - \mathcal{A}f) \\ &= \int (\partial_t - \mathcal{A})\mathcal{S}f\overline{f} - \int \mathcal{S}(\partial_t - \mathcal{A})f\overline{f} \\ &= \int \partial_t \mathcal{S}f\overline{f} + \int(-\mathcal{A}\mathcal{S} + \mathcal{S}\mathcal{A})f\overline{f} \\ &= \int (\partial_t\mathcal{S} + [\mathcal{S},\mathcal{A}])f\overline{f}. \end{aligned}$$

In our case

$$\partial_t \mathcal{S} = -4\mu i R(1-2t)\partial_1 - 4\mu R(x_1 + Rt(1-t)) + 2\mu R^2(1-2t)^2 + \frac{(1+\epsilon)R^2}{8\mu}$$

and

$$[\mathcal{S},\mathcal{A}]f = -8\mu\Delta f + 32\mu^3|x + Rt(1-t)e_1|^2 f - 4\mu Ri(1-2t)\partial_1 f.$$

So we end up with

$$\begin{aligned}(\partial_t\mathcal{S} - [\mathcal{S},\mathcal{A}])f = &- 8\mu\Delta f + 32\mu^3|x + Rt(1-t)e_1|^2 f\\ &- 8\mu Ri(1-2t)\partial_1 f - 4\mu R(x_1 + Rt(1-t))\\ &+ 2\mu R^2(1-2t)^2 + \frac{(1+\epsilon)R^2}{8\mu}.\end{aligned}$$

We assert that this last formula is equal to

$$\begin{gathered}8\mu\sum_{k=2}^{n}\int|\partial_k f|^2 + 8\mu\int|i\partial_1 f - \frac{R(1-2t)}{2}f|^2\\ +32\mu^3\int|x + Rt(1-t)e_1 + \frac{Re_1}{16\mu^2}|^2|f|^2 + \frac{\epsilon}{8\mu}R^2\int|f|^2\end{gathered}$$

which is in turn greater than $\frac{\epsilon}{8\mu}R^2\int|f|^2 = \frac{\epsilon}{8\mu}R^2\|e^{\varphi}g\|^2_{L^2(\mathbb{R}^{n+1})}$, and this gives us the result.

To prove the claim we just need to add up the following formulas

- $32\mu^3|x+Rt(1-t)e_1|^2 - 4\mu R(x_1+Rt(1-t))|f|^2 = 32\mu^3|x+Rt(1-t)e_1 - \frac{Re_1}{16\mu^2}|^2 - \frac{R^2}{8\mu}$
- $\int(-8\mu\Delta f)\bar{f} = 8\mu\sum_{k=1}^{n}\int|\partial_k f|^2$
- $-8i\mu R(1-2t)\partial_1 f\bar{f} + 2\mu R^2(1-2t)f\bar{f} = 8\mu|i\partial_1 f - R\frac{1-2t}{2}f|^2 - 8\mu|\partial_1 f|^2.$

□

Finally we present the proof of the third lemma:

**Lemma 3.1.3.** *Let $u \in C([0,1] : L^2(\mathbb{R}^n))$ be a solution of the Schrödinger equation with potential, such that $\|e^{\gamma|x|^2}u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{\gamma|x|^2}u(1)\|_{L^2(\mathbb{R}^n)} < \infty$ then there exists $N > 0$ such that*

$$\begin{aligned}&\|\sqrt{t(1-t)}e^{\gamma|x|^2}|x|u\|_{L^2(\mathbb{R}^n\times[0,1])} + \|\sqrt{t(1-t)}e^{\gamma|x|^2}\|x\|^2\nabla u\|_{L^2(R^n\times[0,1])}\\ &\leqslant Ne^N\|V\|_{L^2(R^n\times[0,1])}\cdot\sup_{[0,1]}\|e^{\gamma\|x\|^2}u(t)\|_{L^2(R^n\times[0,1])}.\end{aligned}$$

