MULTIPLICITY ONE AND STRICTLY STABLE ALLEN-CAHN MINIMAL HYPERSURFACES

MARCO A. M. GUARACO, FERNANDO C. MARQUES, ANDRE NEVES

Abstract. Allen-Cahn minimal hypersurfaces are limits of nodal sets of solutions to the Allen-Cahn equation. The energy of the solutions concentrates on each connected component of the hypersurface as an integer multiple of its area. Multiplicity greater than one (interface foliation) is known to occur for certain hypersurfaces with unstable double cover.

We show that generically these are the only possible examples of interface foliation. More precisely, we prove that Allen-Cahn minimal hypersurfaces with strictly stable double cover only occur with multiplicity one. In addition, based on recent curvature estimates, we establish the uniqueness of multiplicity one solutions converging to, possibly unstable, non-degenerate hypersurfaces. All results hold in arbitrary dimensions.

1. Introduction

Since Modica-Mortola ([33, 32]) and De Giorgi ([9]), mathematicians expect to find strong connections between minimal hypersurfaces and functions $u \in C^\infty(M)$ satisfying the semilinear elliptic equation

$$\varepsilon^2 \Delta u - W'(u) = 0. \tag{1}$$

Here $\varepsilon > 0$, (M^n, g) is a closed Riemannian manifold, Δ is the Laplace-Beltrami operator of M, $u \in C^2(M)$ and $W(t) = (1-t^2)^2/4$.

Equation (1) is kwon as the (stationary) Allen-Cahn equation ([6]) and its solutions are critical points of the energy

$$E_\varepsilon(u) = \int_M \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}. \tag{2}$$

A special feature of its solutions is that, in a broad sense, their nodal set $\{u = 0\}$ becomes a minimal hypersurface as $\varepsilon \to 0$. This feature goes both ways, being also possible to construct solutions with nodal set converging to suitable minimal hypersurfaces.

In this work, we study the phenomenon of interface foliation for solutions of (1) in arbitrary dimensions. The main result is that no interface foliation occurs around hypersurfaces with strictly stable double cover (Theorem 2.1). The hypothesis of having stable double cover is necessary, as shown in Example 2 below. In addition, we obtain the uniqueness of generic multiplicity one solutions (Theorem 2.2).

Our interest in the subject comes from its connection with the multiplicity problem for min-max constructions of minimal hypersurfaces, which we briefly describe in the next subsection.

1.1. **Multiplicity and min-max minimal hypersurfaces.** The min-max approach for variational problems was first devised by Birkhoff ([4]) in the context of the construction of a closed geodesic. Later, Almgren ([2]) developed the theory of varifolds in order to generalize Birkhoff's ideas to higher dimensions. These efforts culminated in the work of J. Pitts ([38]), who exploited the regularity theory developed by Schoen-Simon-Yau ([41]) to show that a varifold obtained through a one parameter min-max is indeed an embedded hypersurface, when $3 \leq n \leq 6$. The problem of multiplicity already appears in this construction. The hypersurface obtained by Pitt's might be the limit of hypersurfaces that folded into themselves.

Shortly after Pitt's work, Schoen-Simon [40] developed a regularity theory for dimensions $n \geq 7$ that also applied to the min-max construction of Pitts. Summarizing their results, the Almgren-Pitts-Schoen-Simon Theorem guarantees, on an arbitrary closed manifold, the existence of a minimal hypersurface embedded outside of a set of dimension at most $n-8$. This result let Yau ([47]) to conjecture the existence of infinitely many immersed minimal hypersurfaces on an arbitrary closed manifold.

A few years ago, the last two authors proposed a program towards the solution of Yau's conjecture. In ([29]), they developed a technical framework allowing them to extend Almgren-Pitt's ideas to higher parameter min-max families. As a result, they concluded existence of infinitely many minimal hypersurfaces in manifolds with positive Ricci curvature, ([29]). This led to the definition of a non-linear spectrum for the area functional which, together with Liokumovich ([24]), they showed it satisfies a Weyl law. In joint work with Irie ([23]), they obtained the density of minimal hypersurfaces for generic metrics as a consequence of the Weyl Law. Which also led, in joint work with Song ([31]), to the proof of the existence of a equidistributed family of minimal hypersurfaces for generic metrics, solving Yau's conjecture for generic metrics. The general case of Yau's conjecture was later solved by Song ([42]). In this elegant work, Song localized the methods from [29] to prove the existence of infinitely many minimal hypersurfaces on a domain bounded by stable hypersurfaces. As in Pitt's work, it is not known (even for bumpy metrics) if it is possible to rule out high multiplicity directly in any of these constructions.

In a series of papers ([26, 27, 28, 34]), the last two authors also presented a program to obtain a Morse-theoretic description of the set of minimal hypersurfaces for generic metrics. More precisely, they conjectured:

Morse Index Conjecture. *For a generic metric g on M^n, $3 \leq n \leq 7$, there exists a sequence $\{\Sigma_k\}$ of smooth, embedded, two-sided, closed minimal hypersurfaces such that:* $\operatorname{index}(\Sigma_k) = k$ *and* $C^{-1}k^{1/n} \leq \operatorname{area}(\Sigma) \leq Ck^{1/n}$ *for some $C > 0$.*

The proposed program to prove this conjecture was based on three main components: the use of min-max constructions over multiparameter sweepouts to obtain existence results, the characterization of the Morse index of min-max minimal hypersurfaces under the multiplicity one assumption, and a proof of the Multiplicity One Conjecture:

Multiplicity One Conjecture. *For generic metrics on M^n, $3 \leq n \leq 7$, any component of a closed, minimal hypersurface obtained by min-max methods is two-sided and has multiplicity one.*

In a novel work, X. Zhou [48] used a regularization of the area functional (developed by him and Zhu in [49]) to prove the Multiplicity One Conjecture. Finally, the characterization of the Morse index of min-max minimal hypersurfaces under the multiplicity one assumption was obtained by the last two authors in [30], completing the Morse-theoretic program they proposed for the area functional.

1.2. **Min-max Allen-Cahn minimal hypersurfaces.** In his PhD thesis ([19]), the first author proposed a different technical framework for the min-max construction of minimal hypersurfaces. The basic idea is to first use min-max methods to construct solutions of the Allen-Cahn equation having good Morse theoretical properties. Once one constructs solutions, the problem of the convergence of the nodal set towards a minimal hypersurface can be studied separately. In [19], the regularity of the limit set is derived from the assumption of bounded Morse index of the solutions, building on the stable case previously handled by Tonegawa-Wickramasekera in [44]. In this way, the first author was able to obtain a new proof of Almgren-Pitts-Schoen-Simon's Theorem. Later, these ideas were developed further by the first author together with Gaspar [16]. In [16], an Allen-Cahn spectrum is defined which is analogous to the volume spectrum from [29] and [24]. It also known that the index of the limit hypersurface is bounded by the index of the solutions. This was shown by Hiesmayr [21] for two-sided limits, and by Gaspar [15] in the general case. Later, together with Gaspar [17], the first author showed that after suitable modifications, the Allen-Cahn spectrum can replace the volume spectrum in the density [23] and equidistribution [31] arguments.

A stronger regularity theory for stable solutions was subsequently developed for dimension $n = 3$ by Chodosh-Mantoulidis [8]. Their work is based on Ambrosio-Cabré's characterization of entire stable solutions in $\mathbb{R}^3$ [3], as well as on improvements of recent regularity estimates for the nodal set of solutions, obtained by Wang-Wei [46]. Using these estimates they showed that, if multiplicity higher than one occurs, then the limit hypersurface admits a positive Jacobi vector field. In addition, in [8] they showed the Morse index is lower semicontinuous for multiplicity one solutions. This is the Allen-Cahn analogue of the Morse Index Conjecture for $n = 3$. When this regularity theory is applied to the min-max constructions of the first author and Gaspar [16], one obtains an Allen-Cahn analogue to the Multiplicity One Conjecture for $n = 3$.

Finally, we note that parallel ideas to [19] and [16], have been developed for dimension two by Mantoulidis [25], and in the codimension two setting by Stern [43] and Pigati-Stern [37]. In [37] they prove the existence of codimension two integer rectifiable varifolds obtained as the limit interface of min-max critical points of a complex valued functional. As a result, they obtain a new proof of the existence of stationary integral $(n-2)$-varifolds in an arbitrary closed Riemannian manifold. Although this result was already known to Almgren [1], the proof presented in [37] is considerably less technically involved, opening possibilities for new developments in the field.

1.3. **Interface foliation on Allen-Cahn minimal hypersurfaces.** Existence of solutions with nodal set near certain minimal hypersurfaces, has been proven for the multiplicity one case by [36, 35] and more recently [7]. Interface foliation was studied in [12] for non-degenerate separating hypersurfaces Σ satisfying the second order condition $|A|^2 + \mathrm{Ric}(\nu, \nu) > 0$, where A and ν are the second fundamental form and normal vector of the hypersurface. This inequality implies that the Jacobi operator of Σ is unstable. As mentioned in [11], this construction works also when Σ is non-separating. In this case, the multiplicity of the interface must be even.

In [12], it is shown that for any hypersurface satisfying the condition above and any $k \in \mathbb{N}$, there exists a sequence of $\varepsilon \to 0$ and solutions of the Allen-Cahn, whose nodal set is k small graphs accumulating on Σ.

For convenience of the reader, we include a sketch of a different construction which works for some minimal hypersurfaces of $\mathbb{S}^n$, $\mathbb{RP}^n$ and a torus. In these geometries, solutions exist for ε sufficiently small, rather than for just a subsequence going to zero. Example 1 is a rotationally symmetric *antipodal* solution on $\mathbb{S}^n$ converging with multiplicity two towards an equator. Passing to the quotient on $\mathbb{RP}^n$, we obtain in Example 2 a multiplicity two solution converging towards $\mathbb{RP}^{n-1} \subset \mathbb{RP}^n$, which is a strictly stable minimal surface. This example shows the hypothesis of having a stable double cover is necessary in our main theorem. Finally, in Example 3 we discuss the case of a rotationally symmetric torus with having only two minimal slices.

Example 1. Let $\mathbb{S}^n = \{\|x\| = 1 : x \in \mathbb{R}^{n+1}\}$. Given $\tau \in (0,1)$, we partition $\mathbb{S}^n$ into the sets $D_\tau^+ = \mathbb{S}^n \cap \{x_{n+1} \geq \tau\}$, $A_\tau = \mathbb{S}^n \cap \{|x_{n+1}| \leq \tau\}$ and $D_\tau^- = \mathbb{S}^n \cap \{x_{n+1} \leq -\tau\}$. Let Ω be any of these domains. We can minimize the energy E_ε on $W_0^{1,2}(\Omega)$. This produces a solution with zero Dirichlet contidion at $\partial\Omega$. A simple application of the results from [5] (see also [16]), show that there is $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, a minimizer solution will have a sign in A_τ iff τ is far enough from zero, and it will have a sign in $D_\tau^\pm$ iff τ is close enough to one. Moreover, the minimizer is unique [5] and therefore rotationally symmetric. By continuity, one can show that for each $\varepsilon \in (0, \varepsilon_0)$, there is a τ_0 such that the outer derivative of these solutions coincide on $\partial D_\tau^- \cup D_\tau^+ = \partial A_\tau$. If we choose the positive minimizer for A_{τ_0} and the negative for $D_{\tau_0}^\pm$ we obtain a solution with nodal set equal to two parallels equidistant from the equator. This implies a lower bound on the energy (see [16]). For some subsequence of $\varepsilon \to 0$, we can assume the parallels either converge to the equator, or accumulate on two parallels. In the latter, a simple argument by competitors show the convergence happens with multiplicity one. Since the energy is bounded along the sequence, we can apply the results of Hutchinson-Tonegawa [22] to reach a contradiction.

By their result, the energy must accumulate both near the zero level set and near a stationary varifold. Since two parallels are not stationary, we reach a contradiction, meaning the nodal set must converge to the equator.

Example 2. The solutions constructed above are also even with respect to the antipodal map $x \to -x$. In particular, they project well to $\mathbb{RP}^n$ giving solutions accumulating with multiplicity one on a copy of $\mathbb{RP}^{n-1}$. We notice that Example 1 can be reproduced on a bumpy, rotationally and antipodally symmetric ellipsoid. In this case, projecting into the quotient gives a non-orientable strictly stable Allen-Cahn minimal hypersurface with unstable double cover.

Example 3. Let $\mathbb{T}^n$ be a rotationally symmetric torus having only two minimal vertical slices: T_1, which is unstable, and T_0, which is stable. The results of this paper (Theorem 2.1) show that T_0 is not an Allen-Cahn minimal hypersurface. Using this fact, the construction from Example 1 can be adapted to this case in order to show that, for every ε small enough, there is a solution with nodal set two normal graphs over T_1. Therefore, T_1 is an Allen-Cahn minimal hypersurface with multiplicity two. In the non-degenerate case this last statement follows from [12]. In addition, if the union $T_0 \cup T_1$ is non-degenerate then it is an Allen-Cahn minimal hypersurface by [36], since it is separating.

2. Main results and organization

In this paper, we study the accumulation of the nodal set of solutions to (1) around strictly stable minimal hypersurfaces in arbitrary dimensions.

Theorem 2.1. *Let u_k be a sequence of solutions to* (1) *with $\varepsilon = \varepsilon_k \to 0$, whose nodal set $\{u_k = 0\}$ converges in the Hausdorff distance to a separating embedded minimal hypersurface Γ. If Γ is strictly stable, then the convergence is as a smooth graph with multiplicity one.*

By the Morse index estimates of multiplicity one solutions from [8] (see Theorem, [8]), it follows that u_ε is also strictly stable as a critical point of E_ε, for ε small enough. This implies that its energy cannot realize the supremum of any multidimensional minimizing family of sweepouts, e.g. the ones constructed in [16]. In other words, u_ε is not produced by min-max methods.

