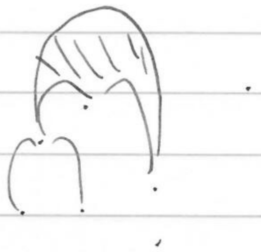
Simplicial sets



- pts
- joined by some 1-simplices
- 2-simplices

For each n, a set X_n of n-simplices.

$\partial_i : X_n \to X_{n-1}$ bd'y maps.
$i=0,\dots,n$.

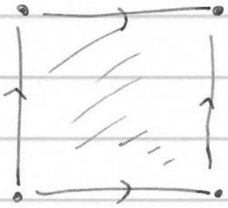
NB: There are also "degeneracy" maps
— I'll come to these in a minute.

If we actually glue together simplices according to the instructions given by $X_\cdot$, we get a top. space $|X_\cdot|$, the "geometric realization" of X.

So simplicial sets are a combinatorial model for top. spaces and (especially) their homotopy theory.

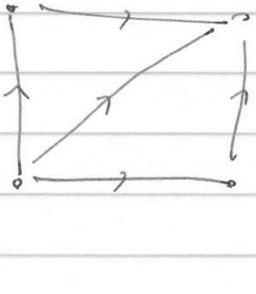
<u>products</u> The interval $\bullet \to \bullet$ is a simplicial set.

Let's form its product with itself:



This doesn't look "simplicial".

Let's triangulate it in the usual way:



We have 3 1-simplices in this picture. ③

If we project them onto each of the axes, only the diagonal actually projects to the 1-simplex in the interval itself.

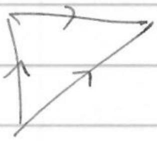
The edges project onto a point in either the horizontal or vertical axis.

This suggests adding "constant" ← or "degenerate" 1-simplices to $\bullet \to \bullet$ (one at each point),

so that all ~~[illegible]~~ 3 1-simplices in the (triangulated) product arise as products of 1-simplices in the two intervals we are taking the product of.

There a two 2-simplices in the (triangulated) product. It we project them onto the two axes, neither ~~[illegible]~~ stays a 2-simplex.

Eg the upper ~~[illegible]~~ left 2-simplex

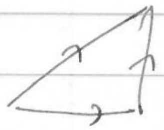


when we project it to the vertical axis, lies ~~[illegible]~~ crushed along the interval

Two if its edges are also ~~[illegible]~~ project to this interval; the

3rd edge project to the "constant" 1-simplex at the top vertex.

Similarly, the lower right 2-simplex

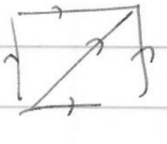


projects to a "crushed" 2-simplex lying along the interval

2 of its edges also project to this interval. The 3rd projects to the constant 1-simplex at the bottom vertex.

So we should add these "crushed", or "degenerate", 2-simplices to • ⟶ • as well:

now the visible 2-simplices in



arise as products of these degenerate 2-simplices in • ⟶ •.

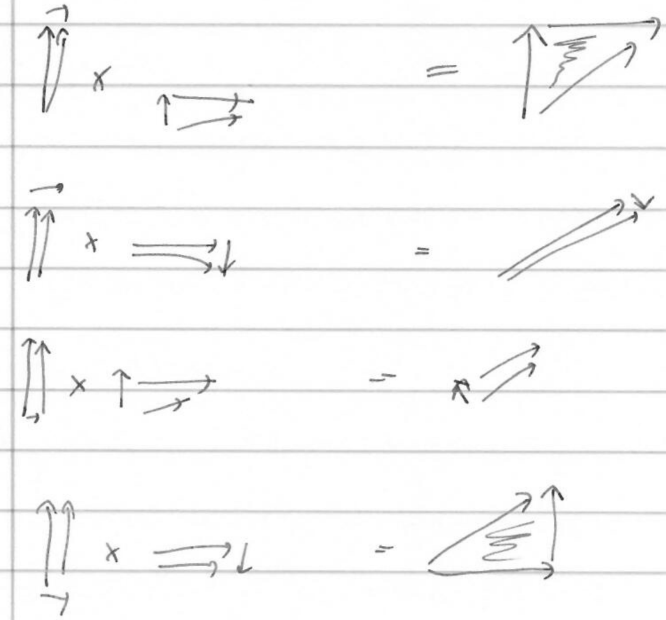
1 non-degen., 2 degen.