*Proof.* As in the formal proof in Chapter 2 define $f \equiv e^{\varphi}u$, where $u$ is a solution of $\partial_t u = i(\Delta u + Vu)$. Now $f$ satisfies

$$\partial_t f = \mathcal{S}f + \mathcal{A}f + iVf$$

where

$$\mathcal{S} = -i\gamma(4x\nabla \cdot +2n\cdot)$$
$$\mathcal{A} = i(4\gamma^2|x|^2 \cdot +\Delta\cdot).$$

As before we compute the second derivative of $H(t) = \langle f, f\rangle$

$$\begin{aligned} H'(t) &= \langle \partial_t f, f\rangle + \langle f, \partial_t f\rangle \\ &= 2\,\mathrm{Re}\langle \partial_t f, f\rangle \\ &= 2\,\mathrm{Re}\langle \partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle + 2\langle \mathcal{S}f, f\rangle \\ &= 2\,\mathrm{Re}\langle \partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle + 2D(t) \end{aligned}$$

where $D(t) = \langle \mathcal{S}f, f\rangle$.

Then

$$H''(t) = 2\partial_t\,\mathrm{Re}\langle (\partial_t \mathcal{S})f + [\mathcal{S}, \mathcal{A}]f, f\rangle + 2D'(t)$$

but

$$\begin{aligned} D'(t) &= \langle (\partial_t \mathcal{S})f, f\rangle + \langle \mathcal{S}\partial_t f, f\rangle + \langle \mathcal{S}f, \partial_t f\rangle \\ &= \langle (\partial_t \mathcal{S})f, f\rangle + \langle \partial_t f, \mathcal{S}f\rangle + \langle \mathcal{S}f, \partial_t f\rangle \\ &= \langle (\partial_t \mathcal{S})f, f\rangle + 2\,\mathrm{Re}\langle \partial_t f, \mathcal{S}f\rangle \\ &= \langle (\partial_t \mathcal{S})f, f\rangle + 2\,\mathrm{Re}\langle \partial_t f - \mathcal{A}f, \mathcal{S}f\rangle + 2\,\mathrm{Re}\langle \mathcal{A}f, \mathcal{S}f\rangle. \end{aligned}$$

And since $2\,\mathrm{Re}\langle \mathcal{A}f, \mathcal{S}f\rangle = \langle [\mathcal{S}; \mathcal{A}]f, f\rangle$ and by the polarization identity we have

$$2\,\mathrm{Re}\langle \partial_t f - \mathcal{A}f, \mathcal{S}f\rangle = \frac{1}{2}\|\partial_t f - \mathcal{S}f + \mathcal{A}f\|_{L^2}^2 - \frac{1}{2}\|\partial_t f - \mathcal{S}f - \mathcal{A}f\|_{L^2}^2$$

then

$$H''(t) = 2\partial_t\,\mathrm{Re}\langle \partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle + 2\langle (\partial_t S)f + [\mathcal{S}; \mathcal{A}]f, f\rangle +$$

$$+\frac{1}{2}\|\partial_t f - \mathcal{S}f + \mathcal{A}f\|_{L^2}^2 - \frac{1}{2}\|\partial_t f - \mathcal{S}f - \mathcal{A}f\|_{L^2}^2.$$

In particular

$$2\langle(\partial_t \mathcal{S})f + [\mathcal{S};\mathcal{A}]f, f\rangle \leqslant H''(t) - 2\partial_t \operatorname{Re}\langle \partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle + \frac{1}{2}\|\partial_t f - \mathcal{S}f - \mathcal{A}f\|_{L^2}^2.$$

Now we multiply this inequality by $t(1-t)$, integrate in $\int_0^1 dt$ and apply integration by parts. Observe that

$$\begin{aligned}
\int_0^1 t(1-t)H''(t)dt &= t(1-t)H'(t)|_{t=0}^1 - \int_0^1 (1-2t)H'(t) \\
&= -(1-2t)H(t)|_0^1 - 2\int_0^1 H(t)dt \\
&= H(0) + H(1) - 2\int_0^1 H(t)dt \\
&\leqslant H(0) + H(1)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 t(1-t)\partial_t \operatorname{Re}\langle \partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle dt &= t(1-t)\operatorname{Re}\langle \partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle\Big|_0^1 \\
&\quad - \int_0^1 (1-2t)\operatorname{Re}\langle \partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle dt \\
&= -\int_0^1 (1-2t)\operatorname{Re}\langle \partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle dt.
\end{aligned}$$

