More on δ-rings and divided powers

The general set-up for divided power envelopes is that we have a base ring A, ~~[illegible]~~ containing an ideal I equipped with divided powers $\gamma_n : I \to A$.

We then consider an A-alg. B, and an ideal J of B s.t. $IB \subseteq J$, and we ~~[illegible]~~ endow J with divided powers in a universal way compatible with the given divided powers on I. This gives the d.p. envelope $D_J(B)$.

Eg. Our base ring will be $\mathbb{Z}_{(p)}$, with ideal (p), and the usual (only) divided powers $\gamma_n(p) = \frac{p^n}{n!}$

Then (changing notation from above) we'll consider a $\mathbb{Z}_{(p)}$-alg. A and an element $a \in A$. We will then want to consider the divided power envelope of (p, a). Since we know how to compute $\gamma_n(p)$, it is just a matter of adjoining the $\gamma_n(a)$.

~~We have the following key lemma:~~

Now recall from last time that the $\gamma_n(a)$ satisfy some basic identities, the upshot of which is that we need only adjoin

$$\gamma_{p^k}(a) = \text{"}\frac{a^{p^k}}{p^{1+\cdots+p^{k-1}}}\text{"} \times \text{unit}.$$

So we have the following guess for

$D_{(p,a)}(A)$, namely

$$A[x_1, x_2, \ldots, x_k, \ldots] \Big/ (px_1 - a^p, \; px_2 - x_1^p, \ldots, \; px_{k+1} - x_k^p, \ldots)$$

I say "guess" b/c we haven't explored whether there are any other less obvious identities we should ask the $\gamma_n(a)$ to satisfy.

However, we have the following key lemma:

<u>Lemma</u> Suppose that A is a p-t.f. (= flat) $\mathbb{Z}_{(p)}$-alg., and that $a \in A$ with the property that a is regular (i.e. a non-zero divisor) on A/pA.

Then we have isomorphisms

$$\begin{tikzcd} D_{(p,a)}(A) & A[x_1, \dots, x_k, \dots] / (px_1 - a^p, px_2 - x_1^p, \dots) \arrow[l, "\cong"'] \arrow[d, "\cong"] \\ & A\left[\frac{a^n}{n!}\right]_{n \geq 0} \subseteq A[\tfrac{1}{p}] = A \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \end{tikzcd}$$

<u>Pf:</u> As already noted, the first arrow is given by

$$x_k \to \text{unit}\ \gamma_{p^k}(a) \in D_{(p,a)}(A).$$

This map is surjective; we don't know if its an $\cong$ b/c we haven't confirmed that <u>all</u> the appropriate relns. b/w the $\gamma_n(a)$ are actually satisfied in $A[x_1, \dots, x_k, \dots] / (px_1 - a^p, px_2 - x_1^p, \dots)$.

The second arrow is given by

$$x_k \longmapsto \text{unit} \times \frac{a^{p^k}}{(p^k)!}.$$

It is again surjective, and evidently, after $-\otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$, it just becomes an iso.

$$A \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \;=\; A \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}.$$

So its kernel is precisely the p-power torsion in $A[x_1, \ldots, x_k, \ldots] / (px_1 - a^p, \ldots)$.

Thus, to prove this arrow is an $\cong$, it suffices to show that $A[x_1, \ldots, x_k, \ldots] / (px_1 - a^p, \ldots)$ is p-t.f.

Furthermore, once we do this, we find that the candidates for $\gamma_n(a)$ in $A[x_1, \ldots, x_k, \ldots]/(\ldots)$ actually $\underline{\text{do}}$ satisfy all the relations they should, b/c they may literally be identified with the elements $\frac{a^n}{n!}$ in $A[\frac{1}{p}]$. Thus we will deduce that the first map is an iso. too.

So we turn to proving the required torsion-freeness.

$$A[x_1,\dots,x_k,\dots]/(px_1-a^p,\ px_2-x_1^p,\dots)$$

$$= \varinjlim_k \ A[x_1,\dots,x_k]/(px_1-a^p,\dots,px_k-x_{k-1}^p)$$

and so it suffices to prove that each term in the direct limit is p-t.f. This is what we do.

