δ-rings and divided powers.

Recall $\mathbb{Z}\{x\}$ = free δ-ring on one variable

$$\cong \mathbb{Z}[x_0, x_1, \dots, x_n, \dots]$$

$x_n = \delta^n(x)$

So $\varphi(x_n) = x_n^p + p x_{n+1}$

Lemma $\varphi : \mathbb{Z}\{x\} \longrightarrow \mathbb{Z}\{x\}$ is faithfully flat.

Pf: Write φ as the direct limit of

$$\mathbb{Z}[x_0, \dots, x_n] \xrightarrow{\varphi_n} \mathbb{Z}[x_0, \dots, x_n, x_{n+1}]$$

$$x_i \longmapsto x_i^p + p x_{i+1}$$

Factor each of these as

$$\mathbb{Z}[x_0, \dots, x_n] \xrightarrow{\cong} \mathbb{Z}[x_0^p, x_1, \dots, x_n]$$

$$x_0 \longmapsto x_0^p + p x_1$$

$$\subseteq \mathbb{Z}[x_0, x_1, \dots, x_n] \xrightarrow[x_1 \mapsto x_1^p + 2x_2]{\cong}$$

f. flat

$$\mathbb{Z}[x_0, x_1^p, x_2, \dots, x_n] \subseteq \mathbb{Z}[x_0, x_1, \dots, x_n]$$
f. flat

$$\cong \quad \dots \subseteq \mathbb{Z}[x_0, \dots, x_n] \subseteq \mathbb{Z}[x_0, \dots, x_n, x_{n+1}]$$
f. flat, f. flat

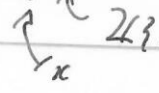
$$\cong \mathbb{Z}[x_0, \dots, x_n^p, x_{n+1}]$$
$$x_n \mapsto x_n^p + p x_{n+1}$$
$$\subseteq \mathbb{Z}[x_0, \dots, x_n]$$
f. flat

Thus φ_n is faithfully flat, and passing to the inductive limit, so is φ.

□

Cor If A is a δ-ring and $a \in A$, $\exists$ a faithfully flat map of δ-rings $A \to B$ s.t. image of a in B equals $\varphi(b)$ $\exists b \in B$.

Pf: Let $B = A \otimes \mathbb{Z}\{x\}$



Then $\varphi(1 \otimes x) = 1 \otimes \varphi(x) = a$. □

We now work over $\mathbb{Z}_{(p)}$. Note that if A is a $\mathbb{Z}_{(p)}$-alg., then $A[\frac{1}{p}] = A \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathbb{Q}$-algebra.

On $\mathbb{Q}$-algebras, we define "divided powers" via

$$\gamma_n(x) = \frac{x^n}{n!} \qquad n \geq 0$$

If A is a p-torsion free (= flat) $\mathbb{Z}_{(p)}$-alg., then $A \hookrightarrow A[\frac{1}{p}]$.

Typically A won't be preserved by the γ_n.

Ex: $\gamma_p(a) = \frac{a^p}{p!} = u \cdot \frac{a^p}{p}$ $\quad u \in \mathbb{Z}_{(p)}^\times$

$\therefore \gamma_p(a) \in A$ iff $a^p \in pA$.

Lemma: If A is a p-t.f. $\mathbb{Z}_{(p)}$-~~alg.~~ S-alg., and if $a \in A$, then TFAE:

(1) $a^p \in pA$
(2) $\varphi(a) \in pA$
(3) $\gamma_p(a) \in A$
(4) ~~$\gamma_n(a) \in A \ \forall n \geq 0$~~ $\gamma_p(a), \gamma_p^2(a) \in A$
(5) $\gamma_n(a) \in A \ \forall n \geq 0$

Proof: First recall p-adic properties of factorials:

$$v_p(n!) = \frac{n - s(n)}{p-1}$$ ← sum of base p digits of n

$\therefore$ if $n = pq + r$, with $0 \le r \le p-1$, we have

$$\gamma_n(a) = u \cdot \gamma_q(\gamma_p(a)) \cdot \gamma_r(a) \quad \text{with } u \in \mathbb{Z}_{(p)}^{\times}$$

$$\left[v_p(n!) = \frac{pq + r - s(pq+r)}{p-1} \right.$$

$$= \frac{pq + r - s(q) - s(r)}{p-1}$$

$$= q + \frac{q - s(q)}{p-1} + \frac{r - s(r)}{p-1}$$

$$\left. = v_p\left((p!)^q \cdot q! \cdot r! \right) \right]$$

Note $\gamma_r(a) = \frac{a^r}{r!} \in A$, for any $a \in A$

$\uparrow$
a unit in $\mathbb{Z}_{(p)}$!

Now to the proof:

Since $\varphi(a) \equiv a^p \pmod{p}$,

we see (1) $\Leftrightarrow$ (2), and we've seen (2) $\Leftrightarrow$ (3).

