

Crystalline comparison

①

$(A, (p))$ - crystalline prism.

$(p) \subseteq I \subseteq A$, I a pd-ideal.

Note that if $a \in \overset{I}{\bullet}$, then $a^p = p \cdot \gamma(a)$
 $\in pA$

$$\begin{array}{ccc} A/I & \xrightarrow{\text{Frob}_{A/I}} & A/I \\ \uparrow & \dots & \uparrow \\ A/pA & \xrightarrow{\text{Frob}_{A/pA}} & A/pA \end{array}$$

admits a unique dotted arrow, which we denote ψ .

If $I = (p)$, then ψ is just $\text{Frob}_{A/pA} : A/pA \rightarrow A/pA$.

~~Let~~ Let R be a smooth A/I -alg.,

write $R^{(1)} = R \otimes_{A/pA} A/pA$, so $\text{Spec } R^{(1)} = \psi$ -pullback of $\text{Spec } R$,

Then $\triangleleft_{R^{(1)}/A} \cong R \Gamma_{\text{crys}}(R/A)$ as E_{∞} - A -algebras compatibly with Frobenius.

Ex: As already noted, $a \in I \Rightarrow a^p \in pA$, so that

$$A/pA \rightarrow A/I \quad \text{is a nilp. thickening.}$$

Thus if R is smooth over A/I , then we may lift R to \tilde{R} smooth over A/pA , i.e. find \tilde{R} smooth over A/pA s.t.

$$R = \tilde{R} \otimes_{A/pA} A/I$$

Then $R^{(1)} := R \otimes_{A/I} A/pA = \tilde{R} \otimes_{A/pA} A/I \otimes_{A/I} A/pA$

$$= \tilde{R} \otimes_{A/pA} A/pA =: \hat{R}^{(1)} \text{ in the usual (Frob. twist) sense.}$$

$$= \tilde{R} \otimes_A A \xleftarrow{A \rightarrow pA} \text{ lift of Frob. on } A$$

We then find that $\Delta_{R^{(1)}/A} \cong \Delta_{(\tilde{R} \otimes_A A)/A}$

$$\xrightarrow{\text{by crys. comparison}} \mathbb{R}P_{\text{crys}}(R/A) \cong \varphi_A^* \Delta_{\tilde{R}/A}$$

by crys. comparison

once we establish base-change for the prismatic complex (as a consequence of the Hodge-Tate comparison theorem)

Ex: If $I = (p)$, then there is no ambiguity in the choice of \tilde{R} ; we just have $\hat{R} = R$,

and so we get $\mathbb{R}P_{\text{crys}}(R/A) = \varphi_A^* \Delta_{R/A}$,

so $\Delta_{R/A}$ is a φ_A -descent of $\mathbb{R}P_{\text{crys}}(R/A)$.

Proof of comparison theorem:

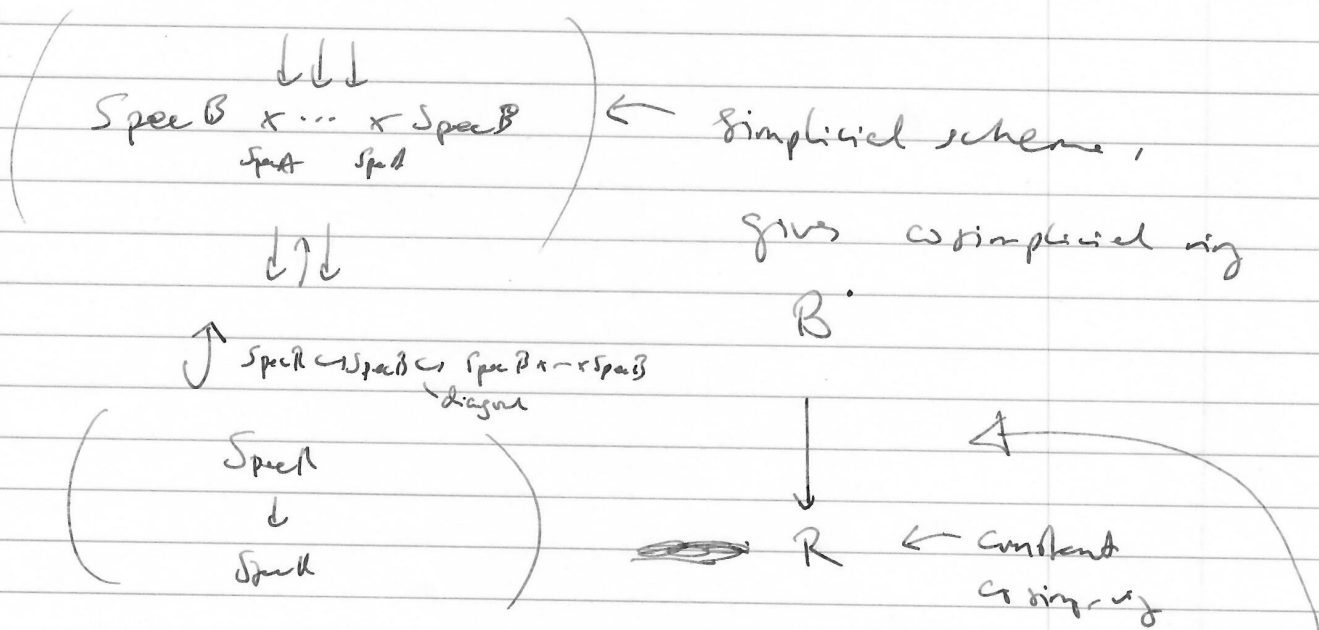
(3)

Consider $B \rightarrow R$ where B is a p -completely ind-smooth S -A- alg .

(eg. write R as the quotient of a free S -A- alg , and take B to be the p -completion of this.)

$$B^\circ \rightarrow R$$

\hookrightarrow
 p -completed Čech nerve of $A \rightarrow B$



$D^\circ = p$ -completed dp env. of B° along kernel of

Then $D^\circ = R \hat{P}_{\text{comp}}(R/A)$.

We now show that the f.a.-alg. structure on B extends to a f.a.-alg. structure on D .

The idea: If B is ind-smooth, say $B = \varinjlim B_i$ with B_i smooth/A, then since R is f.g./A (being smooth over A/pA), we may assume each $B_i \rightarrow R$ is surjective (by dropping some initial segment of B_i 's).

Then each of these lifts to a surjection

$$B_i \rightarrow \tilde{R} \quad (\text{if we choose a smooth lift of } R \text{ to } \tilde{R} \text{ over } A/pA)$$

inducing a surjection $B_i/pB_i \rightarrow \tilde{R}$ of smooth A/pA -alg., whose kernel J_i is Zariski locally given by a regular sequence.

Now if $K_i = \text{kernel of } B_i \rightarrow R$, then

$$K_i = \text{kernel of } B_i \rightarrow R \quad (I, \text{ lift of } J_i), \text{ and since}$$

$$I \text{ has d. power, } D_{B_i}(K_i) = D_{B_i}(I, \text{ lift of } J_i)$$

Zariski locally given by a sequence which is regular on B_i/pB_i .

$$\text{If } K = \text{ker}(B \rightarrow R), \text{ then } K = \varinjlim K_i, \quad D_B(K) = \varinjlim D_{B_i}(K_i).$$

Now we apply our results showing that

if ~~now~~ B is a p -t.f. \mathcal{S} -ring, ^{and} if $(x_1, \dots, x_r) \in B$ is a regular sequence on B/pB , then

$$B \left\{ \frac{\varphi(x_i)}{p} \right\} = \cancel{D_B(B)} \quad D_B((x_i))$$

to conclude that $D_B(K)$ has a \mathcal{S} -structure compatible with that on B .

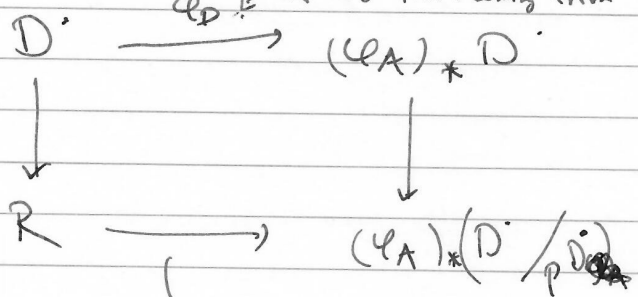
(I still have to think through the details myself, since our regular sequence occurs for B_i , not B , and B_i isn't a \mathcal{S} -ring; only B is.)

p -completing gives $D_B(K)^\wedge$ a \mathcal{S} -ring structure.