We need to show that

$$A[x_1,\dots,x_k]/(\dots) \xrightarrow{p\times} A[x_1,\dots,x_k]/(\dots)$$

is injective, or equivalently, that

$$0 \to A[x_1,\dots,x_k]/(\dots) \xrightarrow{p\times} A[x_1,\dots,x_n]/(\dots)$$

$$\to A/pA[x_1,\dots,x_n]/(\dots) \to 0$$

is short exact.

For this, we replace each term by a resolution, thereby obtaining a sequence of resolutions whose exactness is more evident.

In fact, we consider the sequence of Koszul complexes

$$0 \to \mathrm{Kos}(A[x_1, \dots, x_k] ;\ px_1 - a^p, \dots, px_k - x_{k-1}^p)$$

$$\xrightarrow{p\times} \mathrm{Kos}(A[x_1, \dots, x_k] ;\ px_1 - a^p, \dots, px_k - x_{k-1}^p)$$

$$\to \mathrm{Kos}(A/pA[x_1, \dots, x_k] ;\ px_1 - a^p, \dots, px_k - x_{k-1}^p)$$

$$\to 0$$

Since A is p.t.f., this is a short exact sequence of complexes.

We claim that $px_1 - a^p, \dots, px_k - x_{k-1}^p$ gives a ~~gives a regular sequence on both $A[x_1, \dots, x_k]$ and both~~ regular sequence on $A/pA[x_1, \dots, x_k]$.

~~Actually, we only need to check this for $A/pA[x_1, \dots, x_k]$, where A~~ This is clear, b/c

$$(px_1 - a^p, px_2 - x_1^p, \dots, px_k - x_{k-1}^p)$$

$$\equiv (-a^p, -x_1^p, \dots, -x_{k-1}^p)$$

which is evidently regular on $A/pA[x_1, \dots, x_k]$ (given our assumption on a being regular mod p).

Now consider the two double complexes (one is the "transpose" of the other)

(1)

$$\begin{tikzcd}
\arrow[r] & \mathrm{Kos}(A/pA) \arrow[r] & {} \\
\arrow[r] & \mathrm{Kos}(A) \arrow[r] \arrow[u, two heads] & {} \\
\arrow[r] & \mathrm{Kos}(A) \arrow[r] \arrow[u, hook, "p^*"] & {}
\end{tikzcd}$$

we can compute the cohomology of the total complex via the spectral sequence whose E_1-page is given by taking cohomology in the vertical direction. But, as we've already observed, the vertical sequence is exact, $\therefore$ $E_1^{p,q} = 0$ $\forall_{p,q}$, $\therefore$ the total complex is acyclic.

(2)

$$\begin{tikzcd}
{} & {} & {} \\
\mathrm{Kos}(A) \arrow[r, hook, "p^*"] \arrow[u] & \mathrm{Kos}(A) \arrow[r, two heads] \arrow[u] & \mathrm{Kos}(A/pA) \arrow[u] \\
{} \arrow[u, dash] & {} \arrow[u, dash] & {} \arrow[u, dash]
\end{tikzcd}$$

We can compute the cohomology via the spectral sequence whose E_1-page is given by taking cohom in the vertical direction.

We get that $E_1^{p,q}$ is given by (8)

$$
\begin{array}{c|ccccc}
0 & A[x_1,\dots,x_k]/(\cdots) & \xrightarrow{p^*} & A[x_1,\dots,x_k]/(\cdots) & \longrightarrow & A/p[x_1,\dots,x_k]/(\cdots) \\
-1 & Kos_1(A) & \xrightarrow{p^*} & Kos_1(A) & \longrightarrow & 0 \\
\vdots & & & & & \vdots \\
q & Kos_{-q}(A) & \xrightarrow{p^*} & Kos_{-q}(A) & \longrightarrow & 0 \\
 & \vdots & & \vdots & & \vdots
\end{array}
$$

Here Kos_i denotes the ith homology of the Koszul complex (in cohomological degree $-i$).