Clearly (5) $\Rightarrow$ (4) $\Rightarrow$ (3).

Suppose we show (3) $\Rightarrow$ (4).

Then we can use the identity on p. 4 to show that (3) $\Rightarrow$ (5) by induction.

(To see that $\gamma_n(a) \in A$, it suffices — in the notation of p. 4 — to see that

$$\gamma_q(\gamma_p(a)) \in A.$$

But if we know (3) $\Rightarrow$ (4), then we find that

$$\gamma_p(\gamma_p(a)) = \text{unit} \cdot \gamma_{p^2}(a)$$

$$\in A,$$

and so since $q < n$, we have that $\gamma_q(\gamma_p(a)) \in A$ by induction.)

So it remains to show that

(3) $\Rightarrow$ (a) , or equivalently

(using the equivalence (1) $\Leftrightarrow$ (3) and the formula $\gamma_{p^2}(x) = \text{unit} \cdot \delta_p\left(\frac{x_p}{p}\right)$)

that $\varphi\left(\frac{a^p}{p}\right) \equiv 0 \ (p)$ if $\gamma_p(a) \in A$ i.e. if $\frac{a^p}{p} \in A$.

But indeed $\varphi\left(\frac{a^p}{p}\right) = \frac{\varphi(a)^p}{p}$

$= \frac{(p \cdot b)^p}{p}$ where $\varphi(a) = pb$

$= p^{p-1} b^p$

$\equiv 0 \ (p)$ since $p \geq 2$.

$\square$

Here is the key application of the previous lemma:

Lemma: $\mathbb{Z}_{(p)}\left\{x, \frac{\varphi(x)}{p}\right\}$

$= \mathbb{Z}_{(p)}\{x\}[\gamma_n(x)]_{n\geq 0}$.

Pf: First we show $\mathbb{Z}_{(p)}\left\{x, \frac{\varphi(x)}{p}\right\}$ is p-t.f.; it then lives in

$\mathbb{Z}_{(p)}\{x\}\left[\frac{1}{p}\right] = \mathbb{Q}\{x\}$,

~~and it is in this latter ring~~

$\mathbb{Z}_{(p)}\{x\}[\gamma_n(x)]_{n\geq 0}$ also sits inside $\mathbb{Q}\{x\}$,

claimed

and then the equality makes sense as an equality of subrings of $\mathbb{Q}\{x\}$.

For the p-torsion freeness, we consider the diagram

$$\begin{array}{ccc} \mathbb{Z}_{(p)}\{y\} & \xrightarrow{y \mapsto pz} & \mathbb{Z}_{(p)}\{z\} \\ \downarrow{\scriptstyle y \mapsto \varphi(x)} & & \downarrow \\ \mathbb{Z}_{(p)}\{x\} & \longrightarrow & \mathbb{Z}_{(p)}\{x,z\}/(\varphi(x)-pz)_S =: \mathbb{Z}_{(p)}\{x, \frac{\varphi(x)}{p}\}_S \end{array}$$

This is a pushout diagram, which means, in terminology I'm more comfortable with, that

$$\mathbb{Z}_{(p)}\{x, \tfrac{\varphi(x)}{p}\} = \mathbb{Z}_{(p)}\{x\} \otimes_{\mathbb{Z}_{(p)}\{y\}} \mathbb{Z}_{(p)}\{z\}$$

where $\mathbb{Z}_{(p)}\{y\} \xrightarrow{\varphi(x) \leftarrow y} \mathbb{Z}_{(p)}\{x\}$ and $\mathbb{Z}_{(p)}\{y\} \xrightarrow{y \mapsto pz} \mathbb{Z}_{(p)}\{z\}$.

Now we've seen that $\varphi : \mathbb{Z}_{(p)}\{x\} \to \mathbb{Z}_p\{x\}$, is flat,

$$\begin{array}{ccc} x=y & \to & \| \\ & & \mathbb{Z}_{(p)}\{y\} \end{array} \qquad \begin{array}{c} \nearrow \\ y \mapsto \varphi(x) \end{array}$$

$\therefore$ $\mathbb{Z}_{(p)}\{x, \frac{\varphi(x)}{p}\}$

is flat over $\mathbb{Z}_{(p)}\{z\}$.

The latter is p-t.f., hence so is the former.

Now we know that both sides of the claimed inequality are contained in $\mathbb{Q}_p\{x\}$.

So $\mathbb{Z}_{(p)}\{x, \frac{\varphi(x)}{p}\}$ is the smallest δ-stable subring of $\mathbb{Q}_p\{x\}$ containing $\mathbb{Z}_{(p)}\{x\}[\frac{\varphi(x)}{p}]$.