The multiplicity one from Theorem 2.1 is proven in Theorem 10.1. We do so by a sliding argument after constructing suitable barriers. Then, smoothness of the convergence follows directly from recent curvature estimates as in [45, 46, 8]. In fact, we obtain the following uniqueness result under the multiplicity one assumption.

Theorem 2.2. *Let u_k be a sequence of solutions to* (1) *with $\varepsilon = \varepsilon_k \to 0$, whose nodal set $\{u_k = 0\}$ converges in the Hausdorff distance to an embedded minimal hypersurface Γ. If Γ is non-degenerate and the convergence is with multiplicity one, then for k large enough, u is the one-sheet solution adapted to Γ constructed by Pacard in* [35].

Theorem 2.2 is proven in Section 12, Corollary 12.10. We notice that our computations are done in Fermi coordinates over the limit interface Γ, rather than with respect of the nodal set $\{u = 0\}$ as in [46, 8]. We notice that this choice of coordinates seems to be possible only because we are working on top of the estimates from [46] together with a non-degeneracy assumption. In addition, these of set coordinates have the advantage of being the ones used in the construction of solutions from [35]. This allow us to conclude uniqueness of solutions with multiplicity one from the improved estimates (see Corollary 12.10).

Organization. Section 3, summarizes most of the notation used along the paper. Section 4, presents the fundamental estimates for the canonical solutions as well as its cutoffs. Section 5, discusses Fermi coordinates. Section 6, collects the fundamental injectivity results for the linearized equation. Section 7, describes elementary properties of subsolutions to the Allen-Cahn equation. Section 8, collects standard elliptic estimates suited to our notation. Section 9, describes known characterizations of entire solutions. Section 10, contains the proof of the multiplicity one in Theorem 2.1. Section 11, summarizes curvatures estimates for the multiplicity one case following [45] and [46]. Finally, Section 12, finishes the proof of Theorem 2.1 and 2.2.

3. Notation

We use the big O and little o notation with respect to the variable ε. Let Ω be an open region of a Riemannian manifold M and $\varepsilon_k \to 0$, $k \in \mathbb{N}$. From now on, we omit the reference to the index of the sequence and write $\varepsilon = \varepsilon_k$. In addition, when working with a sequence of functions f_{ε_k}, we also omit the reference to ε, setting $f = f_{\varepsilon_k}$.

Definition 3.1. Given a sequence of functions $f : \Omega \to \mathbb{R}$, we say that

- $f = O(1)$, if $\limsup_{\varepsilon\to 0} \|f\|_{L^\infty(\Omega)} < \infty$,
- $f = o(1)$, if $\lim_{\varepsilon\to 0} \|f\|_{L^\infty(\Omega)} = 0$,
- $f = O(g)$, if $|f| = O(1) \times |g|$, and
- $f = o(g)$, if $|f| = o(1) \times |g|$.

Additionally, we make the following convention: a function $f \in C^\infty(\Omega)$ is said to be of class $o(\varepsilon^{\mathbb{N}})$ in Ω, if all of its derivatives and integrals on the set Ω, decay faster than polynomials on ε, i.e.

- $f = o(\varepsilon^{\mathbb{N}})$, if $\|\nabla^k f\|_{L^p(\Omega)} = o(\varepsilon^m)$, for any fixed values of $k, m \in \{0, 1, 2, \dots\}$ and $p \in [1, \infty]$.

Definition 3.2. Given an open region on Riemannian manifold (Ω, g), $k \in \mathbb{N}$, $\alpha \in [0, 1)$ and $\varepsilon > 0$, we denote by $C^{k,\alpha}_\varepsilon(\Omega)$ the Hölder space $C^{k,\alpha}(\Omega)$ endowed with the (rescaled) norm of $C^{k,\alpha}(\Omega, \varepsilon^{-2} g)$. In other words, given $f \in C^{k,\alpha}(\Omega)$, we define

$$\|f\|_{C^{k,\alpha}_\varepsilon(\Omega)} = |f|_{C^0(\Omega)} + \varepsilon|\nabla f|_{C^0(\Omega)} + \cdots + \varepsilon^k |\nabla^k f|_{C^0(\Omega)} + \varepsilon^{k+\alpha}[|\nabla^k f|]_\alpha,$$

where $[f]_\alpha := \sup_{x,y\in\Omega} \frac{|f(x)-f(y)|}{\mathrm{dist}_\Omega(x,y)^\alpha}$.

3.1. **Index of notations.**

$o(\varepsilon^{\mathbb{N}})$	(see Definition 3.1).
$C^{k,\alpha}_{\varepsilon}$	(see Definition 3.2).
W	denotes the canonical potential $W(x) = \frac{(1-x^2)^2}{4}$.
$Q(u)$	is the Allen-Cahn operator $\varepsilon^2\Delta(u) - W'(u)$.
ψ	denotes the one dimensional solution (see equation (3))
σ_0	is the energy constant $\int_{-1}^{1}\sqrt{W(s)/2}ds$.
σ_1	is the constant $\int_{\mathbb{R}}\psi'$.
σ_2	is the constant $\int_{\mathbb{R}}(\psi')^2$.
$B_R(p)$	denotes the ball of radius R centered at $p \in M$.
sgn	denotes the sign function $\text{sgn} : \mathbb{R} \to \{-1, 0, 1\}$.
$\Gamma(f)$	represents the normal graph of f over a two-sided hypersurface Γ.
$N(h)$	denotes the tubular neighborhood of height h.
ℓ	denotes the 1-D linearized operator $\varepsilon^2\partial_t^2 - W''(\psi(t/\varepsilon)$.
ℓ_0	denotes the 1-D approximate linearized operator $\varepsilon^2\partial_t^2 - W''(\omega)$.
L_0	denotes $\varepsilon^2\Delta^2 - W''(\psi(t/\varepsilon))$.
L	denotes the approximated linearized operator $\varepsilon^2\Delta^2 - W''(\omega)$.
ω	denotes the cutoff of the canonical solution

4. Properties of the canonical solution and its cut-off

We denote by $\psi : \mathbb{R} \to (-1, 1)$ the one dimensional canonical solution, i.e. the unique entire solution to

$$\begin{cases} \psi'' - W'(\psi) = 0 \\ \psi(0) = 0 \\ \psi' > 0, \end{cases}$$

where $W(u) = (1-u^2)^2/4$. It is well known that $\psi(t) = \tanh(t/\sqrt{2})$. By direct differentiation one verifies that $\psi - \text{sgn}$ decays exponentially at infinity together with all of its derivatives. More precisely, there are constants $\sigma > 0$ and c_k, for $k \in \mathbb{N}$, such that $|(\psi - \text{sgn})(t)| \leq c_0 e^{-\sigma|t|}$ and $|(\partial^k\psi)(t)| \leq c_k e^{-\sigma|t|}$.

Often, we will work with the rescaled functions $\psi(t/\varepsilon)$ that satisfies

$$\varepsilon^2(\psi(t/\varepsilon))'' - W'(\psi(t/\varepsilon)) = 0. \tag{3}$$

and the estimates

$$\begin{aligned} |\psi(t/\varepsilon) - \text{sgn}(t)| &\leq c_0 e^{-\sigma|t|/\varepsilon} \\ |(\partial^k\psi)(t/\varepsilon)| &\leq c_k e^{-\sigma|t|/\varepsilon}. \end{aligned} \tag{4}$$

4.1. **A cutoff of the canonical solution.** In later sections, we will use ψ as a model to construct approximate solutions adapted to small tubular neighborhoods of smooth hypersurfaces. For this purpose, we cutoff $\psi(t/\varepsilon)$ on small regions of order greater than $O(\varepsilon)$.

For every $\varepsilon > 0$, we define the cutoff $\omega : \mathbb{R} \to \mathbb{R}$ as

$$\begin{aligned} \omega(t) &= \psi(t/\varepsilon)\chi + (1 - \chi(t))\,\text{sgn}(t) \\ &= \psi(t/\varepsilon) + (\psi(t/\varepsilon) - \text{sgn}(t))(\chi(t) - 1). \end{aligned} \tag{5}$$

where $\chi(t)=\rho(2\varepsilon^{-\delta}|t|-1)$ and $\rho:\mathbb{R}\to\mathbb{R}$ is a smooth bump function satisfying $\rho\geq 0$, $\rho(t)=1$ for $t\leq 0$ and $\rho(t)=0$ for $t\geq 1$.

It follows that

$$\chi(t)=\begin{cases}1 & \text{for } |t|\leq \varepsilon^{\delta}/2\\ 0 & \text{for } |t|\geq \varepsilon^{\delta}\end{cases}$$

and all the derivatives of χ are supported on $\varepsilon^{\delta}/2\leq|t|\leq\varepsilon^{\delta}$ and bounded by rational functions of ε. Namely, for all $k\in\mathbb{N}$, we can assume that

$$|\partial^k\chi|\leq c_k\varepsilon^{-k\delta} \text{ on } [\varepsilon^{\delta}/2,\varepsilon^{\delta}], \tag{6}$$

for some $c_k\in\mathbb{R}$, and $\partial^k\chi=0$, otherwise.

The following are simple consequences of direct differentiation of (5), together with the estimates from (4) and (6).

Lemma 4.1.

(1) $(\psi(t/\varepsilon)-\operatorname{sgn}(t))(\chi(t)-1)=o(\varepsilon^{\mathbb{N}})$.

(2) $\varepsilon^k\partial^k\omega(t)=(\partial^k\psi)(t/\varepsilon)+o(\varepsilon^{\mathbb{N}})$, *for* $k\geq 0$.

(3) $|\omega(t)-\operatorname{sgn}(t)|+|\varepsilon^k\partial^k\omega(t)|=O(e^{|t|/\varepsilon})$.

(4) $\left|\int_{\mathbb{R}} f(\omega-\operatorname{sgn})\right|+\varepsilon^k\left|\int_{\mathbb{R}} f\partial^k\omega\right|=O(\varepsilon)\|f\|_{L^\infty(\mathbb{R})}$.

(5) *If* $\|f\|_{L^\infty([-R,R])}=o(1)$, *for all* $R=O(\varepsilon)$, *then (4) improves to* $\left|\int_{\mathbb{R}} f(\omega-\operatorname{sgn})\right|+\varepsilon^k\left|\int_{\mathbb{R}} f\partial^k\omega\right|=o(\varepsilon)$.

(6) *If* $f=|t|^p$, *then (4) improves to* $\int_{\mathbb{R}}|t|^p|\omega-\operatorname{sgn}|+\varepsilon^k\int_{\mathbb{R}}|t|^p|\partial^k\omega|=O(\varepsilon^{1+p})$, *for all* $p,k\in\mathbb{N}$.

(7) $\int_{\mathbb{R}}\partial^k\omega\partial^m\omega=\begin{cases}c_{k,m}\cdot\varepsilon^{1-k-m}+o(\varepsilon^{\mathbb{N}}) & \textit{if } k \textit{ and } m \textit{ have the same parity}\\ 0 & \textit{otherwise}\end{cases}$
where $c_{k,m}\in\mathbb{R}$ *are universal constants and for* $k=m=1$, $\sigma_2=c_{1,1}>0$.

4.2. **The linearized equation.** We denote by ℓ_0 the linearized Allen-Cahn operator, i.e.

$$\ell_0(\varphi)=\varepsilon^2\varphi''-W''(\psi(t/\varepsilon))\varphi.$$

Direct differentiation of (3), shows that $\psi'(t/\varepsilon)$ is a positive function on the kernel of ℓ_0, i.e. $\ell_0(\psi'(t/\varepsilon))=0$. It is a well known fact, that $\ker\ell_0$ is simple and there is a spectral gap for functions in $(\ker\ell_0)^\perp$. The following result is proven in [35] (see equation (3.15) after Lemma 3.6 in [35]).

Lemma 4.2. *The function* $\psi'(t/\varepsilon)$ *generates* $\ker\ell_0$. *Moreover, there exists* $\gamma>0$ *such that*

$$\gamma\int_{\mathbb{R}}\phi^2\leq-\int_{\mathbb{R}}\phi\ell_0(\phi),$$

$\forall \phi \in (\ker \ell_0)^{\perp}$, *i.e.* $\int_{\mathbb{R}} \phi(t)\psi'(t/\varepsilon) = 0$.

The linearized equation also approximates well by means of the cut-off.

Definition 4.3. We define the approximate one dimensional operator as $\ell(\varphi) = \varepsilon^2\phi'' - W''(\omega)\phi$.

Remark. Notice that $\omega(t)^m = \psi(t/\varepsilon)^m + o(\varepsilon^{\mathbb{N}})$. Since $W''(t) = 3t^2 - 1$ is a polynomial, we obtain $|\ell(\varphi) - \ell_0(\varphi)| = |W''(\psi(\cdot/\varepsilon)) - W''(\omega)||\varphi| = o(\varepsilon^{\mathbb{N}})\|\varphi\|_{L^\infty}$.

Lemma 4.4. *Let* $\varphi \in C^2(\mathbb{R})$. *Then,*

$$\frac{\gamma}{2}\int_{\mathbb{R}} \varphi^2 \leq -\int_{\mathbb{R}} \varphi\ell(\varphi) + O(\varepsilon)\Big[\int_{\mathbb{R}} \varphi\omega'\Big]^2 + o(\varepsilon^{\mathbb{N}})\|\varphi\|_{L^\infty(\mathbb{R})}.$$

Proof. Define $\varphi^{\perp}$ by the formula

$$\varphi(t) = \left(\frac{\int_{\mathbb{R}} \varphi(s)\psi'(s/\varepsilon)ds}{\int_{\mathbb{R}} \psi'(s/\varepsilon)^2 ds}\right)\psi'(t/\varepsilon) + \varphi^{\perp}(t)$$

and notice that $\varphi^{\perp} \in (\ker \ell_0)^{\perp}$.