Then by the previous calculation we get

$$\begin{aligned}
2\langle(\partial_t \mathcal{S})f + [\mathcal{S};\mathcal{A}]f, f\rangle &\leqslant H(0) + H(1) + 2\int_0^1 (1-2t)\operatorname{Re}\langle \partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle \\
&\quad + \int_0^1 t(1-t)\|\partial_t f - \mathcal{S}f - \mathcal{A}f\|^2 dt \\
&\leqslant \|f(0)\|^2 + \|f(1)\|^2 + 2\int_0^1 \|V\|_{L^\infty \mathbb{R}^n}\|f(t)\|^2 dt \\
&\quad + \int_0^1 t(1-t)\|V(t)\|_{L^\infty \mathbb{R}^n}^2 \|f(t)\|^2 dt \\
&\leqslant \|e^{\gamma|x|^2}u(t)\|^2|_{t=0}^1 + 2\int_0^1 \|V\|_{L^\infty(\mathbb{R}^n)}\|e^{\gamma|x|^2}u(t)\|^2 dt \\
&\quad + \int_0^1 t(1-t)\|V(t)\|_{L^\infty(\mathbb{R}^n)}^2 \|e^{\gamma|x|^2}(t)u(t)\|^2 dt
\end{aligned}$$

and by the first technical lemma there is a constant $N > 0$ (depending just on the truncation) such that this quantity is bounded above by

$$N\big(\|e^{\gamma|x|^2}u(0)\|^2 + \|e^{\gamma|x|^2}(t)u(1)\|^2) + \|V\|_{L^\infty(\mathbb{R}^n)}\big).$$

Thus to complete the proof of the theorem it remains to see that

$$2\langle(\partial_t\mathcal{S})f + [\mathcal{S};\mathcal{A}]f, f\rangle \geqslant$$

$$\|\sqrt{t(1-t)}e^{\gamma|x|^2|x|u}\|_{L^2(R^n\times[0,1])} + \|\sqrt{t(1-t)}e^{\gamma\|x\|^2\nabla u}\|_{L^2(R^n\times[0,1])}.$$

Let us first compute explicitly the operator $\partial_t\mathcal{S} + [\mathcal{S};\mathcal{A}]$. Since $\mathcal{S}$ does not depend on $t$ we already know $\partial_t\mathcal{S} = 0$. On the other hand an easy computation shows that

$$\begin{aligned}[\mathcal{S};\mathcal{A}]f &= \gamma[4x\nabla + 2n; 4\gamma^2|x|^2 + \Delta]f \\ &= \gamma[4x\nabla; 4\gamma^2|x|^2]f + \gamma[4x\nabla;\Delta]f \\ &= 32\gamma^3|x|^2 f - 8\gamma\Delta f.\end{aligned}$$

Then

$$\langle\partial_t\mathcal{S}f + [\mathcal{S},\mathcal{A}]f, f\rangle \geqslant 8\gamma\bigg(\int_{\mathbb{R}^n}|\nabla f|^2 + 4\gamma^2\int_{\mathbb{R}^n}|x|^2|f|^2\bigg).$$

We assert the following

**Claim 1.**

$$\int_{R^n}|\nabla f|^2 + 4\gamma^2|x|^2|f|^2 dx = \int_{R^n}e^{2\gamma|x|^2}(|\nabla u|^2 - 2n\gamma|u|^2)dx.$$

**Claim 2.**

$$\int_{R^n}|\nabla f|^2 + 4\gamma^2|x|^2|f|^2 dx \geqslant 2n\gamma\int_{\mathbb{R}^n}e^{2\gamma|x|^2}|u|^2 dx.$$

