Crystalline cohom. — a very short introduction. ①

* Connections : If $\mathcal{E}$ is a v. bundle on X

imagine a manifold or smooth variety $/\mathbb{C}$ for now

Then a connection on $\mathcal{E}$ gives an identification $\mathcal{E}_t \xrightarrow{\cong} \mathcal{E}_{t'}$

fibres of $\mathcal{E}$ at t & t'

(parallel transport) if t, t' are two infinitesimally close pt.

Scheme-theoretically, we can consider T-valued points, where T is some (say affine) test object.

$$t, t' : T \rightrightarrows X.$$

To say that they are inf. close means that

$$t \times t' : T \longrightarrow X \times X \quad \text{lands in } X_1,$$

the first order n.h. of the diagonal $X = X_0$ in $X \times X$,

i.e. $\operatorname{Spec}\left(\mathcal{O}_{X\times X}/J_\Delta^2\right)$, where J_Δ = ideal sheaf of the diagonal.

~~Therefore~~ For all such t, t', we have an iso.

$t^* \mathcal{E} \xrightarrow{\cong} (t')^* \mathcal{E}$, functorial in t, t'; equal to the identity when $t = t'$.

$\therefore$ reduce to the universal case $X_1 \overset{pr_0}{\underset{pr_1}{\rightrightarrows}} X$

$$pr_0^* \mathcal{E} \overset{\alpha}{\cong} pr_1^* \mathcal{E}$$

Now $\Omega^1_X = \mathcal{I}_\Delta / \mathcal{I}_\Delta^2$, $\therefore$

$$0 \to \underset{\underset{\mathcal{I}_\Delta/\mathcal{I}_\Delta^2}{\shortparallel}}{\Omega^1_X} \to \mathcal{O}_{X\times X}/\mathcal{I}_\Delta^2 \to \underset{\underset{\text{diag. copy of } X}{\cup}}{\mathcal{O}_{X_0}} = \mathcal{O}_{X\times X}/\mathcal{I}_\Delta \to 0$$

$$\begin{array}{ccccccccc} \therefore\ 0 & \to & \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X & \to & pr_0^* \mathcal{E} & \to & \mathcal{E} & \to & 0 \\ & & \| \quad \overset{\shortparallel}{(pr_0^* \mathcal{E}) \otimes \mathcal{I}/\mathcal{I}^2} & & \| \wr\, \alpha & & \| & & \\ 0 & \to & \underset{\underset{(pr_1^* \mathcal{E}) \otimes \mathcal{I}/\mathcal{I}^2}{\shortparallel}}{\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X} & \to & pr_1^* \mathcal{E} & \to & \mathcal{E} & \to & 0 \end{array}$$

$\therefore$

$v \mapsto$

$$pr_1^{-1}(v) - \alpha(pr_0^{-1}(v))$$

gives a map (of $\mathbb{C}$-sheaves) $\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^1_X$.

If we recall that $d : \mathcal{O}_X \to \Omega^1_X = \mathcal{J}_\Delta / \mathcal{J}_\Delta^2$

is defined via $df := pr_1^{-1}(f) - pr_0^{-1}(f)$,

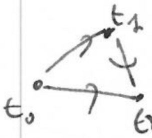
we find that $\nabla(fv) = df . v + f . \nabla v$,

and so ∇ is the ~~[illegible]~~ usual ("Koszul") structure of a connection.

(d is constructed just like ∇, in the special case where $\mathcal{E} = \mathcal{O}_X$ and $\alpha = id :$

$$\mathcal{O}_{X(1)} = pr_0^* \mathcal{O}_X$$
$$\|$$
$$\mathcal{O}_{X(1)} = pr_1^* \mathcal{O}_X .)$$

A connection is flat, or <u>integrable</u> (the alg. geometry adjective of choice) if it has no holonomy, i.e. if



parallel transport around the top of any such triangle agrees with parallel transport along the bottom. This amounts to a cocycle condition on the functorial isomorphisms. It also allows you to extend parallel transport over longer distances: if $t, t' : T \rightrightarrows X$ are s.t.

$$t \times t' : T \longrightarrow X_n \hookrightarrow X \times X \quad , \text{ then}$$

$X_n = \operatorname{Spec} \mathcal{O}_X / \mathcal{J}_\Delta^{n+1}$

n^{th} order nh. of X_0 = diagonal

$t^* \mathcal{E} \cong (t')^* \mathcal{E}$: joint t & t' by paths within 1st order difference from one another; cocycle condition means it doesn't matter how we do this.