Since $\ell_0(\psi'(t/\varepsilon)) = 0$ and ℓ_0 is self-adjoint,

$$\begin{aligned}\gamma\int_{\mathbb{R}} (\varphi^{\perp})^2 &\leq -\int_{\mathbb{R}} \varphi^{\perp}\ell_0(\varphi^{\perp})\\ &= -\int_{\mathbb{R}} \varphi\ell_0(\varphi)\\ &= -\int_{\mathbb{R}} \varphi\ell(\varphi) + o(\varepsilon^{\mathbb{N}})\|\varphi\|_{L^\infty(\mathbb{R})}\end{aligned}$$

Finally, since $\int_{\mathbb{R}} \psi'(t/\varepsilon)^2 = O(\varepsilon)$ we have,

$$\begin{aligned}\int_{\mathbb{R}} |\varphi - \varphi^{\perp}|^2 &= O(\varepsilon^{-1})\Big(\int_{\mathbb{R}} \varphi(t)\psi'(t/\varepsilon)dt\Big)^2\\ &= O(\varepsilon)\Big(\int_{\mathbb{R}} \varphi[\omega' + o(\varepsilon^{\mathbb{N}})]dt\Big)^2.\end{aligned}$$

The result follows from combining both estimates and the properties of $o(\varepsilon^{\mathbb{N}})$ functions. $\square$

5. Fermi Coordinates

Let $\Gamma \subset M$ be a *separating* embedded hypersurface. We will often work using *Fermi coordinates* over Γ, i.e. given a choice of normal frame ∂ on Γ, the coordinates are given by the diffeomorphism $\mathcal{F} : \Gamma \times (-\tau, \tau) \to M$,

$$\mathcal{F}(x, t) = \mathrm{Exp}(x, t\partial(x)),$$

for some small $\tau > 0$ fixed.

For the convenience of the reader, we summarize our notation and several well known facts about Fermi coordinates, in the list below.

- $N(s) = \mathcal{F}(\Gamma \times (-s, s))$ denotes the tubular neighborhood of height $s \in (0, \tau)$.
- From now on, given a function $G : N(s) \to \mathbb{R}$ we will abuse notation and also denote $G \circ \mathcal{F}$ as G.
- $G'(x, t) = (\partial_t G)(x, t)$ denotes the normal derivate of $G \in C^\infty(N(\tau))$ at the point $\mathcal{F}(x, t)$.
- $\Gamma(f) = \{\mathcal{F}(x, f(x)) : x \in \Gamma\}$ denotes the normal graph of $f : \Gamma \to (-\tau, \tau)$.
- ∇_t is the gradient operator of $\Gamma(t)$ with respect to the metric inherited from M.
- Δ_t is the Laplace-Beltrami operator of $\Gamma(t)$ with respect to the metric inherited from M.
- $H_t(x) = H(x, t)$ is the mean curvature of $\Gamma(t)$ at the point $\mathcal{F}(x, t)$ (in the direction of ∂_t). We abbreviate $H = H(\cdot, 0)$, $H_0' = H'(\cdot, 0)$ and $H_0'' = H''(\cdot, 0)$.
- $J = \Delta_0 + H_0'$ is the Jacobi operator of the hypersurface Γ, i.e. the second derivative of the area element in the direction of ∂_t.
- The ambient Laplace-Beltrami operator decomposes through the well-known formula
$$\Delta_g = \Delta_t + \partial_t^2 - H_t \partial_t.$$
- Given a coordinate system ∂_i on Γ we have $(\Delta_t v)(x) = a_{ij}(x, t)(\partial_{ij} v)(x, t) + b_i(x, t)(\partial_i v)(x, t)$ and $(\nabla_t v)(x) = c_i(x, t)(\partial_i v)(x, t)$, for a_{ij}, b_i and c_i, smooth functions on $\Gamma(-\tau, \tau)$.

We record now the following estimates

Lemma 5.1. *Let* $G \in C^{0,\alpha}(N(\tau))$

(1) $\|G(\cdot, t)\|_{C^{0,\alpha}_\varepsilon(\Gamma)} = O(\|G\|_{C^{0,\alpha}_\varepsilon(N(\tau))})$, *for all* $|t| \leq \tau$.

(2) $\|G(x, t + \xi(x))\|_{C^{0,\alpha}_\varepsilon(N(\tau/2))} = O(\|G\|_{C^{0,\alpha}_\varepsilon(N(\tau))})$, *for any* $\|\xi\|_{C^1_\varepsilon(\Gamma)} = O(\varepsilon)$.

(3) $\| \int_{\mathbb{R}} G(\cdot, t) g(t) dt\|_{C^{0,\alpha}_\varepsilon(\Gamma)} = O(\int_{\mathbb{R}} \|G(\cdot, t)\|_{C^{0,\alpha}_\varepsilon(\Gamma)} |g(t)| dt)$, *for any* g *with* $\operatorname{supp} g \subset [-\tau, \tau]$.

(4) $\|\frac{1}{t}(\varepsilon^2 \Delta_t v - \varepsilon^2 \Delta_0 v)\|_{C^{0,\alpha}_\varepsilon(\Omega)} = O(\|v\|_{C^{2,\alpha}_\varepsilon(\Omega)})$, *for any* $v \in C^{2,\alpha}(N(\tau))$ *and* $\Omega \subset N(\tau)$.

(5) $\|\frac{1}{t}(\varepsilon^2|\nabla_t v|^2-\varepsilon^2|\nabla_0 v|^2)\|_{C^{0,\alpha}_\varepsilon(\Omega)} = O(\|v\|^2_{C^{1,\alpha}_\varepsilon(\Omega)})$, *for any* $v \in C^{2,\alpha}(N(\tau))$ *and* $\Omega \subset N(\tau)$.

Proof. (1), (3) and the formula $[G(x,t+\xi(x))]_{0,\alpha} = O([G]_{0,\alpha}(1+|\nabla\xi|^\alpha))$ (from which (2) follows) can be derived directly from the definitions of the Holder norms. For (4), notice that in coordinates we have expressions of the form

$$\varepsilon^2(\Delta_t - \Delta_0)v(x,t) = \varepsilon^2[A_{ij}(x,t)(\partial_{ij}v)(x,t) + B_i(x,t)(\partial_i v)(x,t)] \times t.$$

and

$$\varepsilon^2(|\nabla_t v(\cdot,t)|^2-|\nabla_0 v(\cdot,t)|^2) = \varepsilon^2 C_{ij}(x,t) \times \partial_i v(x,t) \times \partial_j v(x,t) \times t,$$

where $A_{ij}(x,t) = \int_0^1 a'_{ij}(x,ts)ds$, $B_{ij}(x,t) = \int_0^1 b'_i(x,ts)ds$ and $C_{ij}(x,t) = \int_0^1 c'_i(x,ts)ds$ are depending only on the metric and Γ. □

5.1. **CMC near non-degenerate minimal hypersurfaces.** When the Jacobi operator of Γ is invertible, a standard application of the Inverse Function Theorem gives the existence of positive constants $\tau = \tau(M,\Gamma)$ and $C = C(M,\Gamma,\tau)$, such that for all $H \in (-\tau,\tau)$:

- There is a unique hypersurface Γ_H, which is a normal graph over Γ and has constant mean curvature equal to H.
- The graph Γ_H varies smoothly with respect to H.
- The distance function $\mathrm{dist}(\cdot,\Gamma_H)$ is smooth on $N(\tau)$.
- The map $\mathcal{F}_0$ giving Fermi coordinates with respect to Γ_H is a diffeomorphism to $N_H(\tau) = \mathcal{F}_0(\Gamma_H \times (-\tau,\tau))$.
- $C^{-1}\|G\circ\mathcal{F}_0\|_{C^k(\Gamma_H\times(-\tau,\tau))} \le \|G\|_{C^k(N_0(\tau))} \le C\|G\circ\mathcal{F}_0\|_{C^k(\Gamma_H\times(-\tau,\tau))}$, for $k=1,2,3$.

6. Injectivity results

6.1. **The case of a cylinder.** Let (Γ,h) be a closed $(n-1)$-dimensional Riemannian manifold. The Laplace-Beltrami operator of the cylinder $\Gamma\times\mathbb{R} = \{(x,t) : x \in \Gamma, t \in \mathbb{R}\}$, endowed with the product metric, decomposes as $\Delta_{\Gamma\times\mathbb{R}} = \Delta_0 + \partial_t^2$.

In this context, the function $\psi(t/\varepsilon)$, satisfies

$$\varepsilon^2\Delta_{\Gamma\times\mathbb{R}}(\psi(t/\varepsilon)) - W'(\psi(t/\varepsilon)) = 0$$

and its linearized operator at $\psi(t/\varepsilon)$ is given by

$$L_0 = \varepsilon^2\Delta_{\Gamma\times\mathbb{R}} - W''(\psi(t/\varepsilon)).$$

By differentiating the equation for $\psi(t/\varepsilon)$ with respect to the normal direction, we see that $L_0(\psi'(t/\varepsilon)) = 0$, i.e. $\psi'(t/\varepsilon) \in \mathrm{Ker}(L_0)$. In fact, Lemma 3.7 from [35], implies $\mathrm{Ker}(L_0) = \mathrm{span}\langle\psi'(t/\varepsilon)\rangle$.

We say that a function $f \in C^{2,\alpha}(\Gamma \times \mathbb{R})$ on the cylinder is orthogonal to the kernel of L_0 (or orthogonal to $\psi'(t/\varepsilon)$), if

$$\int_{\mathbb{R}} f(x,t)\psi'(t/\varepsilon)dt = 0, \quad \forall x \in \Gamma. \tag{7}$$

Direct computation shows that functions orthogonal to the kernel form an invariant subspace of L_0. The invertibility properties of L_0 on this subspace are summarized in the following statement, which combines Propositions 3.1 and 3.2 from [35].

Proposition 6.1. *Let (Γ, h) be a closed $(n-1)$-dimensional Riemannian manifold. There are positive constants C and ε_0, such that for all $\varepsilon \in (0, \varepsilon_0)$*

(1) *If $v \in C^{2,\alpha}_\varepsilon(\Gamma \times \mathbb{R})$ satisfies (7) then,*
$$\|v\|_{C^{2,\alpha}_\varepsilon(\Gamma\times\mathbb{R})} \leq C\|L_0 v\|_{C^{0,\alpha}_\varepsilon(\Gamma\times\mathbb{R})}.$$
(2) *If $f \in C^{0,\alpha}_\varepsilon(\Gamma\times\mathbb{R})$ satisfies (7), then there exists a unique $v \in C^{2,\alpha}_\varepsilon(\Gamma\times\mathbb{R})$ satisfying (7) such that $L_0 v = f$ and (7).*

The following result, which is proved in Proposition 3.3 of [35], regards the coercive operator that approximates $L_0 = \varepsilon^2\Delta - W''(\psi(t/\varepsilon))$ at infinity. In fact, notice that $W''(\psi(t)) \to 2$ as $t \to \infty$.

Proposition 6.2. *For any $\varepsilon > 0$, the operator $L_\infty = \varepsilon^2\Delta_g - 2$ is an isomorphism $L_\infty : C^{2,\alpha}(M) \to C^{0,\alpha}(M)$ with inverse bounded respect to the rescaled norms, i.e.*
$$\|f\|_{C^{2,\alpha}_\varepsilon(M)} = O(\|L_\infty f\|_{C^{0,\alpha}_\varepsilon(M)}).$$

7. Elementary properties of solutions and subsolutions

The existence of solutions for the problem with Dirichlet boundary data

$$\begin{cases} -\Delta u + W'(u) = 0 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

depends on the region being *large enough*, in the sense that the first eigenvalue for the Laplacian has to be small. The proof the following of proposition can be found in Proposition 2.4 from [16].

Proposition 7.1. *Let $\Omega \subset M$ be a bounded open region with Lipschitz boundary and first eigenvalue $\lambda_1 = \lambda_1(\Omega)$. There exists a unique positive solution of (8) if and only if $\lambda_1 < W''(0)$. Moreover, u and $-u$ are the unique global minima of the energy E in $H^1_0(\Omega)$.*

The following is just Serrin's maximum principle (see [20]).

Proposition 7.2. *Let u be a supersolution of (1) in Ω and v be a subsolution of (1) in Ω, i.e. $\Delta u - W'(u) \leq 0$, and $\Delta v - W'(v) \geq 0$. If $u \geq v$ in Ω, then either $u = v$ or $u > v$ in Ω.*

In our context, this proposition can be applied together with the following standard lemma.

Lemma 7.3. *Let u be a positive subsolution (resp. a negative supersolution) of (1) on Ω. Then, for all $\theta \in (0,1)$, the function θu is a subsolution (resp. supersolution) and for all $\theta \geq 1$, the function θu is a supersolution (resp. subsolution).*

Proof. Take $\theta \in (0,1)$ and x such that $u(x) > 0$. Then

$$\Delta(\theta u(x)) = \theta W'(u(x)) = \theta u(x)\frac{W'(u(x))}{u(x)} \geq W'(\theta u(x)),$$

where the inequality comes from the monotonicity of $W'(t)/t = t^2 - 1$. The proofs for the other cases are analogous. □

Using this fact we obtain the following useful result.

