Sites, topoi, and Čech theory – a rapid overview

A site is a category in which certain classes of morphisms are declared to be "covers".

The basic model is the following:

If X is a top. space, then the category $\mathcal{T}_X$ whose objects are open subsets of X, and for which

$$\mathrm{Hom}(U,V) = \begin{cases} \phi & \text{if } U \not\subseteq V \\ * & \text{if } U \subseteq V \end{cases}$$

$\uparrow$ 1-pt. set

is a site, with covers being defined as usual.

This category is a subcat of Top, the category of top. spaces, but is not full. (There might be lots of maps $U \to V$ that are not related to the inclusion $U \subseteq V$ when we think of U & V as subsets of X).

We can remedy this by thinking about Top/X, the cat. of top. spaces equipped with a map to X.

Then if $U \underset{open}{\subseteq} X$, we regard U as an object of Top/X via the inclusion.

Then τ_X is a full subcat of Top/X

— it is equivalent to the full subcat where objects are $f: Y \to X$ for which Y is an open immersion of top. spaces.

Top/X can also be made into a site:

A collection of morphisms $\{Y_i \to Y\}_{i \in I}$ is a cover if it is an open cover.

τ_X is what is usually called a "small site"

Top/X is a "big site".

On any category $\mathcal{C}$ we have presheaves:

$$\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets}) =: \mathrm{PreSh}(\mathcal{C})$$

On a site we have a full subcategory of sheaves $=: \mathrm{Sh}(\mathcal{C})$.

We call $\mathrm{Sh}(\mathcal{C})$ a (Grothendieck) "topos" — it is a category with lots of properties that make it behave like the cat. of sets.

Very often, representable functors are sheaves: (we say $\mathcal{C}$ has a "subcanonical topology" if this holds)

then $\mathcal{C} \xhookrightarrow{\text{Yoneda}} \mathrm{Sh}(\mathcal{C})$

(Eg. this is true for Top/X — note that this big site has many more representable sheaves than τ_X.) There is a largest subcanonical top., called the "canonical top." on $\mathcal{C}$.

The topos $\mathrm{Sh}(\mathcal{C})$ has a nice feature: the sheaves on $\mathrm{Sh}(\mathcal{C})$ with respect to its canonical topology are <u>precisely</u> the representable sheave

$\therefore \quad Sh(\mathcal{C}) = Sh(Sh(\mathcal{C}))$

↑ sheaves w.r.t. its topology

sheaves w.r.t. the canonical top.

Now we can take cohomology, etc. on a site, just like on a top. space!

On a top. space, X is the final object (of τ_X, or Top/X), so has a categorical meaning, and $\Gamma(\mathcal{F}) = \mathcal{F}(X)$

= value of $\mathcal{F}$ on final object of $\mathcal{C}$.

In a general site, $\mathcal{C}$ may not have a final object, but $Sh(\mathcal{C})$ does: the constant sheaf $\{*\}$.

So define $\Gamma(\mathcal{F}) = \mathcal{F}(\{*\})$

sheaf on $\mathcal{C}$

final object of $Sh(\mathcal{C})$

now think of $\mathcal{F}$ as sheaf on $Sh(\mathcal{C})$

defn of representable sheaf

$= Maps_{Sh(\mathcal{C})}(\{*\}, \mathcal{F})$

- Now $\mathrm{Maps}_{\mathrm{Sh}(\mathcal{C})}(\{*\}, \mathcal{F})$

$$\underset{\varphi}{=} \mathrm{Maps}_{\mathrm{PreSh}(\mathcal{C})}(\{*\}, \mathcal{F})$$

$\mathrm{Sh}(\mathcal{C})$ is full in $\mathrm{PreSh}(\mathcal{C})$

= choice of a section $f \in \mathcal{F}(U)$

for <u>all</u> ~~objects~~ objects $U \in \mathcal{C}$,

compatible with <u>all</u> morphisms $U \to U'$ in $\mathcal{C}$.

Now can define $R\Gamma$ for a sheaf of ab. gp, etc.

Čech theory works quite generally:

if $\{Y_i\}_{i \in I}$ covers the final object $\{*\}$

some collection of sheaves – which in practice will be representable (or maybe Ind-representable)

then we form the Čech resolution

$$\to \prod_{\substack{i_0, \dots, i_n \\ \in I}} \mathcal{F}(Y_{i_0} \times \dots \times Y_{i_n}) \to \cdots$$

and (replacing $\mathcal{F}$ by an injective resolution) we get

$$E_1^{p,q} := \prod_{\substack{i_0, \dots, i_p \\ \in I}} \overset{\substack{RΓ^q \text{ on } \mathcal{C}/Y_{i_0} \times \dots \times Y_{i_p} \\ \|}}{H^q}(Y_{i_0} \times \dots \times Y_{i_p}, \mathcal{F})$$

$$\Rightarrow \underset{\substack{\| \\ RΓ^{p+q} \text{ on } \mathcal{C} \text{ itself}}}{H^{p+q}}(\mathcal{F}).$$

Often we can just replace $\{Y_i\}$ by the single object $\coprod_{i \in I} Y_i =: Y$.

Then $\underbrace{Y \times \dots \times Y}_{p \text{ times}} = \coprod_{i_0, \dots, i_p} Y_{i_0} \times \dots \times Y_{i_p}$,

and get the same complex, just phrased more succinctly as

$$H^q(\underbrace{Y \times \dots \times Y}_{p\text{-times}}, \mathcal{F}) \Longrightarrow H^{p+q}(\mathcal{F}).$$

<u>To elaborate</u>: ~~[illegible]~~ If we let

h_U denote the sheaf represented by U,

then we can replace the cover h_{Y_i} of $\{*\}$

by $\coprod h_{Y_i}$.

Now $\coprod h_{Y_i} = h_{\underbrace{\coprod Y_i}_{\text{coproduct in } \mathcal{C}}}$ exactly if

$\coprod Y_i$ is <u>disjoint</u>, in the sense that

~~[illegible]~~ each $Y_i \longrightarrow \coprod Y_i$ is a monomorphism,

and $Y_i \times_{\coprod Y_i} Y_j = \underbrace{\phi}_{\text{initial object of } \mathcal{C}}$ if $i \neq j$.

Indeed, in this case, the sheaf axiom shows that

$$F(\coprod Y_i) = \prod F(Y_i) = \mathrm{Hom}_{Sh(\mathcal{C})}(\coprod h_{Y_i}, F)$$
$$\| $$
$$\mathrm{Hom}_{Sh(\mathcal{C})}(h_{\coprod Y_i}, F)$$

for any sheaf F,

implying that indeed $h_{\coprod Y_i} = \coprod h_{Y_i}$.

If $\mathcal{C} = \tau_X$ (for example), then $\mathcal{C}$ admits coproducts, but these are simply unions of subsets of X, and so typically not disjoint ($\underset{\substack{\uparrow \\ \text{coproduct} \\ \text{in } \tau_X}}{U \coprod V} = U \cup V$,

$\therefore U \underset{U \coprod V}{\times} V = U \underset{X}{\times} V = U \cap V$, $\neq \phi$ in general),

$\therefore$ can't replace $\{Y_i\}$ by $\coprod Y_i$ in Čech computation.

But if eg. $\mathcal{C} = \mathrm{Top}/X$, then coproducts are just disjoint unions, and so can replace Y by $\coprod Y_i$.

This is why Čech computations are ~~often~~ just phrased as involving Y, $Y \times Y$, ..., $\underbrace{Y \times \cdots \times Y}_{n \text{ times}}$, ... rather than mentioning a cover explicitly.