Since • ⟶ • now has 3 1-simplices, there are 9 product 1-simplices in the product square — 5 of them are the "visible", non-degen 1-simplices; the other 4 are degenerate, or constant, at the 4 vertices of the square.

And since each interval factor now has two 2-simplices, both degenerate, we have 4 2-simplices in the product square.

Two of them are the "visible", "non-degenerate" 2-simplices.

The other 2 are still degenerate:



They are the 2 degenerate 2-simplices lying along the diagonal 1-simplex.

to be consistent / general

But shouldn't we also have degenerate 2-simplices along the 4 edges?

To get these, we need to add *more* degen 2-simplices to each interval — namely a degenerate 2-simplex at each vertex.

∴ each interval actually has 4 2-simplices

∴ the product has 16 2-simplices:

2 nondegen

2×5 = 10 degenerate 2-simplices along the 5 edges

4×1 = 4 degenerate 2-simplices ~~also~~ at each vertex.

—

To be consistent, we should add degenerate 3-simplices, 4-simplices, etc. as well. These products will *all* be degenerate, though, so we don't "see" them in the square.

In general, simplices are determined by an ordered set of vertices, maps b/w them are determined by monotone maps on the vertices:

faces are embeddings $\{n \text{ ordered vertices}\} \hookrightarrow \{n+1 \text{ ordered vertices}\}$

$n-1$ simplex $\hookrightarrow$ n simplex.

In general, a map $\Delta_m \longrightarrow \Delta_n$ factors as

$$\Delta_m \twoheadrightarrow \Delta_k \hookrightarrow \Delta_n$$

degenerate m-simplex supported on a k-face of an n-simplex.

So a simplicial set $X_\cdot$ gives a collection X_n of n-simplices for each n, and if

$$\Delta_m \to \Delta_n \quad , \text{ get } X_n \to X_m$$

each n-simplex gives rise to corresponding m-simplices

In short, if Δ = cat. of finite ordered sets with order-preserving maps, Then a simplicial set is a contravariant functor (= presheaf)

$$X_\bullet : \Delta^{op} \to \text{Sets}.$$

Δ_n itself is the presheaf represented by $\{0, \dots, n\}$.

Yoneda gives $\Delta \hookrightarrow$ S. Sets, s.t.

$$\text{Hom}_{S.\,Sets}(\Delta_n, X_\bullet) = X_n .$$

So it all works out!

Ex: If X is a set, $X_n = X$ is "constant" simp. set, just X as a set of pts., plus all degenerate simplices

$X_\bullet \times Y_\bullet$ is just the product functor,

ie $(X_\bullet \times Y_\bullet)_n = X_n \times Y_n$,

and the above example illustrates how this really matches w/ the intuitive 'geometric' product.

More precisely, if $|\ |$ denote geometric realization, as above, then

$|\ |$ commutes with finite limits.

In particular $|X_\bullet \times Y_\bullet| = |X_\bullet| \times |Y_\bullet|$.

to be precise, work in cat. of compactly gen. Hausdorff spaces

---

If T is a top. space,

$\mathrm{Sing}_\bullet(T)$ = s. set of singular simplices in T

$$\left(\begin{array}{l} \mathrm{Sing}_n(T) := \mathrm{Maps}(|\Delta_n|, T) \\ \quad \| \\ \mathrm{Hom}_{s\text{-sets}}(\Delta_n, \mathrm{Sing}_\bullet(T)) \end{array} \right)$$

In fact, $|\ |$ is left adjoint to Sing.

(The preceding defn is this fact applied to the object $\Delta_n \in$ s. sets)

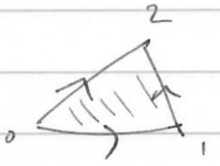
$\therefore$ $|\ |$ preserves arbitrary colimits.

$|\mathrm{Sing}_\bullet(T)| \longrightarrow T$ is a weak equivalence ("CW replacement").

We also have maps $X_\bullet \to \mathrm{Sing}_\bullet(|X_\bullet|)$, which will also be a weak equivalence — a "fibrant replacement" — as I now try to explain.

We Can "do homotopy theory" w/ simplicial sets, but not all s. sets are well-adapted to this.