Corollary 7.4. *Let $\Omega \subset M$ be a bounded open region with smooth boundary. Let u and v be solutions to* (1) *in Ω. If u is continuous and positive on $\overline{\Omega}$ and v has Dirichlet boundary data equal to* 0*, then $u > v$.*

Proof. Since $\overline{\Omega}$ is compact there is $\alpha > 0$ such that $u \geq \alpha$ everywhere. In particular, for small values of $\theta > 0$ one must have $\theta v < u$ on Ω. This inequality also holds for $\theta = 1$. If not, by continuously making $\theta \to 1$ from below, one would find a first point of contact of the graphs of θv and u. Since, by Lemma 7.3, θv is a subsolution for $\theta \in [0,1]$, this would contradict the maximum principle, i.e. Proposition 7.2.

□

8. Standard elliptic estimates

In this section, we summarize the elliptic estimates we will use in the proofs contained in the next sections. In what follows, $\Omega \subset M$ denotes a Lipschitz open region of a fixed closed Riemannian manifold and Lv, a linear elliptic operator of the form

$$L_\varepsilon v = \varepsilon^2 \Delta_g v - c(x)v. \tag{9}$$

Theorem 8.1 (Estimates for weak solutions)**.** *There exists $\varepsilon_0 = \varepsilon_0(M) > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds.*

Assume $f \in L^\infty(\Omega)$ and $\|c\|_{L^\infty(\Omega)} \leq K$, for some $K > 0$. If $v \in W^{1,2}(\Omega)$ is a weak solution of $L_\varepsilon v = f$, we have for any $\Omega' \subset\subset \Omega$ the estimate

$$\begin{aligned}\|v\|_{C^{0,\alpha}_\varepsilon(\Omega')} &\leq C(\varepsilon^{-n/2}\|v\|_{L^2(\Omega)} + \|L_\varepsilon v\|_{L^\infty(\Omega)})\\ &\leq C(\varepsilon^{-n/2}\|v\|_{L^\infty(\Omega)} + \|L_\varepsilon v\|_{L^\infty(\Omega)}),\end{aligned}$$

where $C = C(M, K, \varepsilon^{-1}d', \mathrm{Vol}(\Omega)) > 0$, $\alpha = \alpha(M, \varepsilon^{-1}d')$ and $d' = \mathrm{dist}(\Omega', \partial\Omega)$.

Theorem 8.2 (Schauder estimates)**.** *Given $v \in C^{2,\alpha}(\Omega)$ and $\Omega' \subset\subset \Omega$ we have the estimate*

$$\|v\|_{C^{2,\alpha}_\varepsilon(\Omega')} \leq C(|v|_{C^0(\Omega)} + \|Lv\|_{C^{0,\alpha}_\varepsilon(\Omega)}),$$

where $C = C(M, \|c\|_{C^{0,\alpha}_\varepsilon(\Omega)}, \alpha, \varepsilon^{-1}d') > 0$ and $d' = \mathrm{dist}(\Omega', \partial\Omega)$.

Lemma 8.3 (Exponential decay lemma)**.** *Let $\Omega \subset M$ be a bounded region with smooth boundary and such that* $\mathrm{dist}(\cdot, \partial\Omega)$ *is smooth on* $\{p \in \Omega : \mathrm{dist}(p, \partial\Omega) < 2\rho\}$. *Let L_ε and c be as in* (9) *with the additional assumption that* $\min_\Omega c = c_0 > 0$.

There are positive constants σ and ε_0 (depending only on M, ρ and c_0), such that for $\varepsilon \in (0, \varepsilon_0)$ and $v \in C^{2,\alpha}(\overline{\Omega})$ satisfying $L_\varepsilon v \geq -a$, with $a \geq 0$, we have the estimate

$$v(p) \leq C\|v\|_{L^\infty(\partial\Omega)} \max\{e^{-\sigma\, \mathrm{dist}_{\partial\Omega}(p)/\varepsilon}, e^{-\sigma\rho/\varepsilon}\} + \frac{a}{c_0},$$

for all $p \in \Omega$ and $C = C(M) > 0$.

Proof. The argument is standard and we just summarize it now. Let $t(p) = \mathrm{dist}_{\partial\Omega}(p)$ and $u = v - \frac{a}{c_0}$. By the maximum principle, any positive maximum of u on an open domain $U \subset \Omega$, must belong to ∂U. Therefore, it is enough to prove the inequality for p in the region $t < \rho/\varepsilon$.

Let $f(t) = e^{-\sigma t/\varepsilon}$ and $g(t) = e^{\sigma(t-2\rho)/\varepsilon}$. From $u = v - \frac{a}{c_0}$,

$$\|u^+\|_\infty = \|(v - a/c_0)^+\|_\infty \leq \|v\|_\infty,$$

and $f + g \leq 2f$ for $t < \rho$, it follows that it is enough to prove $u \leq \|u^+\|_\infty(f + g)$, for $t < \rho$. It is a consequence of the maximum principle: let U be the larger region $0 < t < 2\rho$. The inequality holds on ∂U by construction. If it does not hold everywhere, one concludes $u - \|u^+\|_\infty(f + g)$ has a positive maximum in the interior of U. By the maximum principle this cannot happen if we choose $\sigma^2 < c_0$.

This gives us the estimate in the smaller region $t < \rho$ by the observations at the beginning of this paragraph. The constants C and ε_0 depend on the geometry of $\partial\Omega$ and Ω on $t < 2\rho$, and it appears in order to correct first order terms. (Indeed, $C \to 2$ as $\varepsilon \to 0$, but this fact is not necessary for our purposes). □

Lemma 8.4. *If u is a solution of* (1) *that does not vanish on $B(p, r) \subset M$, then $0 \leq 1 - |u(p)| \leq \nu(\varepsilon/r)$, where $\nu : \mathbb{R}_{\geq 0} \to [0, 1)$ is continuous and increasing with $\nu(0) = 0$ and $\lim_{s\to\infty} \nu(s) = 1$.*

Proof. The argument is standard: after blowing up the solutions from a ball of radius r to a ball of radius r/ε, every sequence with $\varepsilon \to 0$ admits a subsequence which converges (in compacts) to an entire solution in $\mathbb{R}^n$. On the other hand, the only entire bounded positive solution is the constant $+1$. □

Corollary 8.5. *Fix $k \geq 0$. There is an integer m_k, such that for every $0 < r + \varepsilon < R < \tau$ and u solution to* (1) *with nodal set $\{u = 0\} \subset N(r)$ we have*

$$\|\mathrm{sgn}(u) - u\|_{C^{k,\alpha}_\varepsilon(M \setminus N(R))} = O(\varepsilon^{-m_k} e^{-(R-r)/\varepsilon}).$$

Proof. On $M \setminus \{u = 0\}$ we can rewrite (1) as $\varepsilon^2 \Delta v - cv = 0$, where $v = \mathrm{sgn}(u) - u$ and $c = |u|^2 + |u|$. From Lemma 8.4, it follows that $|u| \geq 1 - \nu(1)$ on $M \setminus N(r + \varepsilon)$. Therefore, c is uniformly bounded from below in this region. Applying Lemma

8.3 and Lemma 8.4, we get the existence of a $C > 0$ such that

$$\sup_{\partial N(R)} |v| \leq C \sup_{\partial N(r+\varepsilon)} |v| e^{-(R-r)/\varepsilon} \leq C e^{-(R-r)/\varepsilon},$$

for all $R \in (r+\varepsilon, \tau)$. By the maximum principle, $\|v\|_{L^\infty(M \setminus N(R))} \leq \sup_{\partial N(R)} |v|$. It follows from Theorem 8.1 that $\|v\|_{C^{2,\alpha}_\varepsilon(M \setminus N(R))} = O(\varepsilon^{-n/2} e^{-(R-r)/\varepsilon})$.

Finally, the estimate for $C^{k,\alpha}_\varepsilon$ follows from iteratively applying Schauder estimates to derivatives of v. □

9. Characterizations of entire one dimensional solutions

Definition 9.1. An entire solution u of (1) in $\mathbb{R}^n$ is said to be *one dimensional* if there are $p, v \in \mathbb{R}^n$, with $|v| \leq 1$, such that $u(x) = \psi(\varepsilon^{-1}(x-p) \cdot v)$.

A one dimensional solution has parallel planar level sets, with its profile in the orthogonal direction to these planes being a translation of $\psi(t/\varepsilon)$. Characterizing such solutions is the first step in order to obtain curvature estimates for the level set of general solutions. In the late 70s, De Giorgi conjectured that entire monotone bounded solutions in $\mathbb{R}^n$ of (1) should be one dimensional, at least for $n \leq 8$. This is now known to be true for $n = 2, 3$ (see [18] and [3], respectively) and false for $n \geq 9$, see [10]. In dimensions $4 \leq n \leq 8$, it is known under the additional hypothesis of Savin's Theorem (see [39, 45]) which is an analogue of Bernstein's Theorem. At the present time, entire stable solutions of the Allen-Cahn are known to be one-dimensional only when $n = 3$ and they have finite multiplicity at infinity [3].

In this section, we summarize two characterizations of one dimensional solutions. First, they are the only entire solutions with multiplicity one at infinity (see [45], and Theorem 9.2). Second, they are the only entire solutions having its nodal set enclosed in between two parallel planes (see [13] and Theorem 9.3).

For solutions with multiplicity one at infinity, we have the following theorem (see Theorem 11.2, for a local version).

Theorem 9.2 (K. Wang, [45])**.** *There is $\tau_0 \in \mathbb{R}$, such that if u is an entire solution to* (1)*, with $\varepsilon = 1$, in $\mathbb{R}^{n+1}$, then*

$$\lim_{R\to\infty} R^{-n} \int_{B_R} \frac{|\nabla u|^2}{2} + W(u) \leq (1+\tau_0)\omega_n \sigma_0,$$

implies that u is one dimensional.

For solutions with nodal set contained between two parallel planes we use the following version of the well-know Gibbons conjecture, taken from Theorem 1.1, [14].

Theorem 9.3. *Let $u : \mathbb{R}^N \to \mathbb{R}$ be a solution to $\Delta u - W'(u) = 0$. Assume the nodal set $\{u = 0\}$ is contained in a slab $\{x \in \mathbb{R}^n : \langle x, v\rangle \leq K\}$, for* *some $K \geq 0$ and some direction $v \in S^{N-1}$. Then, u is one dimensional, i.e $u(x) = \pm\psi((x - dv)\cdot v)$, for some $|d|\leq K$.*

10. Solutions converging to strictly stable minimal hypersurfaces

In this section, we prove the following bound on the Hausdorff norm of solutions around strictly stable minimal hypersurfaces:

Theorem 10.1. *Let $\Gamma \subset M$ be a separating embedded minimal hypersurface which is the limit interface of a sequence of solutions of (1). If Γ is strictly stable, then the convergence happens with multiplicity one in the sense of measures.*

Theorem 10.1 is derived in several steps. First, we show that the distance from the nodal set $\{u = 0\}$ to the limit interface Γ is of order $O(\varepsilon)$. This follows from a sliding argument once appropriate barriers have been constructed. This implies that ε- blow-ups u near Γ, have nodal set bounded between two horizontal planes. From the characterization in Theorem 9.3 they converge to an entire one-dimensional solution. Finally, we obtain the desired energy estimates combining the analysis near Γ with the exponential decay Lemma 8.3.

The following lemma summarizes the construction of the barriers necessary for the sliding argument.

Lemma 10.2. *There exists $c, \varepsilon_0 > 0$ depending only on M and Γ, such that for each $\varepsilon \in (0, \varepsilon_0)$, there is a continuous one parameter family of functions $v_H \in C^{2,\alpha}(M)$, for $|H|\in (c\varepsilon, \tau]$ such that sign of $\operatorname{sgn}(\varepsilon^2\Delta v_H - W'(v_H)) = -\operatorname{sgn} H$.*

Each v_H decomposes as $v_H = \omega_H + \phi_H$, where $\omega_H(x,t) = \omega(t)$ is the approximate solution in Fermi coordinates with respect to the CMC hypersurface Γ_H and $\|\phi_H\|_{C^{2,\alpha}_\varepsilon(M)}= O(\varepsilon H)$.

In particular, for the extremal values $H = \pm c\varepsilon$, the zero level set of v_H is located in a tubular neighborhood $N(r)$ of Γ with height $r = O(\varepsilon)$.

Assuming this lemma the rest of the argument is short. We present it first and postpone the proof of Lemma 10.2 for later.

Proof of Theorem 10.1 . Let v_H be the family constructed in Lemma 10.2. Notice that for $H \in (c\varepsilon, \tau]$ the function v_H is a subsolution to (1), while for $H \in [-\tau, -c\varepsilon)$ it is a supersolution.

The sliding argument goes as follows. By hypothesis, for $\varepsilon > 0$ small enough, the level set $\{u = 0\}$ is contained in nonzero regions of v_τ. Therefore, we can the apply Corollary 7.4, concluding $u < v_\tau$. By the maximum principle, $u < v_H$, also holds as we slide H from τ to $c\varepsilon$. Arguing analogously for $H \in [-\tau, -c\varepsilon)$, we conclude $v_{-c\varepsilon} < u < v_{c\varepsilon}$, which in turn implies that $\{u = 0\} \subset N(r)$, where $r \in O(\varepsilon)$.