In our set-up B_i is p -completely ind-smooth, not actually ind-smooth, so we have to take that into account as well.

~~But it's not~~ But the upshot is that D^\cdot does admit a \mathcal{S} -A-dg. structure.

Now consider the diagram



analogue of φ ; it is equivariant w.r.t

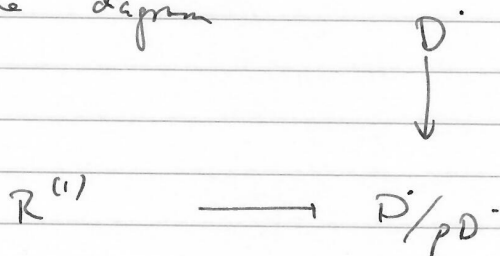
$$\varphi: A/\mathbb{Z} \rightarrow A/pA \cong D^\cdot / pD^\cdot$$

①

∴ The map $R \rightarrow (\varphi_A)_* (D/pD)$

induces $R^{(1)} := R \otimes_{A/pA} A/pA \rightarrow D/pD$
 $\uparrow \varphi_A$

Now the diagram



realize D as a co-simplicial object in $\Delta_{R^{(1)}/A}$.

~~As B ranges over all possible choices,~~

∴ get $\Delta_{R^{(1)}/A} \rightarrow D = \mathcal{R}P_{\text{crys}}(R/A)$.

As B ranges over all possible choices, these maps are compatible. Since the category of $\{B \rightarrow R\}$ is sifted, we get a well-defined map

$$\Delta_{R^{(1)}/A} \rightarrow \bullet \mathcal{R}P_{\text{crys}}(R/A)$$

ind. of choice of B .

To check it's an iso., choose a lift

\hat{R} of R as before, and choose B s.t.

$$B \rightarrow \hat{R} \rightarrow R$$

\downarrow
p-completion of free S/A -alg.

(7)

Then $B \rightarrow \hat{R}$, with kernel J , and our results on prismatic envelopes show that

$$C := B \left\{ \frac{J}{p} \right\}^\wedge \text{ is base-change compatible.}$$

(Again, there is an "ind" argument to be made here, to pass from smooth to ind-smooth.)

(See Cor. 3.14 & Construction 4.16 of Bhatt-Scholze for examples of these sorts of arguments.)

Base-change compatibility shows that $\varphi_A^* C$ computes each nerve of the

$$\Delta_{R^{(n)}}/A$$

(B/c $\varphi_A^* C$ is the prismatic envelope of the p -completely free S -A- \mathcal{A} .

$\varphi_A^* B$, and this prismatic envelope is weakly final, since $\varphi_A^* B$ is p -completely free over A .)

p -complete freeness of B ^{over A} shows that $A \rightarrow B$ is a homotopy equivalence, \therefore

$$\begin{aligned}
\varphi_A^* C &\simeq \varphi_B^* C = \varphi_B^* (B \left\{ \frac{J}{p} \right\}^\wedge) \\
&= B \left\{ \frac{\varphi(J)}{p} \right\}^\wedge && = \ker(B \rightarrow \mathbb{Z}) \\
&&& = (J, I) \\
&\xrightarrow{\quad} D_B \cdot (J)^\wedge = D_B \cdot (K)^\wedge && \uparrow \\
&&& \text{b/c } I \text{ has d-powers}
\end{aligned}$$

This again uses our results about d.p. envelopes and δ -structures, with some Ind-argument thrown in.

⑧

As we know, the last expression computes $\mathbb{R}P_{\text{crys}}(R/A)$,

\therefore we get the required isomorphism

$$\Delta_{R(A)/A} \xrightarrow{\sim} \mathbb{R}P_{\text{crys}}(R/A)$$

□