Eg Δ_2 :

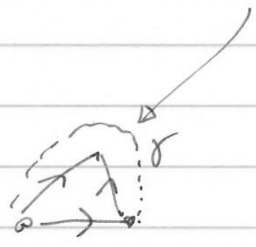


This is much smaller than $\mathrm{Sing}_\bullet(|\Delta_2|)$.

The adjunction gives

$$\Delta_2 \hookrightarrow \mathrm{Sing}_\bullet(|\Delta_2|)$$

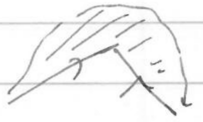
but in the latter, we have a 1-simplex γ



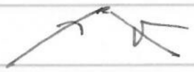
joining 0 & 1 by passing around the top (indeed, many such, depending on parametrization), and degenerate 2-simplices filling in , and of course many more simplices.

A 2-simplex "filling in"

with another edge and 2-face



is called a "filling" of the "horn"



.

Simplicial sets which admit all horn fillings are called *Kan complexes*

Eg. Sing.(T) is always a Kan complex.

Top. spaces have a "model structure" ↙ "Quillen model str."
with weak equivalences being usual weak homotopy equivalences, and fibrations being Serre fibrations.

Note that *any* morphism T → pt. is a Serre fibration, so all objects are fibrant.

∃ a "matching" model structure on s-Sets too.
("Quillen model st" again!)

$| \; | : \text{s-Sets} \longrightarrow \overset{\text{top.}}{\text{spaces}}$

is a Quillen equivalence.

⊛ $X_\cdot \to Y_\cdot$ is a weak equiv iff $|X_\cdot| \to |Y_\cdot|$ is.

⊛ cofibrations are just inclusions $X_\cdot \hookrightarrow Y_\cdot$

⊛ fibrations have to satisfy a homotopy lifting property in terms of "horn-filling":

If $\overset{\text{horn}}{\Lambda} \hookrightarrow \Delta_n$ is a commutative diagram,

$$\begin{tikzcd}
\Lambda \arrow[r, hook] \arrow[d] & \Delta_n \arrow[d] \arrow[dl, dashed] \\
X_\cdot \arrow[r] & Y_\cdot
\end{tikzcd}$$

we should be able to lift as indicated.

In particular, not all objects are fibrant; exactly the Kan complexes are.

NB. There is another natural model str. on s.Sets, the "Joyal model str", which has to do with ∞-cat. theory — in this picture, simplicial sets are related to categories rather than spaces.

~~If we restrict to Kan complexes (full sub. cat. of simpli~~

Internal hom : $\underline{\mathrm{Hom}}(X_\bullet, Y_\bullet)$ is a s-set, defined by

$$\mathrm{Hom}(U_\bullet, \underline{\mathrm{Hom}}(X_\bullet, Y_\bullet))$$

$$:= \mathrm{Hom}(U_\bullet \times X_\bullet, Y_\bullet)$$

eg $\underline{\mathrm{Hom}}(X_\bullet, Y_\bullet)_n := \mathrm{Hom}(\Delta_n \times X_\bullet, Y_\bullet)$

So s-Set is enriched over s-Sets.

If $Y_\bullet$ is Kan, then $\underline{\mathrm{Hom}}(X_\bullet, Y_\bullet)$ is Kan.

fibrant & cofibrant ↓ ; weak equiv. of simp. cats.

So if we restrict to {Kan complexes} $\hookrightarrow$ s-Set., we get a $\underbrace{\text{"locally Kan" simplicial category}}_{\text{one model for } \infty\text{-cats.}}$,

This gives the ∞-cat of "anima" (= animated sets)

Now we consider simplicial gps., ab. gps., rings.

$$X_\bullet : \Delta^{op} \to \text{Groups} / \text{Ab. Groups} / \text{Rings}.$$

Forgetful functor:

$$\begin{tikzcd} s\text{-Gps.} \arrow[r] & s\text{-Sets} \\ s\text{-Ab.} \arrow[ur] & \\ s\text{-Ring} \arrow[uur] & \end{tikzcd}$$

Left adjoint:

$$\begin{tikzcd} & s\text{-Gps} \\ s\text{-Sets} \arrow[ur] \arrow[r] \arrow[dr] & s\text{-Ab.} \\ & s\text{-Rings} \end{tikzcd}$$

free gp. / ab gp. / ring.