Blowing up u in Fermi coordinates over Γ, produces entire solutions with nodal set between two parallel planes. Theorem 9.3 implies that it has to be a one dimensional solution with horizontal level sets. Using as blow-up point a $x_\varepsilon \in \Gamma$ in which $\int_{-\varepsilon R}^{\varepsilon R} \varepsilon\frac{|\nabla u(x_\varepsilon,t)|^2}{2} + \frac{W(u(x_\varepsilon,t))}{\varepsilon} dt$ attains its maximum, shows that

$$\limsup_{\varepsilon\to} \int_{-\varepsilon R}^{\varepsilon R} \varepsilon\frac{|\nabla u(x_\varepsilon,t)|^2}{2} + \frac{W(u(x_\varepsilon,t))}{\varepsilon} dt \leq \sigma_0.$$

From Fubini, it follows that for any $R > 0$,

$$\lim_{\varepsilon\to 0}\int_{N(\varepsilon R)} \varepsilon\frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} \leq \sigma_0|\Gamma|.$$

Moreover, from the local convergence in $N(r)$ together with the exponential decay Lemma 8.3, it follows that for $R > 0$ large enough we have $|\mathrm{sgn}(t) - u(x,t)|\leq Ce^{-\sigma t/\varepsilon}$ on $N(\varepsilon^\delta) \setminus N(\varepsilon R)$ and $u = o(\varepsilon^{\mathbb{N}})$ in $M \setminus N(\varepsilon^\delta)$. This implies

$$\int_{N(\varepsilon^\delta)\setminus N(\varepsilon R)} \varepsilon\frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} = O(|\Gamma|\times e^{-2R})$$

and

$$\int_{M\setminus N(\varepsilon^\delta)} \varepsilon\frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} = o(\varepsilon^{\mathbb{N}}).$$

Combining all the estimates, and since $R > 0$ can be choosen arbitrarily large, we conclude that

$$\lim_{\varepsilon\to 0}\int_{M} \varepsilon\frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} \leq \sigma_0|\Gamma|.$$

□

We now present the setting for the proof of Lemma 10.2.

Let $\Gamma \subset M$ be a separating strictly stable minimal hypesurface (with a choice of normal vector). As in Section 5.1, there is $\tau > 0$, such that for any $H \in (-\tau, \tau)$ there is a unique hypersurface Γ_H of constant mean curvature equal to H, which is also a smooth graph over Γ.

In this section, we seek to construct a subsolution (res. supersolution) to (1) for each $H < 0$ (resp. $H > 0$), whose nodal set is a small perturbation of Γ_H. For this purpose we work on Fermi coordinates (x, t) with respect to Γ_H.

More precisely, denote the Allen-Cahn operator by $Q(v) = \varepsilon^2\Delta_g v - W'(v)$ and (abusing notation) define $\omega(x, t) := \omega(t)$ in Fermi coordinates. CMC hypersurfaces like Γ_H are usually modeled by solutions to $Q(u) + \varepsilon\lambda = 0$. Because of this, we expect to find small ϕ and λ such that $Q(\omega+\phi)+\varepsilon\lambda = o(\varepsilon)$, i.e. so that the sign of $Q(\omega + \phi)$ is controlled by the sign of λ, for small ε.

A short computation shows that the linearization of Q at ω, is given by

$$Q(\omega + \phi) = Q(\omega) + L\phi - \phi^2(3\omega - \phi),$$

where $L(\phi) = \varepsilon^2\Delta_g\phi - W''(\omega)\phi$, depends continuously on H.

The error term ϕ will be constructed separately on regions close to Γ_H and regions far from Γ_H, we decompose it as $\phi = \phi_1 + \phi_2$, where $\operatorname{supp}\phi_2 \subset N(\varepsilon^\delta)$. Substituting into the expression in Fermi coordinates from above, we have the formula

$$Q(\omega + \phi) + \varepsilon\lambda = (f + L\phi_1) + L_0\phi_2 + E_1,$$

where

- $f(t) = Q(\omega) + \varepsilon\lambda$ and

- $E_1 = -\phi^2(3\omega - \phi) + \varepsilon^2(\Delta_t - \Delta_0)\phi_2 - \varepsilon^2 H_t \phi_2'$, with
- $\mathcal{R}(x,t) = \int_0^1 H'(x,ts)ds$, is the reminder of the Taylor expansion of $H(x,t)$ with respect to t

Construction of the barriers amounts to select appropriate λ, ϕ_1 and ϕ_2 and estimate the error terms.

Remark 1. *In what follows, we will consider two different projection operators, one for $\eta \in C^{k,\alpha}(M)$ and the other for $\eta \in C^{k,\alpha}(N \times \mathbb{R})$. In the first case, we define*

$$\eta^\perp := \eta - \left(\frac{\int_{\mathbb{R}} \eta\omega'}{\int_{\mathbb{R}}(\omega')^2}\right)\omega'.$$

In the second case

$$[\eta]_0^\perp := \eta - \left(\frac{\int_{\mathbb{R}} \eta\psi'(t/\varepsilon)}{\int_{\mathbb{R}}(\psi'(t/\varepsilon))^2}\right)\psi'(t/\varepsilon).$$

Notice that $\|\varepsilon\omega'\|_{C^{0,\alpha}_\varepsilon(M)} + \|\psi'(t/\varepsilon)\|_{C^{0,\alpha}_\varepsilon(N\times\mathbb{R})} = O(1)$, so in both cases

- $\|\eta^\perp\|_{C^{0,\alpha}_\varepsilon(M)} = O(\|\eta\|_{C^{0,\alpha}_\varepsilon(M)})$
- $\|[\eta]_0^\perp\|_{C^{0,\alpha}_\varepsilon(N\times\mathbb{R})} = O(\|\eta\|_{C^{0,\alpha}_\varepsilon(N\times\mathbb{R})})$.

These represente, respectively, the projection operators onto the approximate kernel of L and the kernel of L_0.

Proof of Lemma 10.2.

Choice of λ: orthogonality. First, we choose $\lambda = \lambda(\varepsilon, H)$ so that f is orthogonal to ω'. From $Q(\omega) = -\varepsilon^2 H\omega' - \varepsilon^2\mathcal{R}t\omega'$, $\int_{\mathbb{R}} f\omega' = 0$ and Lemma 4.1, this amounts to $\lambda := \frac{\sigma_2}{\sigma_1}H + \varepsilon\int_{\mathbb{R}}\mathcal{R}t(\omega')^2 = \frac{\sigma_2}{\sigma_1}H + O(\varepsilon)$. This choice of λ depends continuously on H. Additionally, we have $\|f\|_{C^{0,\alpha}_\varepsilon(M)} = O(\varepsilon H + \varepsilon^2)$.

Choice of ϕ_1: inverting the operator at infinity. By Proposition 6.2 there is $v_1 \in C^{2,\alpha}_\varepsilon(M)$ with $L_\infty v_1 = -f$ and $\|v_1\|_{C^{2,\alpha}_\varepsilon(M)} = O(\|f\|_{C^{0,\alpha}_\varepsilon(M)})$. Since $\operatorname{supp}\omega' \subset N(\varepsilon^\delta)$, on the complement of $N(\varepsilon^\delta)$ we have the equalities $v_1 = v_1^\perp$ and $L = L_\infty$. Therefore, choosing $\phi_1 := v_1^\perp$, we obtain $\operatorname{supp}(f + L\phi_1) \subset N(\varepsilon^\delta)$ and $\|\phi_1\|_{C^{2,\alpha}_\varepsilon(M)} = O(\varepsilon H + \varepsilon^2)$. Notice that v_1 and ϕ_1 depend continuously on H.

Summarizing, our equation now reads

$$Q(\omega + \phi) + \varepsilon\lambda = g + L_0\phi_2 + E_1,$$

where $g = f + L\phi_1$, satisfies $\|g\|_{C^{0,\alpha}_\varepsilon(M)} = O(\varepsilon H + \varepsilon^2)$ and $\operatorname{supp} g \subset N(\varepsilon^\delta)$.

Choice of ϕ_2: inverting the operator in Fermi coordinates

In this step, we use the invertibility properties of the operator L_0 in order to control $g + L\phi_2$. First, notice that $\operatorname{supp} g \cup \operatorname{supp} g^\perp \cup \operatorname{supp} g^T \subset N(\varepsilon^\delta)$.

This allow us to consider them as functions of $C^{0,\alpha}_\varepsilon(N\times\mathbb{R})$ by means of Fermi coordinates.

Using the projection operator in $C^{0,\alpha}_\varepsilon(N\times\mathbb{R})$ we can write

$$g+L_0\phi_2=g^T+[g^\perp]_0^T+[g^\perp]_0^\perp+L_0\phi_2.$$

By construction we expect $g=f+L\phi_1$ to have a small tangent projection which should control the norm of $g^T+[g^\perp]_0^T$.

First, we estimate the order of the tangent component $g^T=g-g^\perp$. Since $\int_{\mathbb{R}} f\omega'=0$, $\operatorname{supp}\omega'=N(r)$, with $r=O(\varepsilon^\delta)$ and $\ell_0(\omega')=o(\varepsilon^{\mathbb{N}})$ (the latter follows from Lemma 4.1 and the remark before Lemma 4.4), we have

$$\begin{aligned}\int_{\mathbb{R}} g\omega' &= \int_{\mathbb{R}}(L\phi_1)\omega'\\ &=\varepsilon^2\int_{\mathbb{R}}(\Delta_t\phi_1)\omega'+\varepsilon^2\int_{\mathbb{R}}[\phi_1''-W'(\omega)\phi_1]\omega'-\varepsilon^2\int_{\mathbb{R}}H(x,t)\phi_1'\omega'\\ &=\varepsilon^2\Delta_0\int_{\mathbb{R}}\phi_1\omega'+\int_{\mathbb{R}}\phi_1\ell_0\omega'+\int_{\mathbb{R}}[\varepsilon^2(\Delta_t-\Delta_0)\phi_1]\omega'-\varepsilon^2\int_{\mathbb{R}}H(x,t)\phi_1'\omega'\\ &=0+O(\|\phi_1\|_{C^{0,\alpha}_\varepsilon(M)})\times o(\varepsilon^{\mathbb{N}})+\int_{\mathbb{R}}\frac{1}{t}[\varepsilon^2(\Delta_t-\Delta_0)\phi_1]\times t\omega'-\varepsilon^2\int_{\mathbb{R}}H(x,t)\phi_1'\omega'\end{aligned}$$

From the computation above, Lemma 5.1, and (6) of Lemma 4.1 with $p=1$, it follows that the $C^{0,\alpha}_\varepsilon$-norm of this expression is $O(\varepsilon\|\phi_1\|_{C^2_\varepsilon(M)}+\varepsilon\|\phi_1\|_{C^1_\varepsilon(M)})=O(\varepsilon^2H+\varepsilon^3)$. In particular,

$$\|g^T\|_{C^{0,\alpha}_\varepsilon(M)}=O(\varepsilon^2H+\varepsilon^3).$$

Similarly,

$$\begin{aligned}[g^\perp]_0^T(x,t)&:=\frac{\int g(x,s)^\perp\psi'(s/\varepsilon)ds}{\int(\psi'(s/\varepsilon))^2ds}\psi'(t/\varepsilon)\\ &=c\int g(x,s)^\perp\varepsilon^{-1}\psi'(s/\varepsilon)ds\times\psi'(t/\varepsilon)\\ &=c\int g(x,s)^\perp[\varepsilon^{-1}\psi'(s/\varepsilon)-\omega'(s)]ds\times\psi'(t/\varepsilon)\\ &=c\int g(x,s)^\perp[\varepsilon^{-1}\chi\psi'(s/\varepsilon)-\omega'(s)+\varepsilon^{-1}(1-\chi)\psi'(s/\varepsilon)]ds\times\psi'(t/\varepsilon)\\ &=c\int g(x,s)^\perp[\chi'(\psi(s/\varepsilon)-\operatorname{sgn})+\varepsilon^{-1}(1-\chi)\psi'(s/\varepsilon)]ds\times\psi'(t/\varepsilon).\end{aligned}$$

From which it follows that $\|[g^\perp]_0^T\|_{C^{0,\alpha}_\varepsilon(N\times\mathbb{R})}=o(\varepsilon^{\mathbb{N}})$.

Finally, we find ϕ_2 so that $\operatorname{supp}\phi_2\in N(\varepsilon^\delta)$ and $[g^\perp]_0^\perp+L_0\phi_2$ is small. By Proposition 6.1, there is $v_2\in C^{2,\alpha}_\varepsilon(N\times\mathbb{R})$ such that $[g^\perp]_0^\perp=-L_0v_2$ and

$$\|v_2\|_{C^{2,\alpha}_\varepsilon(N\times\mathbb{R})}=O(\|[g^\perp]_0^\perp\|_{C^{2,\alpha}_\varepsilon(N\times\mathbb{R})})=O(\|g\|_{C^{2,\alpha}_\varepsilon(N\times\mathbb{R})})=O(\varepsilon H+\varepsilon^2).$$

We define $\phi_2=\rho v_2$, where ρ is a smooth cutoff function, yet to be determined, and obtain $[g^\perp]_0^\perp+L_0\phi_2=L_0[(\rho-1)v_2]$. We can choose ρ so that $(\rho-1)v_2$ has support far away from $N \times \mathbb{R}$. The $C^{2,\alpha}_\varepsilon$ norm of v_2 to decays exponentially fast in the same region. This gives us

$$\|[g^\perp]^\perp_0 + L_0\phi_2\|_{C^{2,\alpha}_\varepsilon(N\times\mathbb{R})} = o(\varepsilon^{\mathbb{N}})$$

and

$$\|\phi_2\|_{C^{2,\alpha}_\varepsilon(N\times\mathbb{R})} = O(\varepsilon H + \varepsilon^2).$$

Estimating the error. From $\|\phi_1\|_{C^{2,\alpha}_\varepsilon(M)} + \|\phi_2\|_{C^{2,\alpha}_\varepsilon(M)} = O(\varepsilon H + \varepsilon^2)$, we estimate the remaining terms by

$$\begin{aligned}
\|\phi^2(3\omega - \phi)\|_{C^{0,\alpha}_\varepsilon(M)} &= O(\varepsilon^2 H^2 + \varepsilon^3 H + \varepsilon^4)\\
\|\varepsilon^2 H_t \phi_2'\|_{C^{0,\alpha}_\varepsilon(M)} &= O(\varepsilon\|\phi_2\|_{C^1_\varepsilon(M)}) = O(\varepsilon^2 H + \varepsilon^3)\\
\|\varepsilon^2(\Delta_t - \Delta_0)\phi_2\|_{C^{0,\alpha}_\varepsilon(M)} &= \|\varepsilon^2 t a_{ij}(x,t)\partial_{ij}\phi_2 + \varepsilon^2 t b_i(x,t)\partial_i\phi_2\|_{C^{0,\alpha}_\varepsilon(M)}\\
&= O(\varepsilon\|\phi_2\|_{C^{2,\alpha}_\varepsilon(M)})\\
&= O(\varepsilon^2 H + \varepsilon^3).
\end{aligned}$$

Therefore, for $|H| < 1$, we have $\|E_1\|_{C^{0,\alpha}_0(M)} = O(\varepsilon^2 H + \varepsilon^3)$.