The preceding lemma shows that

$$\gamma_n(x) \in \mathbb{Z}_{(p)}\{x, \frac{\varphi(x)}{p}\}$$

$$\therefore \mathbb{Z}_{(p)}\{x\}[\gamma_n(x)]_{n\geq 1} \subseteq \mathbb{Z}_{(p)}\{x, \frac{\varphi(x)}{p}\}$$

To get the reverse inclusion, we first note that $\varphi(x) \equiv x^p \pmod{p\,\mathbb{Z}_{(p)}\{x\}}$

$$\therefore \frac{\varphi(x)}{p} \equiv \gamma_p(x) \pmod{\mathbb{Z}_{(p)}\{x\}}$$

$$\therefore \mathbb{Z}_{(p)}\{x\}[\tfrac{\varphi(x)}{p}] \subseteq \mathbb{Z}_{(p)}\{x\}[\gamma_n(x)]_{n\geq 1}.$$

$$\mathbb{Z}_{(p)}\{x\}[\gamma_p(x)]$$

So we have to show that

$\mathbb{Z}_{(p)}\{x\}[\gamma_n(x)]_{n\geq 1}$ is δ-stable.

Equivalently, we have to show that it is φ-stable, and that

$$\varphi(y) \equiv y^p \mod p\,\mathbb{Z}_{(p)}\{x\}[\gamma_n(x)]_{n\geq 1}$$

$$\forall y \in \mathbb{Z}_{(p)}\{x\}[\gamma_n(x)]_{n\geq 1}.$$

Now $\varphi(\gamma_n(x)) = \gamma_n(\varphi(x))$

$= \gamma_n(x^p + p\,\delta(x))$

$= \gamma_n\left(p\cdot\left[\frac{x^p}{p} + \delta(x)\right]\right)$

$= \frac{p^n}{n!}\left(\frac{x^p}{p} + \delta(x)\right)^n$ where $\frac{x^p}{p}$ = unit $\cdot\, \gamma_p(x)$

$\equiv 0 \quad (p\,\mathbb{Z}_{(p)}\{x\}[\gamma_n(x)]_{n\geq 1})$

(In particular $\mathbb{Z}_{(p)}\{x\}[\gamma_n(x)]_{n\geq 1}$ is φ-stable).

Also, since $\gamma_p(\gamma_n(x)) \in \mathbb{Z}_{(p)}\{x\}[\gamma_n(x)]_{n\geq 1}$,
$\gamma_p(\gamma_n(x))$ = unit $\cdot\, \gamma_{pn}(x)$

we find that

$$f_n(x)^p = \text{unit} \cdot p \cdot \gamma_p(f_n(x))$$

$$\equiv 0 \quad (p\ \mathbb{Z}_{(p)}\{x\}[\gamma_n(x)]_{n\geq 1}).$$

$\therefore$ indeed

$$\varphi(f_n(x)) \equiv \gamma_n(x)^p \quad (p\ \mathbb{Z}_{(p)}\{x\}[f_n(x)]_{n\geq 1})$$

(both are $\equiv 0$)

Since the $\gamma_n(x)$ generate $\mathbb{Z}_{(p)}\{x\}[f_n(x)]_{n\geq 1}$ over $\mathbb{Z}_{(p)}\{x\}$, we indeed have that

$$\varphi(y) \equiv y^p \ (p) \quad \text{on } \mathbb{Z}_{(p)}\{x\}[f_n(x)]_{n\geq 1}.$$

This gives the required δ-stability. □

Concretely, $\mathbb{Z}_{(p)}\{x\} = \mathbb{Z}_{(p)}[x_0, x_1, \dots, x_n, \dots]$

$$\frac{\varphi(x)}{p} = \frac{x_0^p + px_1}{p} = \frac{x_0^p}{p} + x_1,$$

$$\therefore \mathbb{Z}_{(p)}\{x\}\left[\frac{\varphi(x)}{p}\right] = \mathbb{Z}_{(p)}\left[x_0, \frac{x_0^p}{p}, x_1, \dots, x_n, \dots\right]$$

$$\delta\left(\frac{x_0^p}{p}\right) = \frac{1}{p}\left\{\varphi\left(\frac{x_0^p}{p}\right) - \left(\frac{x_0^p}{p}\right)^p\right\}$$

$$= \frac{1}{p}\left\{\frac{\varphi(x_0)^p}{p} - \frac{x_0^{p^2}}{p^p}\right\}$$

$$= \frac{1}{p}\left\{\frac{(x_0^p + px_1)^p}{p} - \frac{x_0^{p^2}}{p^p}\right\}$$

$$= x_0^{p^2}\left(\frac{1}{p^2} - \frac{1}{pp^{p+1}}\right) + \text{elt. of } \mathbb{Z}[x_0, x_1]$$

$$= \frac{x_0^{p^2}}{p^{p+1}} \cdot \text{unit} + \text{elt. of } \mathbb{Z}[x_0, x_1]$$

In general, the Lemma shows that to get δ-stability, we just have to add the $\frac{x_0^n}{n!}$

E.g. we don't need denominators in the x_i for $i \geq 1$.

The preceding computations illustrate this.