We continue with the proof of the theorem assuming the claims.

$$
\begin{aligned}
\langle(\partial_t\mathcal{S}+[\mathcal{S},\mathcal{A}])f,f\rangle &\geqslant 8\gamma\Big(\int_{\mathbb{R}^n}|\nabla f|^2+4\gamma^2\int_{\mathbb{R}^n}|x|^2|f|^2\Big)\\
&\geqslant 2\gamma\times\Big[2\Big(\int_{\mathbb{R}^n}|\nabla f|^2+4\gamma^2\int_{\mathbb{R}^n}|x|^2|f|^2\Big)\Big]+16\gamma^2\int_{\mathbb{R}^n}|x|^2|f|^2\\
&\geqslant 2\gamma\Big(\int_{R^n}e^{2\gamma|x|^2}(|\nabla u|^2-2n\gamma|u|^2)+2n\gamma\int_{\mathbb{R}^n}e^{2\gamma|x|^2}|u|^2\Big)\\
&\quad+16\gamma^2\int_{\mathbb{R}^n}e^{2\gamma|x|^2}|x|^2|u|^2\\
&\geqslant 2\gamma\int_{R^n}e^{2\gamma|x|^2}|\nabla u|^2+16\gamma^2\int_{\mathbb{R}^n}e^{2\gamma|x|^2}|x|^2|u|^2\\
&\geqslant 2\gamma\|e^{\gamma|x|^2}\nabla u\|^2_{L^2(\mathbb{R}^n)}+16\gamma^2\|e^{\gamma|x|^2}|x|u\|^2_{L^2(\mathbb{R}^n)}
\end{aligned}
$$

Then multiplying by $t(1-t)$ and integrating in $\int_0^1 dt$ we obtain extactly what we wanted.

□

## 3.2 The Proof

In this section we prove the Theorem 3.0.2

*Proof of Theorem 3.0.2.* We can assume $\gamma > 1/2$, and choose two positive constants $\mu$ and $\epsilon$ such that

$$\frac{(1+\epsilon)^{3/2}}{2(1-\epsilon)}\leqslant\mu\leqslant\frac{\gamma}{1+\epsilon}.$$

Let $\theta_M \in C_0^\infty(\mathbb{R}^n)$ be a function such that

$$\theta_M(x)=\begin{cases}1 & |x|<M\\ 0 & |x|>2M\end{cases}$$

where $M > R$ and $|\theta'_M(x)| \sim 1/M$.

Let $\eta_R \in C_0^\infty([0,1])$ be a real function such that

$$\eta_R(t)=\begin{cases}1 & t\in[1/R,1-1/R]\\ 0 & t\in[0,1/2R]\cup[1-1/2R,1]\end{cases}$$

whith $|\eta'_R(t)| \sim 1/M$.

Define $g(x,t) = u(x,t)\theta_M(x)\eta_R(t)$. This function has compact support but solves another PDE. In fact

$$\partial_t g - i\Delta g = -u\theta_M\eta_R' + 2i\nabla\theta_M\nabla u\eta_R + iu\Delta\theta_M\eta_R - iVu\theta_M\eta_R$$

Let

$$E_1 = -u\theta_M\eta_R'$$

and

$$E_2 = 2i\nabla\theta_M\nabla u\eta_R + iu\Delta\theta_M\eta_R$$

Then the equation for $g$ is

$$\partial_t g - i\Delta g = E_1 + E_2 + iVu\theta_M\eta_R.$$

Because of the truncation we have

$$\operatorname{supp} E_1 = \{(x,t) : |x| < 2M \text{ and } t \in [1/2R, 1/R] \text{ or } t \in [1-1/R, 1-1/2R]\}$$
$$\operatorname{supp} E_2 = \{(x,t) : M \leqslant |x| \leqslant 2M \text{ and } t \in [1/2R, 1-1/2R]\}$$

And the following bounds for the function $\mu|x+Rt(1-t)e_1|^2$ on each of the supports