Subsolution, supersolution and nodal set. Adding all the estimates together, we have

$$\|Q(\omega + \phi) + \varepsilon\lambda\|_{C^{0,\alpha}_\varepsilon(M)} = O(\varepsilon^2 H + \varepsilon^3).$$

Remember $\lambda = c_0 H + O(\varepsilon)$, for some universal constant $c_0 > 0$. The previous estimate implies that

$$\begin{aligned}
\operatorname{sgn} Q(\omega + \phi) &= \operatorname{sgn}(-\varepsilon\lambda + O(\varepsilon^2 H + \varepsilon^3))\\
&= \operatorname{sgn}(-c_0\varepsilon H + O(\varepsilon^2 H + \varepsilon^2))\\
&= \operatorname{sgn}(-[c_0 + O(\varepsilon)]H + O(\varepsilon))
\end{aligned}$$

This tell us, that there is a universal constant $K > 0$, such that if we restrict H to the domain $|H| \geq K\varepsilon$, then

$$\operatorname{sgn} Q(\omega + \phi) = -\operatorname{sgn} H.$$

It follows that $v = \omega + \phi$ is a subsolution for $H \in [-\tau, -K\varepsilon]$ and a supersolution for $H \in [K\varepsilon, \tau]$.

□

11. Curvature Estimates for multiplicity one solutions

In this section, we present the following curvature estimates

Lemma 11.1. *Let $\Gamma \subset M$ be a non-degenerate minimal hypersurface, which is also the limit interface for a sequence of solutions to* (1) *with multiplicity one. Then, for $\varepsilon = \varepsilon(\Gamma, M)$ small enough, the nodal set of the solutions is a normal graph $\Gamma(f)$, with*

$$\|f\|_{C^2(\Gamma)} + \varepsilon^\alpha \|f\|_{C^{2,\alpha}(\Gamma)} = O(\varepsilon).$$

As in [25, 8] these estimates are derived from the work of Wang and Wang-Wei, combined with a standard point-picking and blow-up argument.

In the computations below, we rely on the following two theorems which were proven for the case of $\mathbb{R}^n$ in [45] and [46], respectively. The proof for general ambient manifolds with bounded curvature tensor, follows the same strategy with minor modifications.

Theorem 11.2 (K. Wang, see [45]). *Let M be a closed Riemannian manifold. There are positive numbers $\varepsilon_0, \tau_0, \alpha_0 \in (0,1)$ and $r_0 < R_0, K_0$, such that the following holds. Let u be a solution of* (1) *with $\varepsilon < \varepsilon_0$ on $B_{R_0}(p) \subset M$ for $p \in M$, if*

$$R_0^{-n} \int_{B_{R_0}} \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} \leq (1+\tau_0)\omega_n \sigma_0,$$

then $\{u=0\} \cap B_{r_0}(p)$ is a normal graph (in exponential coordinates) over some hyperplane in T_pM, with C^{1,α_0} norm bounded by K_0.

Theorem 11.3 (Wang-Wei, see Section 15 of [46]). *Let M be a closed Riemannian manifold. Let $u_i : B_R(p_i) \to \mathbb{R}$ be a sequence of solutions to* (1) *with $\varepsilon = \varepsilon_i \to 0$. Assume that*

i) *$\{u_i = 0\}$ is, in exponential coordinates, a normal graph over a hyperplane $\pi_i \subset T_{p_i}B_R$, with Lipschitz constant uniformly bounded on i and converging to a smooth hypersurface as $i \to 0$.*
ii) *For any $q_i \in \{u_i = 0\}$ the blow-ups $\tilde{u}_i(x) = u_i \circ \exp_{q_i}(\varepsilon x)$ converge to a one dimensional solution in $\mathbb{R}^{n+1}$, and*
iii) *The second fundamental form of $\{u_i = 0\}$ is bounded uniformly on i.*

Then, on a smaller ball $B_r \subset B_R$, the mean curvature of $\{u_i = 0\}$ satisfies

$$|H|_{C^0(\pi_i)} + \varepsilon^{\alpha} |H|_{C^{0,\alpha}(\pi_i)} = O(\varepsilon).$$

Moreover, the $C^{2,\alpha}$ norm of the nodal set as a graph is bounded.

Proof of Lemma 11.1.

Claim 1. *For ε small enough, the nodal set $\{u=0\}$ is an embedded hypersurface and the generalized second fundamental form of u near $\{u=0\}$ is of order $o(\varepsilon^{-1})$.*

By the monotonicity formula ε-rescalings of multiplicity one solutions centered at the nodal set, have multiplicity one at infinity and therefore are one dimensional by Theorem 9.2. This implies $\varepsilon|\nabla u| \neq 0$ on the nodal set. The estimate on the second fundamental form follows from the smoothness of the convergence of the rescalings to the 1-D solution, which has planar level sets.

Claim 2. *For ε small enough, the nodal set is a normal graph $\Gamma(f) = \{u=0\}$ for some $f \in C^\infty(\Gamma)$. Moreover, the Lipschitz norm of f is uniform on ε.*

Theorem 11.2 gives uniform $C^{1,\alpha}$ bounds some plane π for each point of the nodal set. If there is a sequence of π converging to a vertical plane in Fermi coordinates with respect to Γ, then the $C^{1,\alpha}$ bound would imply there is concentration of energy far from Γ, which we are assuming it does not happen. It follows that, for ε small, $\{u=0\}$ is a normal graph over Γ with Lipschitz norm uniform on ε.

Claim 3. *The second fundamental form of* $\Gamma(f)$ *is* $O(1)$.

Let $p \in \{u=0\}$ be the point where the norm of the second fundamental form of the nodal set attains its maximum, which we denote by λ. From Claim 1, we have $\varepsilon\lambda \to 0$. To proceed by contradiction, assume $\limsup \lambda = \infty$. Then, after passing to a subsequence, for any $R>0$ and ε small enough, the rescalings $v(x) := u \circ \exp_p(x/\lambda)$ are solutions to $(\varepsilon\lambda)^2 \Delta v - W'(v) = 0$ in $B_R(0) \subset T_pM$ with respect to the metric $g_\lambda = \lambda^{-2} \exp_p^*(g)$. Moreover, from the monotonicity formula and the multiplicity one assumption, the limit varifold in $\mathbb{R}^n$ (by making $R \to \infty$) has to be a plane. It follows that for a fixed R, we obtain a list of solutions, such that the nodal set is a uniformly bounded Lipschitz graph with second fundamental form bounded from above by 1. Therefore, $\lambda = O(1)$.

Finally, we obtain a contradiction from Theorem 11.3, which implies that the $C^{2,\alpha}$ norm of this graph is universally bounded and therefore it must converge in C^2 to a plane. This contradicts that the norm of the second fundamental form at the origin is exactly 1.

Claim 4. $\|f\|_{C^{2,\alpha}(\Gamma)} + \varepsilon^\alpha \|f\|_{C^{2,\alpha}(\Gamma)} = O(\varepsilon)$.

Finally, we can apply Theorem 11.3 to our original sequence of solutions and conclude that its mean curvature satisfies $|H|_{C^0(\pi_i)} + \varepsilon^\alpha |H|_{C^{0,\alpha}(\pi_i)} = O(\varepsilon)$. Since the minimal hypersurface is invertible, the same bounds must hold for f, i.e.

$$|f|_{C^2(\pi_i)} + \varepsilon^\alpha |f|_{C^{2,\alpha}(\pi_i)} = O(\varepsilon).$$

□

12. Uniqueness of multiplicity one solutions around non-degenerate interfaces

Let u be a solution of (1) converging to a non-degenerate minimal hypersurface Γ with multiplicity one. From Section 11 we know that for ε sufficiently small, $\Gamma(f) = \{u=0\}$ with $\|f\|_{C^2(\Gamma)} + \varepsilon^\alpha \|f\|_{C^{2,0}(\Gamma)} = O(\varepsilon)$.

Definition 12.1. Given $\xi \in C(\Gamma)$ with $|\xi| = o(1)$, we denote $\omega_\xi(x,t) = \omega(x, t - \xi(x))$. Similarly, $\omega'_\xi(x,t) = \omega'(t-\xi(x))$, $\omega''_\xi(x,t) = \omega''(t-\xi(x))$ and so on.

Remark 2. Together with Proposition 9.2, the estimates above imply that, for any fixed $R>0$, the $C^2_\varepsilon(N(\varepsilon R))$ norm of $\|u - \omega_f\|_{C^2_\varepsilon(N(\varepsilon R))} = o(1)$.

We begin this section by looking for a perturbation of f of the form $\xi = f + h$, and such that the error $\phi = u - \omega_\xi$ is orthogonal to the approximate kernel ω'_ξ in the following sense:

Definition 12.2. A smooth function $\phi : M \to \mathbb{R}$ is said to be *orthogonal to the approximate kernel* ω'_ξ if for all $x \in \Gamma$,

$$\int_{\mathbb{R}} \phi(x,t)\omega'_\xi(t)dt = 0. \tag{10}$$

Remark 3. As in [46] and [8], equation (10) allows for the following procedure. First, L^2 estimates for ϕ are obtained from Lemma 4.4 and (10). Then, these are improved to estimates of the $C^{2,\alpha}_\varepsilon$-norm, using Theorem 8.1, Theorem 8.2 and the $C^{0,\alpha}_\varepsilon$ norm of $\varepsilon^2\Delta\phi - W''(\omega_\xi)\phi$.

In this section, we carry out an argument following the lines described in Remark 3.

Proposition 12.3. *There exists $\xi \in C^\infty(\Gamma)$ such that the error $\phi = u - \omega_\xi$ is orthogonal to the approximate kernel ω'_ξ. In addition,*

$$\|\phi\|_{C^{2,\alpha}_\varepsilon(M)} = o(1) \quad \textit{and} \quad \|\nabla_0^k \xi\|_{C^{0,\alpha}_\varepsilon(\Gamma)} = O(\varepsilon + \varepsilon^{1-k}\|\phi\|_{C^{k,\alpha}_\varepsilon(M)}),$$

for $k = 0, 1, 2$ and α as in Lemma 11.1.

Proof of Proposition 12.3. Let $U = \{h \in C(\Gamma) : |h| < \tau/2\}$ and F be the map $F : U \to C(\Gamma)$, given by

$$F(h)(x) := \varepsilon \int_{\mathbb{R}} [u(x,t) - \omega_{f+h}(x,t)]\omega'_{f+h}(x,t)dt.$$

From Remark 2 and Lemma 4.1-(6) we obtain

Claim 1. $F(0) = o(\varepsilon)$.

Similarly, we can estimate $|DF(h)|$ from below when $\|h\|_{C(\Gamma)}$ is small. Denote by $B(f, r) \subset C(\Gamma)$, the ball of radius $r > 0$ centered at $f \in C(\Gamma)$, with respect to the supremum norm. Let $r = o(\varepsilon)$ and $h \in B(0, r)$.

Claim 2. *For ε small enough, $DF(h)(v) = cv, \forall v \in C(\Gamma)$, where $c = c(h) \geq \sigma_2/2$. In particular, $B(F(0), \frac{c}{2}r) \subset F(B(0, r))$.*

Indeed, from Lemma 4.1 (6) and (7) we get

$$\begin{aligned} DF(h)(v) &= \frac{d}{ds}F(h + sv)|_{s=0} \\ &= v \cdot \varepsilon \left[\int_{\mathbb{R}} (\omega'_{f+h})^2 - \int_{\mathbb{R}} [u - \omega_{f+h}]\omega''_{f+h} \right] \\ &= v \cdot [\sigma_1 + o(\varepsilon^{\mathbb{N}}) + o(1)], \end{aligned}$$

which implies the claim for ε small enough.

Claim 3. *There exists $h \in C^\infty(\Gamma)$, satisfying $\|h\|_{C(\Gamma)} = o(\varepsilon)$ and $F(h) \equiv 0$.*

To see this, choose $r = o(\varepsilon)$ such that $F(0) = o(r)$, e.g. $r = \sqrt{\varepsilon F(0)}$. Since $F(0) = o(\varepsilon)$, the last claim implies $0 \in B(F(0), \frac{c}{2}r)$, for ε sufficiently small. Therefore, $0 \in F(B(0, r))$.

Claim 4. *Let $\xi = f + h$, where h is as in the previous claim. Then, $|\nabla_0^k \xi| = o(\varepsilon^{1-k})$, for $k = 1, 2, 3$. In particular, $\|\phi\|_{C^k_\varepsilon(M)} = o(1)$, for $k = 1, 2, 3$.*

Now that we have guaranteed the existence of h with $\|h\|_{C(\Gamma)} = o(\varepsilon)$, its smoothness follows from applying the Implicit Function Theorem and the nondegeneracy of $DF(h)$ to the function $\tilde{F} : \Gamma \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ given by

$\tilde{F}(x,h) := F(h)(x)$. Then, the estimate $|\nabla_0^k \xi| = o(\varepsilon^{1-k})$, for $k = 1,2,3$, follows by recursively differentiating $F(h)(x) = 0$ with respect to ∇_0^k and estimating the norm of the result, each time using Lemma 4.1.

