δ-rings

$\varphi : A \xrightarrow{\text{homo.}} A$ is a Frobenius lift

if $\varphi(x) \equiv x^p \mod pA \quad \forall x \in A$.

$\delta : A \to A$ (not a homomorphism!!)

gives A a δ-ring structure if

δ satisfies the obvious identities necessary

to ensure that $\varphi(x) := x^p + p\,\delta(x)$

is a Frob. Lift

i.e. • $\delta(0) = \delta(1) = 0$

• $\delta(x+y) = \delta(x) + \delta(y) - \text{"}\dfrac{(x+y)^p - x^p - y^p}{p}\text{"}$

i.e. $= \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} \cdot x^i y^{p-i}$

(2)

- $\delta(xy) = x^p \delta(y) + \delta(x)\varphi(y)$

$$\left(= \varphi(x)\delta(y) + \delta(x) y^p \right.$$

the equality of the two expressions follows directly from the formula for φ).

A δ-ring is a ring with extra structure, and so the forgetful functor

$$\delta\text{-rings} \longrightarrow \text{rings}$$

has a left adjoint $A \mapsto A^\delta$, and δ-rings admits all limits; these are compatible with limits on the underlying rings (or, indeed, on the underlying sets).

If S is a set, we can form the free ring $\mathbb{Z}[S]$ on S (a polynomial ring with variables being the elements of S) and then also form $\mathbb{Z}\{S\} := \mathbb{Z}[S]^{\delta}$, the free δ-ring on S.

Eg. $\mathbb{Z}\{x\}$ is the polynomial ring

$$\mathbb{Z}[x, \delta x, \delta^2 x, \ldots, \delta^n x, \ldots].$$

(φ is defined by $\varphi(\delta^n x) = (\delta^n x)^p + p\,\delta^{n+1} x$)

$\mathbb{Z}\{S\}$ has an analogous description.

It is less obvious, but true, that the forgetful functor from δ-rings to rings also preserves all colimits (in particular, δ-rings is cocomplete). Thus it is a left adjoint as well as a right adjoint. Its right adjoint functor is denoted W : rings $\longrightarrow$ δ-rings, the "Witt vectors" functor.

Since the left adjoint of W is the forgetful functor, which preserves all limits and reflects isomorphisms, we see that W is <u>comonadic</u>, i.e. if A is a ring, then giving

A a δ-ring structure is the same as giving a "co-action map"

$$A \xrightarrow{k} W(A) \qquad \text{(a homomorphism of rings)}$$

s.t. $A \xrightarrow{k} W(A) \xrightarrow{\text{counit}} A$ is the identity,

and s.t.

$$\begin{tikzcd}
A \arrow[r, "k"] \arrow[d, "k"'] & W(A) \arrow[d, "\text{comonadic comultiplication}"] \\
W(A) \arrow[r, "W(k)"'] & W(W(A))
\end{tikzcd}$$

commutes.

Now $W(A)$ admits a description:

$$W(A) = \mathrm{Map}_{\mathrm{Sets}}(\{*\}, W(A))$$

$$= \mathrm{Map}_{\delta\text{-rings}}(\mathbb{Z}\{x\}, W(A))$$

$$= \mathrm{Map}_{\mathrm{rings}}(\mathbb{Z}\{x\}, A)$$

$$= \mathrm{Map}_{\mathrm{rings}}(\mathbb{Z}[x, \delta x, \delta^2 x, \dots], A)$$

$$= \mathrm{Map}_{\mathrm{Sets}}(\mathbb{N}, A)$$

$$= A^{\mathbb{N}}$$

The ring structure on $W(A) = A^{\mathbb{N}}$ isn't immediately clear from this description, but we see that $\delta(a_0, \dots, a_n, \dots)$

$$= (a_1, a_2, \dots, a_n, \dots)$$

For any δ-ring B, we have

$$B = \mathrm{Map}_{\mathrm{Sets}}(\{*\}, B)$$

$$= \mathrm{Map}_{\delta\text{-rings}}(\mathbb{Z}\{x\}, B)$$

$\therefore$ $\mathbb{Z}\{*\}$ is a "co-δ-ring" object in δ-rings, since $\mathrm{Maps}_{\delta\text{-rings}}(\mathbb{Z}\{x\}, -)$ is a δ-ring, not just a set.

Eg

$$\begin{tikzcd} \delta^* : \mathbb{Z}\{x\} \arrow[r] \arrow[d, equal] & \mathbb{Z}\{x\} \arrow[d, equal] \\ \mathbb{Z}[x_0, x_1, \ldots] & \mathbb{Z}[x_0, x_1, \ldots] \end{tikzcd}$$

maps x_i to x_{i+1}

To describe the other "co-operations" on $\mathbb{Z}\{x\}$, we need to consider coproducts in δ-rings.
Since the forgetful functor respects colimits, these are just coproducts in rings, i.e. tensor product.