Quillen adjunction: * Right adjoint preserves fibrations (so we can test fibrations on underlying s-sets)

* Left adjoint preserves cofibrations (so only certain objects, namely free objects, are known to be cofibrant a priori)

⑮

<u>Lemma</u>: Any simplicial gp. is a <u>Kan complex</u>.

So <u>all</u> objects are fibrant. But not all objects are cofibrant.

If $X_\bullet$ consists of free objects term-by-term, then $X_\bullet$ <u>is</u> cofibrant (at least in ab. gp. / ring case).

Using "free resolutions", we can replace any $X_\bullet$ by a weakly equivalent cofibrant object $U_\bullet \xrightarrow[\text{w.e.}]{\sim} X_\bullet$

↑ cofibrant

both fibrant & cofibrant

$\therefore$ {cofibrant} $\xhookrightarrow[\text{w.e. of s-cats.}]{}$ s-Ab gp. / s-rings

↳ (see over)

This is again a "locally Kan" simplicial category, ∴ an ∞-cat.

"Animated abelian gps."
"Animated rings"

Brief
Explanation of

Simplicial cat structure:

$X_\cdot, Y_\cdot$ s-gps or rings

then if $U_\cdot$ is any s-set,

$\mathrm{Hom}(U_\cdot, Y_\cdot)$ is not just a set, but is a gp/ring.

and $\underline{\mathrm{Hom}}(U_\cdot, Y_\cdot)$ is a s-gp. or s-ring.

(Just as Maps from any space to a top gp or ring forms a gp or ring, via [illegible] pointwise operation).

$$\therefore \ \mathrm{Hom}_{s\text{-sets}}(U_\cdot, \underline{\mathrm{Hom}}_s(X_\cdot, Y_\cdot))$$

test s-set

simplicial hom of simp. gps/rings

$$= \mathrm{Hom}(U_\cdot \times X_\cdot, Y_\cdot)$$

s-sets, gp. or ring hom in 2nd variable

$$= \mathrm{Hom}(X_\cdot, \underline{\mathrm{Hom}}(U_\cdot, Y_\cdot))$$

s-gps. or s-rings

s-sets

a s-gp. or s-ring

define $\underline{\mathrm{Hom}}(X_\cdot, Y_\cdot)$ in terms of things we already know.

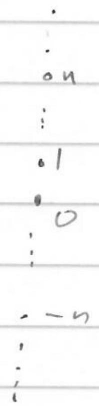
Shift to examples

$\mathbb{Z}/n\mathbb{Z}$. Simple ex. of an (non-free!) ab. gp

hom. alg. p.o.v. $0 \to \mathbb{Z} \xrightarrow{n\times} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$

simplicial p.o.v.:

$\mathbb{Z}$:



set of 0-simplices

Want to identify 0 & n, ∴ introduce a 1-simplex

Perform gp. operations on γ to get $\mathbb{Z}\cdot\gamma$

$$\begin{tikzcd} \mathbb{Z}\cdot\gamma \arrow[r, shift left, "\partial_1"] \arrow[r, shift right, "\partial_0"'] & \mathbb{Z} \end{tikzcd}$$

$\partial_1(\gamma)=n$
$\partial_0(\gamma)=0$

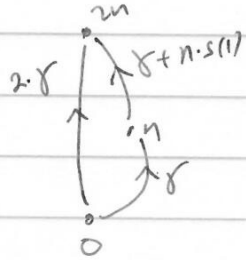
degenerate 1-simplices:

$$s : \underset{\underset{\mathbb{Z}\cdot 1}{\shortparallel}}{\mathbb{Z}} \longrightarrow \mathbb{Z}\cdot s(1)$$

So actually, need $\mathbb{Z}\langle \gamma, s(1)\rangle \underset{\partial_0}{\overset{\partial_1}{\rightrightarrows}} \mathbb{Z}$

$\partial_1 \gamma = n \quad \partial_1 s(1) = 1$

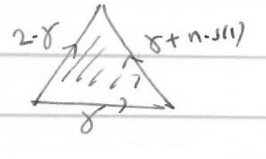
$\partial_0 \gamma = 0 \quad \partial_1 s(1) = 1$.



a "loop"

We are trying to make a cofibrant s-ab. gp.
That is w.equiv. to $\mathbb{Z}/n\mathbb{Z}$ (discrete gp),
so we need to ~~fill~~ [illegible] fill in this loop.