For every $(x,t)$ on the support of $E_1$ we have

$$\begin{aligned}
\mu|x+Rt(1-t)e_1|^2 &\leqslant \mu\big(|x|^2 + 2Rt(1-t)|x| + R^2t^2(1-t)^2\big)\\
&\leqslant \mu\big(|x|^2 + 2R|x|\frac{1}{R} + R^2\frac{1}{R^2}\big)\\
&\leqslant \mu\big(|x|^2 + 2|x| + 1\big)\\
&\leqslant \mu|x|^2 + \mu\big(\epsilon|x|^2 + \frac{1}{\epsilon}\big) + 1\big)\\
&\leqslant \mu\big(1+\epsilon)|x|^2 + \mu(1+\frac{1}{\epsilon}\big)\\
&\leqslant \gamma|x|^2 + \frac{\gamma}{\epsilon}
\end{aligned}$$

and for every $(x,t)$ on the support of $E_2$

$$\begin{aligned}\mu|x+Rt(1-t)e_1|^2 &\leqslant \mu\big(|x|^2+2R|x|+R^2)\big)\\ &\leqslant \mu\left(|x|^2+\left(\epsilon|x|^2+\frac{R^2}{\epsilon}\right)+R^2\right)\\ &\leqslant \mu(1+\epsilon)|x|^2+\mu\left(1+\frac{1}{\epsilon}\right)R^2\\ &\leqslant \gamma|x|^2+\frac{R^2}{\epsilon}.\end{aligned}$$

Where we have used the inequality $2|x|\sqrt{\epsilon}\frac{1}{\sqrt{\epsilon}} \leqslant \epsilon|x| + \frac{1}{\epsilon}$.

Now, since our function $g$ is compact supported, we can apply the technical lemmas. Call $\varphi = \gamma|x|^2 - \frac{(1+\epsilon)R^2t(1-t)}{16\mu}$. Then we obtain

$$R\sqrt{\frac{\epsilon}{8\mu}}\|e^{\varphi}g\|_{L^2(\mathbb{R}^n\times[0,1])} \leqslant \|e^{\varphi}Vg\|_{L^2(\mathbb{R}^n\times[0,1])} + \|e^{\varphi}E_1\|_{L^2(\mathbb{R}^n\times[0,1])} + \|e^{\varphi}E_2\|_{L^2(\mathbb{R}^n\times[0,1])}.$$

Observe that the term $\|e^{\varphi}Vg\|_{L^2(\mathbb{R}^n\times[0,1])}$ is bounded by $\|V\|_{L^\infty(\mathbb{R}^n\times[0,1])}\|e^{\varphi}g\|_{L^2(\mathbb{R}^n\times[0,1])}$, but we can take $R \leqslant 0$ as large as we want, so we can pass this last term to the left hand side paying by two, and simpliying the inequality to

$$R\sqrt{\frac{\epsilon}{8\mu}}\|e^{\varphi}g\|_{L^2(\mathbb{R}^n\times[0,1])} \leqslant 2\left(\|e^{\varphi}E_1\|_{L^2(\mathbb{R}^n\times[0,1])} + \|e^{\varphi}E_2\|_{L^2(\mathbb{R}^n\times[0,1])}\right)$$

We can also get rid of the term involing $E_2$ because

$$\begin{aligned}\|e^{\varphi}E_2\|_{L^2(\mathbb{R}^n\times[0,1])} &\leqslant e^{\gamma\frac{R^2}{\epsilon}}\|e^{\gamma|x|^2-\frac{(1+\epsilon)R^2t(1-t)}{16\mu}}E_2\|_{L^2(\mathbb{R}^n\times[0,1])}\\ &\leqslant \frac{C_{\gamma,R,\epsilon}}{M}\|e^{\gamma|x|^2-\frac{(1+\epsilon)R^2t(1-t)}{16\mu}}(|u|+|\nabla u|)\|_{L^2(\mathbb{R}^n\times[0,1])}\end{aligned}$$