If
$$\begin{tikzcd} A \arrow[r] \arrow[dr] & B \\ & C \end{tikzcd}$$
are homos. of δ-rings, then the δ on $B \otimes_A C$ is given by

$$\delta(b \otimes c) = b^p \otimes \delta(c) + \delta(b) \otimes \varphi(c)$$
$$\left(= \varphi(b) \otimes \delta(c) + \delta(b) \otimes c^p\right)$$

Now

$$\begin{array}{ccc} +^* : \mathbb{Z}\{x\} & \longrightarrow & \mathbb{Z}\{x\} \otimes_{\mathbb{Z}} \mathbb{Z}\{x\} \\ \| & & \| \\ \mathbb{Z}[x_0, x_1, \ldots] & & \mathbb{Z}[y_0, y_1, \ldots] \otimes \mathbb{Z}[z_0, \ldots] \\ & & \| \\ & & \mathbb{Z}[y_0, z_0, y_1, z_1, \ldots] \end{array}$$

$\mathbb{Z}$ is the initial δ ring, with $\varphi = \mathrm{id}$, $\delta(n) = \dfrac{n - n^p}{p}$.

is given by

$$x_0 \longmapsto y_0 + z_0$$

$$\begin{array}{ccl} x_1 & \longmapsto & \delta(y_0 + z_0) \\ \| & & \| \\ \delta(x_0) & & \delta(y_0) + \delta(z_0) + \dfrac{y_0^p + z_0^p - (y_0+z_0)^p}{p} \\ & & \| \\ & & y_1 + z_1 + \dfrac{y_0^p + z_0^p - (y_0+z_0)^p}{p} \end{array}$$

etc.

$$X^* : \mathbb{Z}\{x\} \longrightarrow \mathbb{Z}\{x\} \otimes_{\mathbb{Z}} \mathbb{Z}\{x\}$$

$$\begin{array}{ccc} \mathbb{Z}\{x\} & & \mathbb{Z}\{x\} \otimes_{\mathbb{Z}} \mathbb{Z}\{x\} \\ \| & & \| \\ \mathbb{Z}[x_0, x_1, \dots] & & \mathbb{Z}[y_0, y_1, z_0, z_1, \dots] \end{array}$$

is given by $x_0 \longmapsto y_0 z_0$

$$\begin{aligned} x_1 &\longmapsto \delta(y_0 z_0) \\ \| \\ \delta(x_0) \end{aligned}$$

$$\begin{aligned} \delta(y_0 z_0) &= y_0^p \delta(z_0) + \delta(y_0) \varphi(z_0) \\ &= y_0^p z_1 + y_1 (z_0^p + p z_1) \\ &= y_0^p z_1 + y_1 z_0^p + p z_1 y_1 \end{aligned}$$

etc.

From this, returning to the formula

$$W(A) = \mathrm{Maps}_{\delta\text{-rings}}(\mathbb{Z}\{x\}, W(A))$$

$$= \mathrm{Maps}_{\text{rings}}(\mathbb{Z}[x_0, x_1, \ldots], A)$$

$$= A^{\mathbb{N}}$$

we compute at least the "beginning" of the ring structure on $W(A)$:

$$(a_0, a_1, \ldots) + (b_0, b_1, \ldots)$$

$$= \left(a_0+b_0,\ a_1+b_1+\frac{a_0^p+b_0^p-(a_0+b_0)^p}{p}, \ldots\right)$$

$$(a_0, a_1, \ldots)\cdot(b_0, b_1, \ldots)$$

$$= (a_0b_0,\ a_0^pb_1+a_1b_0^p+pa_1b_1, \ldots)$$

The inclusion $\mathbb{Z}[x_0, \ldots, x_{n-1}] \hookrightarrow \mathbb{Z}[x_0, \ldots, x_{n-1}, \ldots]$

induces a projection

$$\begin{array}{ccl} (a_0, a_1, \ldots) & \in A^{\mathbb{N}} = & \mathrm{Map}_{\mathrm{rings}}(\mathbb{Z}[x_0, \ldots, x_{n-1}, \ldots], A) \\ \Big\downarrow & & \quad \Big\downarrow \\ (a_0, \ldots, a_{n-1}) & A^n = & \mathrm{Map}_{\mathrm{rings}}(\mathbb{Z}[x_0, \ldots, x_{n-1}], A) \end{array}$$

The above inclusion is in fact an inclusion of "co-rings" (though not of "co-δ-rings") and so $A^{\mathbb{N}} \to A^n$ is a homomorphism of rings, which we ~~also~~ denote

$$W(A) \longrightarrow W_n(A).$$

Above, we computed $W_2(A)$ explicitly.

Of course, $W_1(A) = A$ itself,

and $W(A) \to W_1(A) = A$ is the counit.

Now one checks that giving a section $A \to W_2(A)$ of the augmentation $W_2(A) \to A$ is <u>equivalent</u> to giving A a δ-ring structure: the formula is

$$\begin{array}{ccc} a & \longmapsto & (a, \delta(a)) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ A & & W_2(A) \end{array}$$

This a more finitistic incarnation of the comonadic description of δ-rings, and using this is one way of proving that the forgetful functor commutes with colimits (and thus of establishing this comonadic description, the existence of the adjoint W, etc.) in the first place.