Add △ [illegible] a 2-simplex



More general set-up :

$$\begin{tikzcd} B \arrow[r, shift left=3, "\partial_1"] \arrow[r, shift right=3, "\partial_0"'] & A \arrow[l, "s"'] \end{tikzcd} \qquad \partial_1 s = \partial_0 s = \mathrm{id}_A$$

ab. gps.

(For us, $B = \mathbb{Z}\langle \gamma, s(1) \rangle$, $A = \mathbb{Z}$)

Given $b_1, b_2 \in B$, we get a "loop"

$$\begin{tikzcd} \partial_0(b_1) + \partial_0(b_2) \arrow[rr, "b_1 + b_2"] \arrow[dr, "b_1 + s\partial_0(b_2)"'] & & \partial_1(b_1) + \partial_1(b_2) \\ & \partial_1(b_1) + \partial_0(b_2) \arrow[ur, "s\partial_1(b_1) + b_2"'] & \end{tikzcd}$$

(Previous example was $b_1 = b_2 = \gamma$)

Add a 2-simplex labelled by (b_1, b_2), that fills in this loop.

Actually the loop only depends on $(b_1, b_2) \in B \underset{A}{\oplus} B$. (i.e. (b_1, b_2) & $(b_1 + s(a), b_2 - s(a))$ give same loop, if $a \in A$)

∴ define

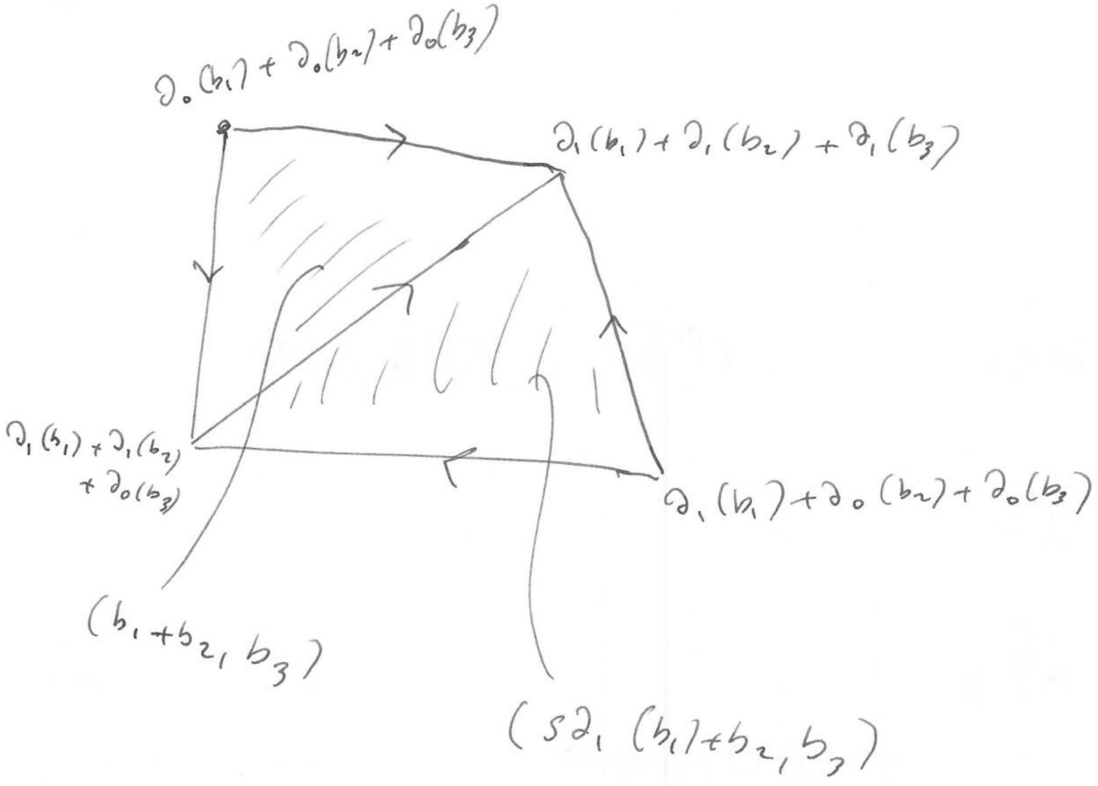
$$B \underset{A}{\oplus} B \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; B$$

via

$$\begin{aligned} (b_1, b_2) &\longmapsto b_1 + s\partial_0(b_2) \\ &\longmapsto b_1 + b_2 \\ &\longmapsto s\partial_1(b_1) + b_2 \end{aligned}$$

But now, having added all these 2-simplices, we get unwanted 2-spheres that have to be filled in with 3-cells, and so on.