(Eg. $\mathbb{C}[x,h]/(h^3) \hookrightarrow \mathbb{C}[x,a,b]/(a^2,b^2)$

$\mathbb{C}[x]\langle 1,h,h^2 \rangle \hookrightarrow \mathbb{C}[x]\langle 1,a,b,ab \rangle$

$h \longmapsto a+b$
$h^2 \longmapsto 2ab$

classifies pairs of pts. $(x, x+h)$ s.t. $h^3 = 0$ $=$ $(x, x+a+b) \longleftarrow$ classifies triples $(x, x+a, x+a+b)$ $a^2=0, b^2=0$)

If $\underset{\text{affine}}{T} \longrightarrow X$, then $\begin{array}{c} T \\ \uparrow \text{thickening} \\ T_{red} \longrightarrow X \end{array}$.

Conversely, suppose

$$\begin{tikzcd} T \\ T_{red} \arrow[u, hook, "\text{thickening}"] \arrow[r] & X \end{tikzcd}$$

affine

Write it as a square

$$\begin{tikzcd} T_{red} \arrow[r] \arrow[d, hook] & X \arrow[d, "\text{smooth}"] \\ T \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbb{C} \end{tikzcd}$$

The infinitesimal ~~[illegible]~~ lifting characterization of smoothness lets us fill in the dotted arrow to some $t : T \to X$.

Define $\mathcal{E}_T := t^* \mathcal{E}$. The flat connection on $\mathcal{E}$ shows that $\mathcal{E}_T$ is well-defined ind. of choice of t.

If $f : T_1 \to T_2$

$$\begin{tikzcd} T_1 \arrow[r] & T_2 \\ (T_1)_{red} \arrow[u, hook] \arrow[r] & (T_2)_{red} \arrow[u, hook] \arrow[r] & X \end{tikzcd}$$

then $f^* \mathcal{E}_{T_2} \cong \mathcal{E}_{T_1}$

So $(\mathcal{E}_T)$ is a <u>crystal</u> on the

site of $\left(\begin{array}{ccc} T & & \\ \cup & & \\ T_{red} & \longrightarrow & X \end{array} \right)$ ← infinitesimal site.

To compute cohomology: try to find a final object.

well, any object $\begin{pmatrix} T \\ \cup \\ T_{red} \end{pmatrix}$ maps to $\begin{pmatrix} X \\ \parallel \\ X \end{pmatrix}$.

So X is at least "cofinal" with any $\begin{pmatrix} T \\ \cup \\ T_{red} \end{pmatrix}$.

— X is "weakly final".

So compute cohom. using X, $X \times X$, ..., $X \times \cdots \times X$, ...

What are these? $X \times X$ classifies pairs $t, t' : T \longrightarrow X$ lifting the given map $T_{red} \longrightarrow X$. This pair (t,t') is equiv. to $T \longrightarrow X_n$ (if T is an n^{th} order thickening).

So $X \times X = \text{``}\varinjlim\text{''}\, X_n$, analogously for higher products.

The complex $\mathcal{E}(X) \rightrightarrows \mathcal{E}(X\times X) \overset{\to}{\rightrightarrows} \cdots$

$$" \varprojlim \mathcal{E}(X_n) "$$

when totalized, amounts to the global sections of the $\mathcal{O}$-linearization of the de Rham complex $\mathcal{E} \xrightarrow{\nabla} \mathcal{E}\otimes\Omega^1 \xrightarrow{\nabla} \mathcal{E}\otimes\Omega^2 \to \cdots$.

$\therefore$ The cohom. of the crystal arising from $\mathcal{E}$ on the infinitesimal site $\left(\begin{array}{c} T \\ \cup \\ T_{red} \to X \end{array}\right)$ is the usual de Rham cohom. of $\mathcal{E}$.

