Perfect δ-rings and prisms

If R is a perfect $\mathbb{F}_p$-algebra, then $W(R)$ has the following description:

- $W(R)$ is p-adically complete & p-torsion free,

  so $W(R) = \varprojlim W(R)/p^n W(R)$

$$\begin{array}{c} \\ \Vert \\ W_n(R) \end{array}$$

- $W_1(R) = W(R)/pW(R) = R$

- $\varphi : W(R) \circlearrowleft$ is an automorphism

  (i.e. $W(R)$ is perfect).

We have the canonical lift

$[\ ] : R \to W(R)$, a multiplicative homo.,

defined as follows:

$W(R)$

If $\bar{x} \in R$, choose y_n lifting $\bar{x}^{\frac{1}{p^n}}$, define $x_n = y_n^{p^n}$.

Then $[\bar{x}] := \lim_{n \to \infty} x_n$

(If $a \equiv b \bmod p$, then $a^{p^n} \equiv b^{p^n} \bmod p^{n+1}$,
$\therefore$ since $y_{n+1}^p \equiv y_n \bmod p$
we have $x_{n+1} \equiv x_n \bmod p^{n+1}$,
and the limit exists.)

Also $\varphi([\bar{x}]) = [\bar{x}^p] = [\bar{x}]^p$.

($\varphi(y_{n+1}) \equiv y_{n+1}^p \equiv y_n \bmod p$,

$\therefore$ $\varphi(x_{n+1}) \equiv x_n^p \bmod p^{n+2}$.
Since φ is p-adically continuous, $\varphi([\bar{x}]) = [\bar{x}]^p$.)

$$W(R) = \left\{ \sum_{n=0}^{\infty} [\bar{x}_n] p^n \;\middle|\; \bar{x}_n \in R \right\}$$

(Given $x \in W(R)$, let $\bar{x} = x \bmod p \in R$; then $x - [\bar{x}] \in pW(R)$, and we obtain the required expansion recursively.)

Eg. To compute $[\bar{x}]+[\bar{y}]$:

mod p, this becomes ~~$\bar{x}+\bar{y}$~~ $\bar{x}+\bar{y}$,

$\therefore \quad [\bar{x}]+[\bar{y}] = [\bar{x}+\bar{y}] + p\,? + p^2\,? + \cdots$

To find the next coefficient in the expansion:

$$[\bar{x}^{\frac{1}{p}}] + [\bar{y}^{\frac{1}{p}}] \equiv [(\bar{x}+\bar{y})^{\frac{1}{p}}] \mod p$$

$\therefore$ Raising both sides to p^{th} power gives

$$[\bar{x}] + [\bar{y}] + p \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} [\bar{x}^{\frac{i}{p}} \bar{y}^{\frac{p-i}{p}}]$$

$$\equiv [\bar{x}+\bar{y}] \mod p^2$$

$$\therefore [\bar{x}]+[\bar{y}] = [\bar{x}+\bar{y}] + p\left[\frac{\bar{x}+\bar{y}-(\bar{x}^{\frac{1}{p}}+\bar{y}^{\frac{1}{p}})^{p}}{p}\right]$$

$$+ p^2 \cdots$$

So the comparison with $W_2(R) = R \times R$ from the δ-ring notes is that

$$\underset{\substack{\uparrow \\ \text{general description} \\ \text{of } W_2(R)}}{(\bar{x}_0, \bar{x}_1)} \longleftrightarrow \underbrace{[\bar{x}_0] + p[\bar{x}_1^{\frac{1}{p}}]}_{\substack{\text{description of } W_2(R) \text{ for} \\ \text{perfect } \mathbb{F}_p\text{-algebra}}}$$

<u>Thm</u>: If A is a perfect p-adically complete δ-ring, then $\exists$ a canonical $\cong$ of δ-rings.

$$A \xrightarrow{\cong} W(A/pA)$$

So $A \longmapsto A/pA$ induces an equivalence of categories

$$\left\{ \begin{array}{c} \text{perfect } p\text{-complete} \\ \delta\text{-rings} \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{perfect} \\ \mathbb{F}_p\text{-algebras} \end{array} \right\}$$

<u>Pf</u>: By Witt vector adjunction, the canonical surjection

$A \longrightarrow A/pA$ induces a morphism of δ-rings

$$A \longrightarrow W(A/pA)$$

which by construction is an iso. mod p.

Since A is p-adically complete, while $W(A/pA)$ is p-adically complete & p-torsion free, this is necessarily an iso.

$\square$

As a corollary, one finds that a perfect p-complete δ-ring is automatically p-torsion free.

One can also see this directly (w/o any completeness assumption):

<u>Lemma</u>: If $x \in A$ (a δ-ring) and $px = 0$, then $\varphi(x) = 0$.

$\therefore$ perfect δ-rings are p-torsion free.

<u>Pf</u>: $\quad 0 = \delta(px) = p^p \delta(x) + \delta(p)\varphi(x)$.

Now $\quad \varphi(x) = x^p + p\,\delta(x)$

$$\begin{aligned} \therefore \quad 0 &= \varphi(px) \\ &= p\varphi(x) = p \cdot x^p + p^2 \delta(x) \\ &= p^2 \delta(x) \qquad (\text{since } px = 0) \end{aligned}$$

$\therefore$ As $p \geq 2$ (!), $p^p \delta(x) = 0$

$$\begin{aligned} \therefore \quad 0 &= \delta(p)\varphi(x) \\ &= (1 - p^{p-1})\varphi(x). \end{aligned}$$

Since $p\varphi(x) = 0$ as well, we get

$\varphi(x) = 0$. $\square$

We now want to analyze *perfect prisms*, (6)
i.e. prisms whose underlying δ-ring is perfect.

If (A, I) is a perfect prism, we have just seen that it is p-torsion free, $\therefore$ not only derived p-adically complete, but classically p-complete.

Thus $A = W(R)$ for a perfect $\mathbb{F}_p$-alg. R.

b/c $\varphi(I)A$ is , and φ is an auto.

Also I is principal, say $I = (d)$, with d distinguished and a non-zero divisor.

So to classify perfect prisms, we have to understand distinguished elements in $W(R)$.