Finally, we have to argue for $\|\phi\|_{C^k_\varepsilon(M)} = o(1)$. From Corollary 8.5, for every $\rho > 0$ we can choose $R = O(\varepsilon)$ such that $\|\phi\|_{C^k_\varepsilon(M\setminus N(R))} < \rho$, for ε small enough. In exponential coordinates on points of $N(R)$, the function ω_ξ rescales as $\psi(t - \tilde{\xi}(x))$, where $\tilde{\xi}(x) = \xi(\varepsilon x)/\varepsilon$. From the previous estimate it follows that $\nabla_0^k \tilde{\xi} = o(1)$, for $k = 1,2,3$. This implies $\psi(t - \tilde{\xi}(x))$ converges to the canonical solution in all C^k norms. The same is true for u from Remark 2 and putting both estimates together we conclude that for any $\rho > 0$, there is an $\varepsilon_0 > 0$ such that $\|\phi\|_{C^k_\varepsilon(M)} \leq \rho$ for $k = 1,2,3$ and $\varepsilon \in (0, \varepsilon_0)$.

Claim 5. *$\|\nabla_0^k h\|_{C^{0,\alpha}_\varepsilon(\Gamma)} = O(\varepsilon^{1-k})\|\phi\|_{C^{2,\alpha}_\varepsilon(M)}$, for $k = 1,2$ and $\alpha \in [0,1)$.*

The estimates $\|\nabla_0^k h\|_{C^{0,\alpha}_\varepsilon(\Gamma)} = O(\varepsilon^{1-k})\|\phi\|_{C^{2,\alpha}_\varepsilon(M)}$ are obtained recursively by differentiating $0 = u(x, f(x)) = \omega_{f+h}(-h(x)) + \phi(x, f(x))$.

□

Directly from the definition of ϕ and (1), using the notation for Fermi coordinates from Section 5 we can write the following equation for ϕ and ξ

$$\begin{aligned} \varepsilon^2 \Delta_0 \phi + \ell_0(\phi) + E_1 &= \varepsilon^2 \Delta_g \phi - W''(\omega_\xi)\phi \\ &= \phi^2(2\omega_\xi + u) + \varepsilon^2 J[\xi]\omega'_\xi + E_2 \end{aligned} \tag{11}$$

where

- $\ell_0(\phi) = \varepsilon^2 \phi'' - W''(\omega_\xi)\phi$,
- $J[\xi] = \Delta_0 \xi - H'_0 \xi$ is the Jacobi operator of Γ,
- $E_1 = -\varepsilon^2 H_t \phi' + \varepsilon^2(\Delta_t - \Delta_0)\phi$,
- $E_2 = \varepsilon^2 \omega'''_\xi |\nabla_t \xi|^2 + \varepsilon^2[(\Delta_0 - \Delta_t)\xi]\omega'_\xi + \varepsilon^2 H'_0(t-\xi)\omega'_\xi + \varepsilon^2 \mathcal{R} t^2 \omega'_\xi$ and
- $\mathcal{R} = \mathcal{R}(x,t) = \int_0^1 H''(x, t\cdot s)(s-1)ds$.

Remark. Although this equation involves both ϕ and ξ, it follows from Proposition 12.3 that the right hand side of all the estimates can be presented in terms of norms of ϕ. This is what we do in the rest of this section.

First, we compute the estimates for $J[\xi]$.

Proposition 12.4. $\|\varepsilon J[\xi]\|_{C^{0,\alpha}_\varepsilon(M)} = O(\varepsilon^3 + \varepsilon\|\phi\|_{C^{2,\alpha}_\varepsilon(M)} + \|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)})$

Proof. We project onto Γ by integrating against ω'_ξ along every vertical direction.

For the first term, notice that from (7) we have the expression,

$$\Big\| \int_{\mathbb{R}} \varepsilon^2 (\Delta_0 \phi)\omega'_\xi \Big\|_{C^{0,\alpha}_\varepsilon(\Gamma)} = \Big\| -2\varepsilon^2 \nabla_0 \xi \int_{\mathbb{R}} (\nabla_0 \phi)\omega''_\xi - \varepsilon^2 |\nabla_0 \xi|^2 \int_{\mathbb{R}} \phi \omega'''_\xi + \varepsilon^2 \Delta_0 \xi \int_{\mathbb{R}} \phi \omega''_\xi \Big\|_{C^{0,\alpha}_\varepsilon(\Gamma)}$$
$$= O(\|\nabla_0 \xi\|_{C^{0,\alpha}_\varepsilon(M)} + \varepsilon \|\nabla_0^2 \xi\|_{C^{0,\alpha}_\varepsilon(M)}) \|\phi\|_{C^{2,\alpha}_\varepsilon(M)}.$$

For the second term, by the formula in the introduction, we have

$$\Big\| \int_{\mathbb{R}} \ell(\phi)\omega'_\xi \Big\|_{C^{0,\alpha}_\varepsilon(\Gamma)} = \Big\| \int_{\mathbb{R}} [\ell_0(\phi) + o(\varepsilon^{\mathbb{N}})]\omega'_\xi \Big\|_{C^{0,\alpha}_\varepsilon(\Gamma)} = o(\varepsilon^{\mathbb{N}}).$$

Next,

$$\Big\| \int_{\mathbb{R}} \varepsilon^2 H_t \phi' \omega'_\xi \Big\|_{C^{0,\alpha}_\varepsilon(\Gamma)} = O(\varepsilon^2) \|H_t\|_{C^1_\varepsilon(M)} \|\phi'\|_{C^1_\varepsilon(M)} = O(\varepsilon \|\phi\|_{C^2_\varepsilon(M)})$$

Now, from the Fermi Coordinates section, we know that

$$\Big\| \int_{\mathbb{R}} [\varepsilon^2 (\Delta_t - \Delta_0)\phi(\cdot, t)]\omega'_\xi \Big\|_{C^{0,\alpha}_\varepsilon(\Gamma)} = O(1) \int_{\mathbb{R}} \|\varepsilon^2 (\Delta_{t+\xi} - \Delta_0)\phi(\cdot, t+\xi)\|_{C^{0,\alpha}_\varepsilon(\Gamma)} |\omega'|$$
$$= O(1) \|\phi\|_{C^{2,\alpha}_\varepsilon(M)} \int_{\mathbb{R}} (|t| + \varepsilon)|\omega'|$$
$$= O(\varepsilon) \|\phi\|_{C^{2,\alpha}_\varepsilon(M)}.$$

Since $\|u\|_{C^{2,\alpha}_\varepsilon(M)} + \|\omega_\xi\|_{C^{2,\alpha}_\varepsilon(M)} = O(1)$, we have

$$\Big\| \int_{\mathbb{R}} \phi^2 (2\omega_\xi + u)\omega'_\xi \Big\|_{C^{0,\alpha}_\varepsilon(\Gamma)} = O(\|\phi\|^2_{C^1_\varepsilon})$$

Similarly,

$$\Big\| \int_{\mathbb{R}} \varepsilon^2 J[\xi](\omega'_\xi)^2 \Big\|_{C^{0,\alpha}_\varepsilon(\Gamma)} = (\varepsilon \sigma_2 + o(\varepsilon^{\mathbb{N}})) \|J[\xi]\|_{C^{0,\alpha}_\varepsilon(\Gamma)}$$

It remains to estimate the error E_2,
First, since ω''' is an odd function

$$\Big\| \int_{\mathbb{R}} \varepsilon^2 |\nabla_t \xi|^2 \omega'''_\xi \Big\|_{C^{0,\alpha}_\varepsilon(\Gamma)} = \Big\| \int_{\mathbb{R}} \varepsilon^2 (|\nabla_t \xi|^2 - |\nabla_0 \xi|^2)\omega'''_\xi \Big\|_{C^{0,\alpha}_\varepsilon(\Gamma)}$$
$$= \int_{\mathbb{R}} \varepsilon^2 \||\nabla_{t+\xi} \xi|^2 - |\nabla_0 \xi|^2\|_{C^{0,\alpha}_\varepsilon(\Gamma)} |\omega'''|$$
$$= \varepsilon^2 \|\nabla_0 \xi\|^2_{C^{0,\alpha}_\varepsilon(\Gamma)} \int_{\mathbb{R}} (|t| + \varepsilon)|\omega'''|$$
$$= O(\varepsilon) \|\nabla_0 \xi\|^2_{C^{0,\alpha}_\varepsilon(\Gamma)}$$

Next, we have

$$\begin{aligned}\left\| \int_{\mathbb{R}} (\varepsilon^2(\Delta_t - \Delta_0)\xi)(\omega_\xi')^2 \right\|_{C^{0,\alpha}_\varepsilon(\Gamma)} &= O(1) \int_{\mathbb{R}} \|\varepsilon^2(\Delta_{t+\xi} - \Delta_0)\xi\|_{C^{0,\alpha}_\varepsilon(\Gamma)} (\omega_\xi')^2 \\ &= O(\|\nabla_0\xi\|_{C^{0,\alpha}_\varepsilon(\Gamma)} + \|\nabla_0^2\xi\|_{C^{0,\alpha}_\varepsilon(\Gamma)}) \int_{\mathbb{R}} (|t| + \varepsilon) \times (\varepsilon\omega_\xi')^2 \\ &= O(\varepsilon^2)(\|\nabla_0\xi\|_{C^{0,\alpha}_\varepsilon(\Gamma)} + \|\nabla_0^2\xi\|_{C^{0,\alpha}_\varepsilon(\Gamma)}).\end{aligned}$$

Since $t(\omega')^2$ is an odd function,

$$\left\| \int_{\mathbb{R}} \varepsilon^2 H_0'(t - \xi)(\omega_\xi')^2 \right\|_{C^{0,\alpha}_\varepsilon(\Gamma)} = \left\| \varepsilon^2 H_0' \int_{\mathbb{R}} t(\omega')^2 \right\|_{C^{0,\alpha}_\varepsilon(\Gamma)} = 0.$$

Finally, since $\|t + \xi\|_{C^{0,\alpha}_\varepsilon(M)} = O(|t| + \|\xi\|_{C^{0,\alpha}_\varepsilon(M)}) = O(|t| + \varepsilon)$, we have

$$\begin{aligned}\left\| \int_{\mathbb{R}} \varepsilon^2 \mathcal{R} t^2 (\omega_\xi')^2 \right\|_{C^{0,\alpha}_\varepsilon(\Gamma)} &= O(\varepsilon^2)\|\mathcal{R}\|_{C^{0,\alpha}_\varepsilon(M)} \int_{\mathbb{R}} (|t|^2 + \varepsilon^2)(\omega')^2 \\ &= O(\varepsilon^2) \int_{\mathbb{R}} (|t/\varepsilon|^2 + 1)(\varepsilon\omega')^2 \\ &= O(\varepsilon^3).\end{aligned}$$

Combining all the estimates, we obtain

$$\begin{aligned}\|\varepsilon J[\xi]\|_{C^{0,\alpha}_\varepsilon(\Gamma)} &= O(\|\nabla_0\xi\|_{C^{0,\alpha}_\varepsilon(M)})(\varepsilon^2 + \varepsilon\|\nabla_0\xi\|_{C^{0,\alpha}_\varepsilon(M)} + \|\phi\|_{C^{2,\alpha}_\varepsilon(M)}) \\ &\quad + O(\|\nabla_0^2\xi\|_{C^{0,\alpha}_\varepsilon(M)})(\varepsilon^2 + \varepsilon\|\phi\|_{C^{2,\alpha}_\varepsilon(M)}) \\ &\quad + O(\varepsilon^3 + \varepsilon\|\phi\|_{C^{2,\alpha}_\varepsilon(M)} + \|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)})\end{aligned} \tag{12}$$

Finally, substituting $\|\nabla_0^k\xi\|_{C^{0,\alpha}_\varepsilon(\Gamma)} = O(\varepsilon + \varepsilon^{1-k}\|\phi\|_{C^{2,\alpha}_\varepsilon(M)})$ into (12) we get

$$\|\varepsilon J[\xi]\|_{C^{0,\alpha}_\varepsilon(\Gamma)} = O(\varepsilon^3 + \varepsilon\|\phi\|_{C^{2,\alpha}_\varepsilon(M)} + \|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)}).$$

□

Now we compute estimates for the value of the approximated linearized operator at ϕ.

Proposition 12.5. $\|\varepsilon^2\Delta_g\phi - W''(\omega_\xi)\phi\|_{C^{0,\alpha}_\varepsilon(M)} = O(\varepsilon^2 + \|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)})$.

Proof. Remember

$$\varepsilon^2\Delta_g\phi - W''(\omega)\phi = \phi^2(2\omega_\xi + u) + \varepsilon J[\xi]\dot{\omega}_\xi + E_2 \tag{13}$$

where

- $J[\xi] = \Delta_0\xi - H_0'\xi$ is the Jacobi operator of Γ
- $E_2 = W'(\omega_\xi)|\nabla_t\xi|^2 + \varepsilon^2[(\Delta_t - \Delta_0)\xi + H_0'(t - \xi) + t^2\mathcal{R}]\omega_\xi'$ and
- $\mathcal{R} = \mathcal{R}(x, t) = \int_0^1 H''(x, ts)(s - 1)ds$.