Crystalline site: X smooth over perfect field k of char. p

$$\begin{tikzcd} T & \operatorname{Spec} k \\ T_{red} \arrow[u, hook] \arrow[r] & X \arrow[u] \end{tikzcd}$$

but now T may "thicken up" in the p direction, so

$$\begin{tikzcd} T \arrow[r] & \operatorname{Spec} W(k) \supseteq \operatorname{Spec} k \\ T_{red} \arrow[u, hook] \arrow[r] & X \arrow[u] \end{tikzcd}$$

so to compute cohom., need to choose a lift

$\tilde{X}$ of X over $W(k)$:

$$\begin{tikzcd} T \arrow[rr] \arrow[drr, dashed] & & \operatorname{Spec} W(k) \\ T_{red} \arrow[u, hook] \arrow[r] & X \arrow[r, hook] & \tilde{X} \arrow[u, "\text{smooth}"'] \end{tikzcd}$$

Also, in order to integrate a connection D over thickenings, we need divided powers. (& maybe need the connection to be "nilpotent" if the D.P. structures are not.)

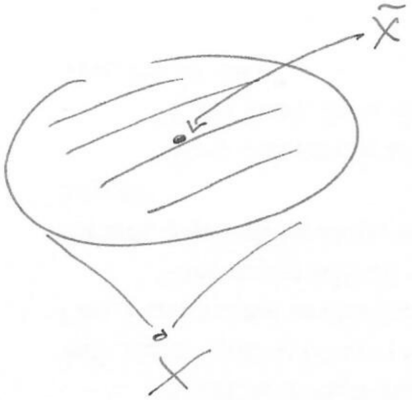
<u>Crystalline site</u>: $\left(\begin{tikzcd} T \\ T_{red} \arrow[u, hook, "\text{divided power thickenings}"'] \arrow[r] & X \end{tikzcd} \right)$

If $(\tilde{\mathcal{E}}, D)$ = bundle w/ flat connection on $\tilde{X}$ $\longleftrightarrow$ crystal $\tilde{\mathcal{E}}_T$,

$R\Gamma_{cris}(X_{/W(k)}, \tilde{\mathcal{E}}_T)$ = de Rham cohomology of $\tilde{\mathcal{E}}$ on $\tilde{X}$.

Constant coeffs: $R\Gamma_{cris}(X_{/W(k)}, \mathcal{O})$ = de Rham cohom. of $\tilde{X}$.

Why is it ind. of $\tilde{X}$? Consider def. space of X



Different choices of $\tilde{X}$ are different $W(k)$-pts. of this def. space.

Gauss-Manin connection gives an $\cong$ b/w their de Rham cohomologies.

Don't need to use $\tilde{X}$ lifting X as the weakly final object. Consider any Y smooth (or even ind-smooth) over $W(k)$ with

$$X \underset{\text{closed}}{\hookrightarrow} Y_0 = Y \times_{W(k)} k.$$

If T d.p. thickening $T_{red} \to X$, we can find

$$\begin{tikzcd} T \arrow[r, "t"] & Y \\ T_{red} \arrow[u, hook] \arrow[r] & X \arrow[u, hook] \end{tikzcd}$$

by smoothness of Y, and actually t lands in D.P.-envelope of X in Y: call it $\hat{Y}$.

But $t, t' : T \rightrightarrows Y$ are two such,

$t \times t' : T \longrightarrow Y \times_{W(k)} Y$ lands in DP-env. of

$X \xhookrightarrow{\text{diag.}} Y_0 \times_k Y_0 \hookrightarrow Y \times Y$,

and similarly for higher powers.

Now interpreting the resulting Čech complex via de Rham theory, we get Thm. 1.6 from Bhatt's lecture 6:

$$\begin{tikzcd}
X = \overset{V(J) \text{ in } \operatorname{Spec} P}{\overset{=}{\operatorname{Spec} R}} \arrow[r, hook] \arrow[drr] & \operatorname{Spec} P/pP \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} P = Y \arrow[d] \\
 & \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} W(k)
\end{tikzcd}$$

$$R\Gamma_{cry}(X/W(k)) \simeq \Omega^\bullet_{P/W(k)} \hat{\otimes}_P \underbrace{D_J(P)}_{\text{DP-env. of } X \text{ in } \operatorname{Spec} Y.}$$

The previous discussion was $J = (p)$ s.t. $R = P/pP$;
in this case we already have D.P. structure on J, so $D_J(P) = P$.
And can replace $W(k) \twoheadrightarrow k = W(k)/pW(k)$ by $A \twoheadrightarrow A/I$ with I a P.D. ideal containing p in a p-torsion free ring A.