# Yet more divided power & $\delta$-ring material. (1)

Let $B$ be an $\mathbb{F}_p$-alg. with $x \in B$ a non-zero divisor.

Then $D_{(x)}(B) = B[y_1, \dots, y_k, \dots] / (py_1 - x^p, \dots, py_{k+1} - y_k^p, \dots)$

$y_k = \gamma_{p^k}(x) \cdot$ appropriate unit

$= \text{“} \dfrac{x^{p^k}}{(p^k)!} \text{”} \cdot$ unit

$= \text{“} \dfrac{x^{p^k}}{p^{p^{k-1} + \dots + p + 1}} \text{”}$

$$= B/(x^p)[y_1, \dots, y_k, \dots] / (y_1^p, \dots, y_k^p, \dots)$$

(We saw the analogous statement when $B$ is p.t.f. and $x$ is a non-zero divisor on $B/pB$.)

$\therefore$ If $B$ is furthermore an $A$-alg., and if $B/(x)$ is $A$-flat, then $D_{(x)}(B)$ is $A$-flat.

(Enough to see that $B/(x^p)$ is $A$-flat: for this, consider

$$\begin{array}{ll} B & \\ \cup & \cong B/(x) \\ (x) & \\ \cup & \cong B/(x) \\ (x^2) & \\ \cup & \\ \vdots & \\ \cup & \\ (x^{p-1}) & \\ \cup & \cong B/(x) \\ (x^p) & \end{array}$$

b/c $x$ is a non-zero divisor on $B$ )

As usual, this generalizes to $(x_1, \dots, x_r)$ being a regular sequence in $B$:

i.e. if $x_1, \dots, x_r \in B$ is a reg. sequence, and if $B/(x_1, \dots, x_r)$ is $A$-flat, then $D_{(x_1, \dots, x_r)}(B)$ is $A$-flat.

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We can rephrase this as follows: if $x_1, \dots, x_r \in B$ is regular relative to $A$, then $D_{(x_1, \dots, x_r)}(B)$ is $A$-flat.

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This rephrasing lets us promote this statement as follows:

if $A.$ is an animated ring,

if $A. \to B.$ is $p$-completely flat,

if $(x_1, \dots, x_r) \in \pi_0(B.)$ is $p$-comp. regular relative to $A.$,

then $D_{(x_1, \dots, x_r)}(B.)^{\wedge}$ ← derived $p$-completion

is $p$-comp. flat over $A$.

<u>Pf</u>: To check $p$-complete flatness, have to form

$$\mathrm{Kos}(A;p) \overset{L}{\underset{A.}{\otimes}} - \quad \text{and check flatness}$$

and can check this after $\pi_0(\mathrm{Kos}(A;p)) \overset{L}{\underset{\mathrm{Kos}(A;p)}{\otimes}} -$

Altogether, this means we can check after $\pi_0(A. \overset{L}{\otimes} \mathbb{F}_p) \underset{A.}{\otimes} -$

This base change undoes the derived $p$-completion, and reduces us to considering

$$\pi_0(A_\cdot \overset{\mathbb{L}}{\otimes} \mathbb{F}_p) \longrightarrow D_{(x_1,\dots,x_r)}\big( \underbrace{\pi_0(A_\cdot \overset{\mathbb{L}}{\otimes} \mathbb{F}_p) \underset{A_\cdot}{\otimes} B_\cdot}_{} \big)$$

This now a flat $\pi_0(A_\cdot \overset{\mathbb{L}}{\otimes} \mathbb{F}_p)$-alg.,

with $(x_1,\dots,x_r)$ being a regular sequence relative to $\pi_0(A_\cdot \overset{\mathbb{L}}{\otimes} \mathbb{F}_p)$

so the theorem follows from our earlier result $\square$

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Recall that we proved: if $A$ is a $p$-t.f. $\delta$-ring, and if $f_1,\dots,f_r \in A$ forms a regular sequence on $A/pA$, then

$$A\left\{\frac{\varphi(f_1)}{p},\dots,\frac{\varphi(f_r)}{p}\right\} \cong D_{(f_1,\dots,f_r)}(A).$$

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We combine this with the preceding result to prove the following:

If $A_\cdot$ is a $p$-complete animated $\delta$-ring, if $B_\cdot$ is a $p$-completely flat animated $\delta$-$A_\cdot$-alg., and if $x_1,\dots,x_r \in \pi_0(B_\cdot)$ is $p$-completely regular relative to $A_\cdot$,

then $C_\cdot := B_\cdot\left\{\frac{x_1}{p},\dots,\frac{x_r}{p}\right\}^{\wedge}$ derived $p$-completion is $p$-completely flat over $A_\cdot$.

<u>Proof</u>: <u>Two steps</u>: (i) Consider

$$C'_\bullet := B_\bullet\left\{\frac{\varphi(x_1)}{p},\dots,\frac{\varphi(x_r)}{p}\right\}^\wedge$$

$\|$ ⟵ by the comparison of δ-rings and D. P. envelopes that we recalled

$$D_{(x_1,\dots,x_r)}(B_\bullet)^\wedge,$$

which is p-completely flat over $A_\bullet$ by what we've already proved.

(ii) Replace the $x_i$ by the $\varphi(x_i)$:

$$\begin{tikzcd}
A_\bullet \arrow[r] \arrow[d, equal] & A_\bullet\{x_1,\dots,x_r\} \arrow[r] \arrow[d, "\varphi"] & B_\bullet \arrow[r] \arrow[d] & B_\bullet\{\frac{x_1}{p},\dots,\frac{x_r}{p}\} \arrow[d] \\
A_\bullet \arrow[r] & A_\bullet\{x_1,\dots,x_r\} \arrow[r] & B'_\bullet \arrow[r] & B'_\bullet\{\frac{\varphi(x_1)}{p},\dots,\frac{\varphi(x_r)}{p}\}
\end{tikzcd}$$

map of δ-rings defined by $x_i \mapsto \varphi(x_i)$, which is faithfully flat

The middle and right squares are both pushout (i.e. $\otimes^{L}$-product) squares, ∴ all vertical arrows are faithfully flat

derived p-completing gives:

$$\begin{tikzcd}
A_\bullet \arrow[r] \arrow[d, equal] & C_\bullet \arrow[d, "\text{p-completely f.flat}"] \\
A_\bullet \arrow[r, "\text{p-comp. flat}"'] & C'_\bullet
\end{tikzcd}$$

∴ $A_\bullet \to C_\bullet$ is p-comp. flat, as required. □