Here is part of one such:



(The other 2 faces are labelled by

(b_1, b_2+b_3) & $(b_1, b_2+s\partial_0(b_3))$)

In general, we add n-cells labelled by elts. of $B_A^{\otimes n}$

$$\begin{tikzcd} B_A^{\oplus n} \arrow[r, shift left=3, "d_n"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "d_0"'] & B_A^{\oplus (n-1)} \end{tikzcd}$$

$$d_0 : (b_1, \dots, b_n) \longmapsto (s\partial_1(b_1) + b_2, \dots, b_n)$$

$$d_i : (b_1, \dots, b_n) \longmapsto (b_1, \dots, b_i + b_{i+1}, \dots, b_n)$$

$1 \leq i \leq n-1$

$$d_n : (b_1, \dots, b_n) \longmapsto (b_1, \dots, b_{n-1} + s\partial_0(b_n))$$

deg. maps $$\begin{tikzcd} B_A^{\oplus (n-1)} \arrow[r, shift left=3] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3] & B_A^{\oplus n} \end{tikzcd}$$

s_i for $i = 1, \dots, n$

given by inserting "0" in the i^{th} position.

Now $B = s(A) \oplus \ker \partial_0$

$\therefore \; B^{\otimes_A n} = s(A) \oplus (\ker \partial_0)^{\otimes n}$

so we can unwrap all this pretty explicitly.

Lemma If $\partial_1|_{\ker \partial_0} : \ker \partial_0 \to A$ is injective, then

$$\overrightarrow{\rightrightarrows} \; B^{\otimes n}_A \; \rightrightarrows \; - - \; B \rightrightarrows A$$

is weakly equiv. to $A / \partial_1(\ker \partial_0)$

How to check such a thing?

Thm If $X_\cdot$ is a simplicial ab. gp.,

let $\mathrm{tot}(X_\cdot)$ = chain complex constructed from $X_\cdot$.

Then $H_i(\mathrm{tot}(X_\cdot)) = \pi_i(X_\cdot)$

In our example, with $X_n = B_A^{\oplus n}$, we find that

$$\mathrm{tot}(X_\bullet) \xrightarrow[q.i]{\sim} \left[\ker \partial_0 \xrightarrow{\partial_1} A \right].$$

Pf.:

$$\begin{tikzcd}
(\ker \partial_0^{\oplus n})_{n \geq 0} \arrow[d, hook] & \leftarrow \text{very simple differentials!} & (x_1, \dots, x_n) \mapsto \left\{ \begin{array}{l} (x_2, \dots, x_n) \\ \vdots \\ (x_1, \dots, x_i + x_{i+1}, \dots) \\ (x_1, \dots, x_{n-1}) \end{array} \right. \\
(B_A^{\oplus n})_{n \geq 0} = X_\bullet & & \\
A_\bullet \arrow[u, hook] \arrow[d, phantom, "\wr"] & & \\
\text{constant simplicial ob.} & &
\end{tikzcd}$$

Taking total complexes & applying Snake lemma gives

$$\begin{tikzcd}
0 \arrow[r] & H_1(\mathrm{tot}(X_\bullet)) \arrow[r] & \ker \partial_0 \arrow[r, "\partial_1"] & A \arrow[d, bend left] \\
& & & H_0(\mathrm{tot}(X_\bullet)) \to 0
\end{tikzcd}$$

& all higher $H_i(\mathrm{tot}(X_\bullet)) = 0$. □

This proves the lemma, given the theorem. In particular, it gives us a cofibrant simplicial ab. gp. model for $\mathbb{Z}/n\mathbb{Z}$.

The thm. is related to the

Dold-Kan equivalence

$$X_\cdot \longmapsto \mathrm{tot}(X_\cdot)$$

gives an equivalence of ∞-cats

$$\text{"animated ab. gps."} \longleftrightarrow D_{\geq 0}(Ab)$$

(∞-cat arising from simplicial ab gps)

derived cat of non-negatively graded chain complexes of ab. gps.

