Another δ-ring Lemma

This is essentially Lemma 2.34 of Bhatt-Scholze

Lemma Let A be a δ-ring s.t. A/pA is reduced.

~~• A is p-torsion free~~

~~• A/pA is reduced~~

Then if $d \in A$ is distinguished and

$$dx = p^2 y \quad \exists\, x, y \in A,$$

we have $p \mid x$ in A.

Pf: Applying δ to both sides gives

$$\delta(d)\varphi(x) + d^p \delta(x) = p(1-p^{2p-1})\cdot y^p + p^2 \cdot \delta(y)$$

$$\equiv 0 \quad (p)$$

∴ multiplying by x, recalling that $dx \equiv 0 \ (p)$, and that $\varphi(x) \equiv x^p \bmod p$,

we obtain

$$\delta(d)\cdot x^{p+1} \equiv 0 \quad (p)$$

Since $\delta(d)$ is a unit (as d is distinguished by assumption)

and A/pA is reduced, we find

$x \equiv 0 \ (p)$ as claimed. □

<u>Corollary</u> If A is a δ-ring s.t.

- A is p-torsion free & p-adically separated
- A/pA is reduced

Then any distinguished element in A is a non-zero divisor.

Pf: Suppose $dx = 0$. By the lemma,

$x = px_1$ $\exists x_1$, then

$pdx_1 = 0$

$\therefore dx_1 = 0$ b/c A is p-torsion free

Continuing, we find $x = p^n x_n$ $\forall n \geq 0$,
$\therefore x = 0$,
b/c A is p-adically separated. □

Corollary If A is a δ-ring s.t. A is p-torsion free and A/p is reduced, and if $d \in A$ is distinguished, then

$$(A/dA)[p^\infty] = (A/dA)[p]$$

Pf: It suffices to show that $(A/dA)[p^2]$

$$= (A/dA)[p].$$

So suppose that

$y \in A$ s.t.

$$p^2 y \equiv 0 \pmod{d},$$

ie $dx = p^2 y \quad \exists\, x \in A.$

By the Lemma, $x = pz \ \exists\, z \in A$

$\therefore \ pdz = p^2 y$

$\therefore \ dz = py$ b/c A is p-torsion free

so in fact $py \equiv 0 \ (d)$, as claimed.

$\square$

Corollary Perfect prisms are bounded.

Pf: We've seen that perfect prisms are of the form

$(A, I) = (W(R), (d))$ where R is a perfect $\mathbb{F}_p$-algebra and d is distinguished.

In particular, A is p-torsion free and p-adically separated, and $A/pA = R$ is reduced (being a perfect $\mathbb{F}_p$-algebra).

The preceding corollary shows that
$$(A/I)[p^\infty] = (A/I)[p] .$$
$\square$

Corollary: If (A, I) is a bounded prism, then A/I is classically p-complete.

Pf: I is invertible, i.e. locally free of rank one over A, $\therefore$ ~~[illegible]~~ derived p-complete, since A is.

Then A/I, the cokernel of $I \hookrightarrow A$, is derived p-complete (Actually, if A is derived p-complete, so is any f.p. A-module, and A/I is f.p.) Since A/I has bounded p-power torsion by assumption, it is classically p-complete. $\square$