Then, since everything is compact supported, we can use the technical lemma to know that this last $L^2$ norm is bounded, so the term involving $E_2$ goes to zero when $M$ goes to infinity, so we end up with

$$R\sqrt{\frac{\epsilon}{8\mu}}\|e^{\varphi}g\|_{L^2(\mathbb{R}^n\times[0,1])} \leqslant 2\|e^{\varphi}E_1\|_{L^2(\mathbb{R}^n\times[0,1])}.$$

Now since $R$ is large enough we can assume $g = u$ on $B(\frac{\epsilon(1-\epsilon)^2R}{4}) \times [\frac{1-\epsilon}{2}, \frac{1+\epsilon}{2}]$. Also on this domain

$$\begin{aligned}\varphi &= \gamma|x|^2 - \frac{(1+\epsilon)R^2 t(1-t)}{16\mu} \\ &\geqslant \mu\left(\frac{R(1-\epsilon)^2}{4} - \frac{\epsilon(1-\epsilon)^2 R}{4}\right)^2 - \frac{(1+\epsilon)R^2(1+\epsilon)^2}{4\cdot 16\mu} \\ &\geqslant \frac{R^2}{64\mu}\big(16\mu^2(1-\epsilon)^6 - (1+\epsilon)^3\big)\end{aligned}$$

and we know by [] that this quantity is strictly positive.

On the domain above we have

$$\begin{aligned}&R\sqrt{\frac{\epsilon}{8\mu}}e^{\frac{R^2}{64\mu}\left(16\mu^2(1-\epsilon)^6-(1+\epsilon)^3\right)}\|u\|_{L^2(B(\frac{\epsilon(1-\epsilon)^2R}{4})\times[\frac{1-\epsilon}{2},\frac{1+\epsilon}{2}])} \\ &\qquad\leqslant \|e^{\varphi}E_1\|_{L^2(\mathbb{R}^n\times(1/R,1-1/R))} \\ &\qquad\leqslant \|e^{\gamma|x|^2+\frac{\gamma}{\epsilon}}Ru\|_{L^2(\mathbb{R}^n\times(1/R,1-1/R))} \\ &\qquad\leqslant Re^{\gamma/\epsilon}\left\{\|e^{\gamma|x|^2}|x|u\|_{L^2(\mathbb{R}^n\times(1/R,1-1/R))} + e^{\gamma}\|u\|_{L^2(\mathbb{R}^n\times(1/R,1-1/R))}\right\}.\end{aligned}$$

Where we used

$$|e^{\gamma|x|^2}u| \leqslant \begin{cases}|e^{\gamma|x|^2}u| & \text{if } |x|\geqslant 1\\ |e^{\gamma}u| & \text{if } |x|\leqslant 1\end{cases}.$$

Also observe that on this domain both $t$ and $1-t$ are greater than $1/2R$. Then $t > 2R$ and $1-t > 2R$. We use this and the formula above to conclude

$$\begin{aligned}&R\sqrt{\frac{\epsilon}{8\mu}}e^{\frac{R^2}{64\mu}\left(16\mu^2(1-\epsilon)^6-(1+\epsilon)^3\right)}\|u\|_{L^2(B(\frac{\epsilon(1-\epsilon)^2R}{4})\times[\frac{1-\epsilon}{2},\frac{1+\epsilon}{2}])} \\ &\leqslant 2R^2e^{\gamma/\epsilon}\|\sqrt{(}t(1-t))e^{\gamma|x|^2}u\|_{L^2(\mathbb{R}^n\times[0,1])} + Re^{\gamma+\gamma/\epsilon}\|u\|_{L^2(\mathbb{R}^n\times[0,1])}.\end{aligned}$$

Now observe that the right hand side grows like and exponential in $R$ while the left hand side grows like $R^2$. Since the formula is valid for every choice of the constants, we can conclude that the nondecreasing (as a function on $R$) quantity $\|u\|_{L^2(B(\frac{\epsilon(1-\epsilon)^2R}{4})\times[\frac{1-\epsilon}{2},\frac{1+\epsilon}{2}])}$ is identicaly zero, and then $u\equiv 0$.

□

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