The 3rd column has all its higher homologies $= 0$ b/c of our regular sequence observation.

Passing to the E_2 page, we get

$$
\begin{array}{c|ccc}
 & \text{kernel of } p \text{ on } A[x_1,\dots,x_k]/(\cdots) & 0 & 0 \\
0 & \bullet & \bullet & \bullet \\
-1 & * & * & 0 \\
\vdots & & & \vdots \\
q & * & * & 0
\end{array}
$$

Now the E_2 and all higher differentials on the ~~...~~ $E_2^{0,0}$ term must vanish, b/c

their targets are equal to 0.

Thus $E_2^{0,0}$ survives to E_∞.

But we've seen that the total complex is acyclic, $\therefore$

$$E_2^{0,0} = 0 .$$

This gives that $A[x_1, \dots, x_n]/(px_1 - a^p, px_2 - x_1^p, \dots, px_n - x_{n-1}^p)$

is p-t.f., ~~...~~ as required.

(Note: we didn't actually show or need that

$(px_1 - a^p, \dots, px_n - x_{n-1}^p)$ is a regular

sequence on $A[x_1, \dots, x_n]$, and so strictly speaking, we didn't show that

$\mathrm{Kos}(A[x_1, \dots, x_n]; px_1 - a^p, \dots, px_n - x_{n-1}^p)$
is a resolution. But this turned out not to matter.) $\square$

Corollary: Let $A \to B$ be a morphism of p-t.f. $\mathbb{Z}_{(p)}$-algebras, let $a \in A$ with image $b \in B$, and suppose that a is regular on A/pA and that b is regular on B/pB.

Then the natural map

$$B \otimes_A A\left[\frac{a^n}{n!}\right]_{n \geq 0} \to B\left[\frac{b^n}{n!}\right]_{n \geq 0}$$

(a priori surjective) is an $\cong$.

(Note: if we invert p, i.e. $-\otimes \mathbb{Q}$, then this just becomes the isomorphism $B \otimes_A A \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \xrightarrow{\cong} B \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$.

The point is that the domain may contain p-torsion. But it actually doesn't!!)

Proof: By the lemma, the corollary follows from the evident isomorphism

$$B \otimes_A A[x_1, \ldots, x_n, \ldots] / (px_1 - a^p, px_2 - x_1^p, \ldots, px_{n+1} - x_n^p, \ldots)$$

$$\xrightarrow{\cong} B[x_1, \ldots, x_n, \ldots] / (px_1 - b^p, px_2 - x_1^p, \ldots, px_{n+1} - x_n^p, \ldots)$$

$\square$

Combining this base change property of d.p. envelopes with our key lemma from last time, we obtain

<u>Lemma</u> If A is a p-torsion free δ-$\mathbb{Z}_{(p)}$-alg., if $a \in A$ s.t. a is regular on A/pA, then

$$A\left\{\frac{\varphi(a)}{p}\right\} = A\left[\frac{a^n}{n!}\right]_{n \geq 0} \quad \left(\text{in } A\left[\tfrac{1}{p}\right] = A \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}\right).$$

Pf: Map $\mathbb{Z}_{(p)}\{x\} \to A$ via $x \mapsto a$.

Then
$$A\left\{\frac{\varphi(a)}{p}\right\} = A \otimes_{\mathbb{Z}_{(p)}\{x\}} \mathbb{Z}_{(p)}\left\{x, \frac{\varphi(x)}{p}\right\}$$
$$= A \otimes_{\mathbb{Z}_{(p)}\{x\}} \mathbb{Z}_{(p)}\{x\}\left[\frac{x^n}{n!}\right]_{n \geq 0} \quad (\text{last time})$$
$$= A\left[\frac{a^n}{n!}\right]_{n \geq 0} \quad (\text{by the corollary}).$$
$\square$

All the results we've proved generalize to the context of a sequence $(a_1, \ldots, a_r)$ of elts. of A which is a regular sequence on A/pA (just use longer Koszul complexes).