We estimate each term of $\|E_2\|_{C^{0,\alpha}_\varepsilon(M)}$ separately

$$\begin{aligned}\|W'(\omega_\xi)|\nabla_t\xi|^2\|_{C^{0,\alpha}_\varepsilon(M)} &= O(\|\nabla_0\xi\|^2_{C^{0,\alpha}_\varepsilon(M)})\\ &= O(\varepsilon+\|\phi\|_{C^1_\varepsilon(M)})^2\\ &= O(\varepsilon^2+\|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)}).\end{aligned}$$

$$\begin{aligned}\|\varepsilon^2(\Delta_t-\Delta_0)\xi\omega'_\xi\|_{C^{0,\alpha}_\varepsilon(M)} &= O(\varepsilon^2\|\nabla_0^2\xi\|_{C^{0,\alpha}_\varepsilon(M)}+\varepsilon^2\|\nabla_0\xi\|_{C^{0,\alpha}_\varepsilon(M)})\|t\omega'_\xi(x,t)\|_{C^{0,\alpha}_\varepsilon(M)}\\ &= O(\varepsilon^2\|\nabla_0^2\xi\|_{C^{0,\alpha}_\varepsilon(M)}+\varepsilon^2\|\nabla_0\xi\|_{C^{0,\alpha}_\varepsilon(M)})\|(t/\varepsilon)\varepsilon\omega'_\xi(x,t)\|_{C^{0,\alpha}_\varepsilon(M)}\\ &= O(\varepsilon^2\|\nabla_0^2\xi\|_{C^{0,\alpha}_\varepsilon(M)}+\varepsilon^2\|\nabla_0\xi\|_{C^{0,\alpha}_\varepsilon(M)})\\ &= O(\varepsilon^2)(\varepsilon+\varepsilon^{-1}\|\phi\|_{C^{2,\alpha}_\varepsilon(M)})\\ &= O(\varepsilon^3+\varepsilon\|\phi\|_{C^{2,\alpha}_\varepsilon(M)}).\end{aligned}$$

$$\|\varepsilon^2H'_0(t-\xi)\omega'_\xi\|_{C^{0,\alpha}_\varepsilon(M)} = O(\varepsilon^2)\|H'_0\|_{C^{0,\alpha}_\varepsilon(M)}\|(t-\xi)\omega'_\xi\|_{C^{0,\alpha}_\varepsilon(M)}= O(\varepsilon^2).$$

$$\|\varepsilon^2t^2\mathcal{R}\omega'_\xi\|_{C^{0,\alpha}_\varepsilon(M)} = O(\varepsilon^2)\|\mathcal{R}\|_{C^{0,\alpha}_\varepsilon(M)}\|t^2\omega'_\xi\|_{C^{0,\alpha}_\varepsilon(M)}= O(\varepsilon^3).$$

Collecting all these estimates with the ones for $J[\xi]$ from the last section, we conclude

$$\begin{aligned}\|\varepsilon^2\Delta_g\phi-W''(\omega)\phi\|_{C^{0,\alpha}_\varepsilon(M)} &= \|\phi^2\|_{C^{0,\alpha}_\varepsilon(M)}+\|\varepsilon J[\xi]\|_{C^{0,\alpha}_\varepsilon(M)}+\|E_1\|_{C^{0,\alpha}_\varepsilon(M)}\\ &= O(\varepsilon^2+\|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)})\end{aligned}$$

□

The following three lemmas estimate the L^∞ norm of ϕ.

Lemma 12.6 (L^∞-norm estimate far from Γ)**.** *There exist positive constants σ, R_0 such that, for all $R\geq R_0$,*

$$\|\phi\|_{L^\infty(M\setminus N(\varepsilon R))}= O(e^{-\sigma R}\|\phi\|_{C^{2,\alpha}_\varepsilon(M)}+\varepsilon^2+\|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)}).$$

Proof. Since $|\xi|= O(\varepsilon)$, there are positive constants R_0 and ε_0, such that $W''(\omega_\xi)>1$ on $M\setminus N(\varepsilon R/2)$ for $\varepsilon\in(0,\varepsilon_0)$. In particular, we can apply Lemma 8.3 with $\|\varepsilon^2\Delta_g\phi-W''(\omega_\xi)\phi\|_{C^{2,\alpha}_\varepsilon(M)}= O(\varepsilon^2+\|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)})$, $\rho=\tau-\varepsilon R$, $\Omega=M\setminus N(\varepsilon R/2)$, $c=W''(\omega_\xi)$, $c_0=1$ and $a=O(\varepsilon^2+\|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)})$. In this way

$$|\phi(p)|= O(\|\phi\|_{L^\infty(\partial N(\varepsilon R))}\times\max\{e^{-\sigma\,\mathrm{dist}(p,\partial N(\varepsilon R/2))/\varepsilon},e^{-\sigma(\tau/\varepsilon-R/2)}\}+\varepsilon^2+\|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)}).$$

Finally, we obtain the desired estimate when $p=(x,t)\in M\setminus N(\varepsilon R)$. □

Lemma 12.7 (L^2-norm estimate near Γ)**.** *For any fixed $R>0$, we have*

$$\varepsilon^{-n/2}\sup_{p\in M_{\varepsilon R}}\|\phi\|_{L^2(B(p,\varepsilon L))}= O\left(\varepsilon^2+\|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)}+e^{-\sigma R}\|\phi\|_{C^{2,\alpha}_\varepsilon(M)}\right).$$

Proof. We start by computing the equation satisfied by the L^2-norm along vertical directions of Γ, i.e. $V(x) = \int_J |\phi(x,t)|^2 dt = \|\phi(x,\cdot)\|^2_{L^2(J)}$, where $J = [-2R\varepsilon, 2R\varepsilon]$.

From the equation satisfied by ϕ, derived above, we get

$$\begin{aligned}\frac{\varepsilon^2}{2}\Delta_0 V &= \int_J \phi(\varepsilon^2\Delta_0\phi) + \int_J \varepsilon^2|\nabla_0\phi|^2\\ &\geq -\int_J \phi\ell(\phi) + \int_J \phi(E_2 - E_1) + O(\phi^3) + \varepsilon^2 J[\xi]\omega'_\xi\phi\\ &\geq -\int_J \phi\ell(\phi) - \frac{\gamma}{4}\int_J \phi^2 + \frac{4}{\gamma}\int_J E_1^2 + E_2^2 + \varepsilon^4(J[\xi]\omega'_\xi)^2.\end{aligned}$$

Define $\tilde\phi = \phi\rho$, where $\operatorname{supp}\rho \subset [-\tau,\tau]$ and $\rho \equiv 1$ on $[-\tau/2,\tau/2]$. Notice we have $\int_{\mathbb{R}} \tilde\phi\omega' = \int_{\mathbb{R}} \phi\omega' = 0$. Therefore, by the Lemma in the cutoff section we have

$$\begin{aligned}-\int_J \phi\ell(\phi) &= -\int_J \tilde\phi\ell(\tilde\phi)\\ &= -\int_{\mathbb{R}} \tilde\phi\ell(\tilde\phi) + \int_{|t|\geq\varepsilon R} \tilde\phi\ell(\tilde\phi)\\ &\geq \frac{\gamma}{2}\int_{\mathbb{R}} \tilde\phi^2 + o(\varepsilon^{\mathbb{N}}) - \|\phi\|_{C^2_\varepsilon(M)}\int_{|t|\geq\varepsilon 2R} |\phi|\rho\\ &\geq \frac{\gamma}{2}V_0 + o(\varepsilon^{\mathbb{N}}) - \|\phi\|^2_{C^2_\varepsilon(M)}\int_{|t|\geq\varepsilon 2R} ce^{-\sigma t/\varepsilon}dt\\ &\geq \frac{\gamma}{2}V_0 + O(\varepsilon e^{-\sigma 2R}\|\phi\|^2_{C^2_\varepsilon(M)}) + o(\varepsilon^{\mathbb{N}}).\end{aligned}$$

From the estimates we computed before we have

$$|E_2| + |\varepsilon^2 J[\xi]\omega'_\xi| = |E_2| + |\varepsilon J[\xi]| = O(\varepsilon^2 + \|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)}),$$

while, for $|E_1|$, since $H = 0$, we have

$$|E_1| = O(|\varepsilon^2 H_t\phi'| + |\varepsilon^2(\Delta_t - \Delta_0)\phi|) = O(\|\phi\|_{C^{2,\alpha}_\varepsilon(M)}) \times |t|,$$

Since $|J| = 2\varepsilon R$, for any fixed R it follows

$$\int_{-\varepsilon 2R}^{\varepsilon 2R} |E_1|^2 + |E_2|^2 + |\varepsilon^2 J[\xi]\omega'|^2 = O(\varepsilon^5 + \varepsilon\|\phi\|^4_{C^{2,\alpha}_\varepsilon(M)} + \varepsilon^3\|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)}).$$

Combining all the estimates the inequality for V_0 reads,

$$\frac{\varepsilon^2}{2}\Delta_0 V - \frac{\gamma}{4}V_0 \geq O(\varepsilon^5 + \varepsilon\|\phi\|^4_{C^{2,\alpha}_\varepsilon(M)} + \varepsilon^3\|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)} + \varepsilon e^{-\sigma 2R}\|\phi\|^2_{C^2_\varepsilon(M)}).$$

From the maximum principle we have

$$\|\phi(x,\cdot)\|^2_{L^2(-\varepsilon 2R,\varepsilon 2R)} = O(\varepsilon^5 + \varepsilon\|\phi\|^4_{C^{2,\alpha}_\varepsilon(M)} + \varepsilon^3\|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)} + \varepsilon e^{-\sigma 2R}\|\phi\|^2_{C^2_\varepsilon(M)}).$$

Finally, notice that since $R>0$ is fixed, for $p=(x,t)$ with $t=O(\varepsilon)$ we have

$$\begin{aligned}\|\phi\|^2_{L^2(B(p,\varepsilon R))} &= O(\varepsilon^{n-1})\times\|\phi(x,\cdot)\|^2_{L^2(-\varepsilon 2R,\varepsilon 2R)}\\ &= O(\varepsilon^{4+n}+\varepsilon^n\|\phi\|^4_{C^{2,\alpha}_\varepsilon(M)}+\varepsilon^{2+n}\|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)}+\varepsilon^n e^{-\sigma 2R}\|\phi\|^2_{C^2_\varepsilon(M)}),\end{aligned}$$

from which the estimate follows. □

Lemma 12.8 (L^∞-estimate near Γ)**.** *For any fixed* $R>0$ *and, we have*

$$\sup_{p\in M_{\varepsilon R}}\|\phi\|_{R^\infty(B(p,\varepsilon R/2))}=O\left(\varepsilon^2+\|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)}+e^{-\sigma R}\|\phi\|_{C^{2,\alpha}_\varepsilon(M)}\right).$$

Proof. The proof is an immediate consequence of Lemma 8.1 and the estimates for $\|\phi\|_{R^2(B(p,\varepsilon R))}$ and $\|\varepsilon^2\Delta_g\phi-W''(\omega_\xi)\phi\|_{L^\infty(M)}$. □

Finally, we obtain the main technical result of this section.

Corollary 12.9. $\|\xi\|_{C^2(\Gamma)}+\varepsilon^\alpha\|\xi\|_{C^{2,\alpha}(\Gamma)}+\|\phi\|_{C^{2,\alpha}_\varepsilon(M)}=O(\varepsilon^2+\|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)})$.

Proof. Combining both estimates we have the existence of R_0, such that for any fixed $R>R_0$, we have

$$\|\phi\|_{L^\infty(M)}=O\left(\varepsilon^2+\|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)}+e^{-\sigma R}\|\phi\|_{C^{2,\alpha}_\varepsilon(M)}\right).$$

Moreover, by the estimates from Proposition 12.5 and Lemma 8.2, we are able to bound the $C^{2,\alpha}_\varepsilon(M)$-norm of ϕ. We conclude by choosing R big enough, which allow us to absorb the term $e^{-\sigma R}\|\phi\|_{C^{2,\alpha}_\varepsilon(M)}$ on the lefthand side. This proves the bound for ϕ. From Proposition 12.4 and the invertibility of J it follows that $\|\xi\|_{C^2(\Gamma)}+\varepsilon^\alpha\|\xi\|_{C^{2,\alpha}(\Gamma)}=O(\varepsilon^2+\|\phi\|^2_{C^{2,\alpha}_\varepsilon(M)})$. □

Remark 4. Finally, we notice that the estimates obtained for the perturbation ξ and the error ϕ are in Fermi coordinates with respect to Γ. They have the same degree of homogeneity (with respect to ε) as in Pacard, [35]. Only the error term is presented in a different format. We have found ϕ so that $\int_{\mathbb{R}}\phi(x,t)\omega'(t-\xi(x))dt=0$. Denote by D_ξ a diffeomorphism which in Fermi coordinates corresponds to $D_\xi(x,t)=(x,t+\xi(x))$ for $(x,t)\in N(\tau/2)$ and that interpolates smoothly to the identity in $M\setminus N(\tau)$ as $|t|\to\tau$. Let $v=\phi\circ D_\xi$. Then, $\int_{\mathbb{R}}v\omega'=0$. If we define $v^\sharp=[\chi_1 v]_0^\perp$ and $v^\flat=v-\chi_4 v^\sharp$ we obtain the desired functions. Pacard [35] obtained solutions by means of a contraction mapping argument. This implies the uniqueness of solutions presenting these asymptotics (see the paragraph after the proof of Lemma 3.9 from [35]). This allow us to conclude:

Corollary 12.10. *If a sequence of solutions to* (1) *converges with multiplicity one to a non-degenerate minimal hypersurface then, for* ε *small enough, the solutions must be the ones constructed by Pacard in* [35].

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Marco A. M. Guaraco - The University of Chicago
E-mail address: guaraco@math.uchicago.edu