Prisms

A δ-pair is a pair (A, I) with A a δ-ring and $I \subseteq A$ an ideal.

<u>Def.</u> A <u>prism</u> is a δ-pair (A, I) s.t. I is ~~Zariski locally principal~~ invertible, i.e. Zariski locally generated by a non-zero divisor, A is derived (p, I)-complete, and $p \in (I, \varphi(I))$.

<u>Types of prisms</u>

- perfect : $\varphi : A \xrightarrow{\simeq} A$
- bounded : A/I has bounded p-power torsion, i.e. $(A/I)[p^\infty] = (A/I)[p^N]$ $\exists N \gg 0$
- Crystalline : $I = (p)$.

<u>Remark:</u> The condition $p \in (I, \varphi(I))$ can be promoted to $p \in (I^p, \varphi(I))$.

Indeed, since A is derived (p, I) ~~adic~~-complete, we have $(p, I) \subseteq \mathrm{Rad}(A)$, so our results on distinguished elements apply to show that $\exists\ A \to A'$ faithfully flat, and a morphism of δ-rings, s.t. (2)

$\underline{I'} = IA' = (d)$ with d distinguished, i.e. with $\delta(d) \in A^\times$. (We also can ensure $(p, d) \subseteq \mathrm{Rad}(A')$, which we don't need now, but will use later.)

Then $\varphi(d) = d^p + p \underbrace{\delta(d)}_{\text{a unit}}$, $\therefore p \in (d^p, \varphi(d)) = (I'^p, \varphi(I'))$

By $\underline{\text{faithfully flat}}$ descent, $p \in (I^p, \varphi(I))$.

We need another distinguished element fact:

$\underline{\text{Lemma}}$ (Lemma 1.7 of Bhatt's lecture 3)

If $d \in A$ (a δ-ring) is distinguished, if $(f, p) \subseteq \mathrm{Rad}(A)$, and if $d = fg$, then g is a unit, $\therefore (d) = (f)$.

$\underline{\text{Pf:}}$ $\delta(d) = \delta(f) \cdot g^p + \varphi(f) \cdot \delta(g)$

Since $f \in \mathrm{Rad}(A)$, so is $\varphi(f)$ while $\delta(d) \in A^\times$ by assumption.

$$\text{Thus } \delta(f) \cdot g^p = \underbrace{\underset{\text{unit}}{\delta(d)} - \underset{\text{elt. of } \mathrm{Rad}(A)}{\varphi(f) \cdot \delta(g)}}_{\in A^\times}$$

$\therefore g \in A^\times$. $\square$

We now have the following lemma:

<u>Lemma</u> If $d \in A$ (δ-ring) is distinguished,

if $(p,d) \subseteq \mathrm{Rad}(A)$, and if $p = x\cdot d^p + y\varphi(d)$, then $y \in A^\times$

<u>Remark</u>: of course, $\varphi(d) = d^p + p\cdot\delta(d)$,

$\therefore\ p = -\delta(d)^{-1}\cdot d^p + \delta(d)^{-1}\varphi(d)$,

so in this case $y = \delta(d)^{-1}$ is a unit.
The lemma ensures that this is the general behavior of the coefficient y.

<u>Pf</u>: If y is a unit mod (p,d), then it is a unit (since $(p,d) \subseteq \mathrm{Rad}(A)$), $\therefore$ by way of obtaining a contradiction, assume that (p,d,y) is a proper ideal, and let B = localization of A along this ideal.
Then B is a non-zero δ-ring with $(p,d,y) \subseteq \mathrm{Rad}\, B$, and

$$p = x\cdot d^p + y\varphi(d) = x\cdot d^p + y\cdot d^p + yp\,\delta(d)$$

$$\therefore\ p\underbrace{(1 - y\,\delta(d))}_{\in B^\times,\ \text{since } y \in \mathrm{Rad}(B)} = d\,(x+y)\cdot d^{p-1}$$

Now p is distinguished (in <u>any</u> δ-ring!!)

and $p, d \in \mathrm{Rad}\, B$, $\therefore$ by the preceding lemma (Bhatt's lemma 1.7) we have

$(x+y) \cdot d^{p-1}$ is a unit. Since $p \geq 2$,

we find that d is a unit, contradicting that $d \in \mathrm{Rad}(B)$ (since $B \neq 0$).