The previous theorem says that $\pi_i \longleftrightarrow H_i$ under this equivalence.

Remark: If we replace $\mathrm{tot}(X_\cdot)$ by a certain quasi-isomorphic subcomplex, we even get an equiv. b/w simplicial ab. gps. and non-negative chain complexes of ab. gps.

This is more elementary to state, and not so hard to prove, and gives the above ∞-cat. version by inverting weak equivalences on each side. But ∞-version seems more natural to me.

Ex: Suppose A is an ab. gp, place in homo. degree n.

Then the corresponding $X_\bullet$ is a simplicial ab. gp. whose underlying s. set is $K(A,n)$.

Why should $K(A,n)$ come from an ab gp object? B/c it is an ∞-loop space!

$$\left(K(A,n) = \Omega K(A,n+1) \right).$$

Recall how to make a $K(A,1)$:

Fix a pt.: this will be x_0 (i.e. X_0 is a singleton), i.e. trivial gp.

Add loops labelled by elts. of A :

so $$X_1 = A. \underset{0}{\overset{0}{\rightrightarrows}} X_0 = 0$$

Now we want $[a+b] \sim [a] + [b]$,

∴ add Δ_2's labelled by pair of elts,

so $$X_2 = A \oplus A \quad , \quad \ldots$$

—

In fact we are in ~~[illegible]~~ a special case of the context we considered above! $A \rightrightarrows 0$

$$X_n = A^{\oplus n}$$

and so $H_0(\operatorname{tot}(X_\cdot))$

$$= H_0\left(\underset{1}{A} \to \underset{0}{0} \right)$$

$$= \begin{cases} A & \text{in deg. } 1 \\ 0 & \text{in all other degrees} \end{cases}$$

This example illustrates how to prove the Dold-Kan equivalence, and why

$\pi_\cdot \longleftrightarrow H_\cdot$ ...

For full proof, can combine this basic idea w/ Postnikov tower techniques (write a simplicial set/ch. gp. as an iterated fibration by $K(\pi, n)$'s)

~~If~~ ~~$\{X(n)_\cdot\}_{n\geq 0}$ is an inverse sequence~~

~~Let~~

If $X_\cdot \to Y_\cdot$ is a morphism of simp. ab. gps, it is a fibration iff the induced map $\underbrace{X^\circ_\cdot}_{\text{conn. comp. of identity}} \to Y^\circ_\cdot$ is surjective term by term.

(This holds if eg. $X_\cdot \to Y_\cdot$ is itself surjective term by term.)

If $\{X(n)_\bullet\}_{n\geq 0}$ is an inverse sequence of simplicial ab. gps., we can replace it by a sequence w/ fibration transition maps:

$$\begin{tikzcd}
Y(n+1)_\bullet \arrow[r, "\sim"] \arrow[d, "\text{fibration}"] & X(n+1)_\bullet \arrow[d] \\
Y(n)_\bullet \arrow[r, "\sim", "\text{weak equiv.}"'] & X(n)_\bullet
\end{tikzcd}$$

and now the transition maps for the $Y(n)_\bullet$'s are "almost" surjective (only possible failure of surjectivity is on π_0's),

and so $\mathrm{tot}\left(\varprojlim Y(n)_\bullet\right) = \tau_{\geq 0} R\varprojlim^{\mathrm{tot}}(X(n)_\bullet)$

$\|$ homotopy inverse limit

← using homological conventions, s.t. $R^1\lim$ is in degree -1.

So if $X(n)_\bullet$ is an inverse sequence for which $R\varprojlim \text{tot}(X(n)_\bullet)$ is in degree ≥ 0, then we can compute the $R\varprojlim$ in the simplicial world as a homotopy inverse limit.

Note: $R\varprojlim$ satisfies a universal property on $D(Ab)$ (as a limit in an ∞-categorical sense), and so $\tau_{\geq 0} R\varprojlim$ does also, on $D_{\geq 0}(Ab)$.

Similarly, the homotopy inverse limit computed by replacing the $X(n)_\bullet$ by $Y(n)_\bullet$ computes an ∞-categorical limit. Thus the matching of the homotopy inverse limit with $\tau_{\geq 0} R\varprojlim$ under the Dold-Kan equivalence also holds for general conceptual reasons.