Thus y must have been a unit in A, as claimed. $\square$

<u>Corollary</u> (Lemma 3.5 of Bhatt's 3rd lecture)

If (A, I) is a prism, then $\varphi(I)A$ is principal and generated by a distinguished element.

<u>Pf:</u> Since $p \in (I^p, \varphi(I))$, we can write

$$p = a + b \qquad a \in I^p, \ b \in \varphi(I) \cdot A$$

Choose faithfully flat $A \to A'$ as above.

Then $I \cdot A' = (d)$, $a = x \cdot d^p$ $b = y\, \varphi(d)$ $x, y \in A'$.

The preceding lemma shows that $y \in (A')^\times$, $\therefore$ $b \cdot A' = \varphi(d) \cdot A' = \varphi(I) \cdot A'$.

$\therefore$ by faithfully flat descent, we have

$$b \cdot A = \varphi(I) \cdot A .$$

$\square$

<u>Rigidity</u>: If $(A, I) \to (B, J)$ is a morphism of prisms, then $IB = J$.

<u>Pf</u>: Choose $A \to A'$ as above. Consider

$$\begin{tikzcd} A' \arrow[r] & A' \otimes_A B \arrow[r] & B' \arrow[r] & B'' \\ A \arrow[u] \arrow[r] & B \arrow[u] \arrow[ur] & & \end{tikzcd}$$

localization analogous to $A \to A'$

δ-ring

localization of $A' \otimes_A B$ along (p, J).

This is flat, $\operatorname{Spec} B' \to \operatorname{Spec} B$ contains $V(p, J)$, $\therefore$ faithfully flat, since $(p, J) \subseteq \operatorname{Rad} B$

We have $B \to B''$ faithfully flat, with $(p, J) \subseteq \operatorname{Rad} B''$ and $J \cdot B''$ gen'd by distinguished element.

$A \to B''$ factors through A', $\therefore$ $I \cdot B''$ also gen'd by distinguished element, and $I \cdot B'' \subseteq J B''$.

Lemma 1.7 then shows that

$$I \cdot B'' \xrightarrow{\sim} J \cdot B'',$$

and so by faithfully flat descent, we have

$$I \cdot B = J, \quad \text{as claimed.} \quad \square$$

If (A, I) is a prism, and B is a δ-A-algebra that is also (p, I)-derived complete, then (B, IB) will be a prism provided that IB is invertible.

For this, we need $I \otimes B \xrightarrow{\sim} IB$, which holds precisely when $B[I] = 0$.

(For any A-module M, the inclusion $M[I] \hookrightarrow M$ induces an inclusion

$$M[I] \otimes I \hookrightarrow M \otimes I$$

(b/c I is flat over A, being invertible)

$$M[I] \otimes I \cong M[I] \otimes I/I^2$$

and in fact $M[I] \otimes I/I^2 \hookrightarrow \ker(M \otimes I \to I \cdot M)$

Claim: $M[I] \otimes I/I^2 \xrightarrow{\sim} \ker(M \otimes I \to IM)$

Pf: Check locally over Spec A, $\therefore$ assume I is principal, say $I = (f)$; in which case it is clear: it amounts to the claim that $M[f] = \mathrm{Kernel}(M \xrightarrow{\times f} M)$.)

If A is a bounded prism, then A is classically (p,I)-adically complete.

Pf: By assumption, A is derived (p,I)-complete

i.e.
$$A \simeq R\varprojlim_{m,n} \left(\begin{tikzcd} I^m \arrow[r, hook] & A \\ I^m \arrow[r, hook] \arrow[u, "p^n"] & A \arrow[u, "p^n"'] \end{tikzcd} \right)$$

here we use that I is locally principal, generated by a non-zero divisor

$$\simeq R\varprojlim_{m,n} \left(A/I^m \xrightarrow{p^n} A/I^m \right)$$

$$\simeq R\varprojlim_{m,n} \left(A/(I^m, p^n) \right)$$

b/c A/I^m has bounded p-power torsion

$$\simeq \varprojlim_{m,n} A/(I^m, p^n)$$ b/c the transition maps are surjective.

□