Crystalline comparison

$(A, (p))$ – crystalline prism.

$(p) \subseteq I \subseteq A$, I a pd-ideal.

Note that if $a \in I$, then $a^p = p \cdot \gamma_p(a) \in pA$

$$\therefore \quad \begin{tikzcd} A/I \arrow[r, "\mathrm{Frob}_{A/I}"] & A/I \\ A/pA \arrow[u] \arrow[r, "\mathrm{Frob}_{A/pA}"] & A/pA \arrow[u] \\ \end{tikzcd}$$

admits a unique dotted arrow $A/I \dashrightarrow A/pA$, which we denote φ.

If $I = (p)$, then φ is just $\mathrm{Frob}_{A/pA} : A/pA \to A/pA$.

Let R be a smooth A/I-alg.,

write $R^{(1)} = R \otimes_{A/I, \varphi} A/pA$, so $\mathrm{Spec}\, R^{(1)} = \varphi$-pullback of $\mathrm{Spec}\, R$.

Thm $\Delta_{R^{(1)}/A} \simeq R\Gamma_{\mathrm{crys}}(R/A)$ as E_∞-A-algebras compatibly with Frobenius.

Ex: As already noted, $a \in I \Rightarrow a^p \in pA$, so that

$$A/pA \twoheadrightarrow A/I \quad \text{is a nilp. thickening.}$$

Thus if R is smooth over A/I, then we may lift R to $\tilde{R}$ smooth over A/pA, i.e. find $\tilde{R}$ smooth over A/pA s.t.

$$R = \tilde{R} \otimes_{A/pA} A/I .$$

Then

$$R^{(1)} := R \otimes_{A/I, \varphi} A/pA = \tilde{R} \otimes_{A/pA} A/I \otimes_{A/I, \varphi} A/pA$$

$$= \tilde{R} \otimes_{A/pA \to \mathrm{Frob}_{A/pA}} A/pA =: \tilde{R}^{(1)} \quad \text{in the usual (Frob. twist) sense.}$$

$$= \tilde{R} \otimes_{A, \varphi_A} A \quad \leftarrow \text{lift of Frob. on } A$$

We then find that

$$\Delta_{R^{(1)}/A} = \Delta_{(\tilde{R} \otimes_{A, \varphi} A / A)}$$

$$\Delta_{R^{(1)}/A} = R\Gamma_{crys}(R/A)$$

by crys. comparison

$$= \varphi_A^* \Delta_{\tilde{R}/A}$$

once we establish base-change for the prismatic complex (as a consequence of the Hodge-Tate comparison Thm)

Ex: If $I = (p)$, then there is no ambiguity in the choice of $\tilde{R}$; we just have $\tilde{R} = R$,

and so we get $R\Gamma_{crys}(R/A) = \varphi_A^* \Delta_{R/A}$,

so $\Delta_{R/A}$ is a φ_A-descent of $R\Gamma_{crys}(R/A)$.

Proof of comparison theorem:

Consider $B \twoheadrightarrow R$ where B is a p-completely ind-smooth δ-A-alg.

(e.g. write R as the quotient of a free δ-A-alg., and take B to be the p-completion of this.)

$B^\bullet \to R$

$\wr$

p-completed Čech nerve of $A \to B$

$$\left(\begin{array}{c} \downarrow\downarrow\downarrow \\ \operatorname{Spec} B \underset{\operatorname{Spec} A}{\times} \cdots \underset{\operatorname{Spec} A}{\times} \operatorname{Spec} B \end{array} \right) \leftarrow \text{simplicial scheme,}$$

gives cosimplicial ring $B^\bullet$

$\downarrow\uparrow\downarrow$

$\operatorname{Spec} R \hookrightarrow \operatorname{Spec} B \hookrightarrow \underbrace{\operatorname{Spec} B \times \cdots \times \operatorname{Spec} B}_{\text{diagonal}}$

$$\left(\begin{array}{c} \operatorname{Spec} R \\ \downarrow \\ \operatorname{Spec} R \end{array} \right) \qquad \begin{array}{c} B^\bullet \\ \downarrow \\ R \end{array} \leftarrow \text{constant cosimp. ring}$$

$D^\bullet$ = p-completed dp env. of $B^\bullet$ along kernel of

Then $D^\bullet \simeq R\Gamma_{\text{crys}}(R/A)$.

We now show that the δ-A-alg. structure on $B^{\cdot}$ extends to a δ-A-alg. structure on $D^{\cdot}$.

The idea: If B is ind-smooth$/A$, say $B = \varinjlim_i B_i$ with B_i smooth$/A$, then since R is f.g.$/A$ (being smooth over A/pA), we may assume each $B_i \to R$ is surjective (by dropping some initial segment of B_i's).

Then each of these lift to a surjection

$B_i \twoheadrightarrow \tilde{R}$ (if we choose a smooth lift of R to $\tilde{R}$ over A/pA),

inducing a surjection $B_i/pB_i \twoheadrightarrow \tilde{R}$ of smooth A/pA-alg., whose kernel J_i is Zariski locally gen'd by a regular sequence.

Now if K_i = kernel of $B_i \twoheadrightarrow R$, then

$K_i = (I, \text{lift of } J_i)$, and since

I has d. powers, $D_{B_i}(K_i) = D_{B_i}(\text{lift of } J_i)$

Zariski locally gen'd by a sequence which is regular on B_i/pB_i.

If $K = \ker(B \twoheadrightarrow R)$, then $K = \varinjlim_i K_i$, $D_B(K) = \varinjlim D_{B_i}(K_i)$.

Now we apply our results showing that

if ~~[scribble]~~ B is a p-t.f δ-ring, and if $(x_1, \dots, x_r) \in B$ is a regular sequence on B/pB, then

$$B\left\{\frac{\varphi(x_i)}{p}\right\} = D_B((x_i))$$

to conclude that $D_B(K)$ has a δ-structure compatible with that on B.

(I still have to think through the details myself, since our regular sequence occurs for B_i, not B, and B_i isn't a δ-ring; only B is.)

p-completing gives $D_B(K)^\wedge$ a δ-ring structure.

In our set-up $B^\cdot$ is p-completely ind smooth, not actually ind-smooth, so we have to take that into account as well.

But the upshot is that $D^\cdot$ does admit a δ-A-alg. structure.

Now consider the diagram

$$\begin{tikzcd}
D^\cdot \arrow[r, "\varphi_D \text{ : lift of Frob. coming from } \delta\text{-structure}"] \arrow[d] & (\varphi_A)_* D^\cdot \arrow[d] \\
R \arrow[r] & (\varphi_A)_* (D^\cdot / pD^\cdot)
\end{tikzcd}$$

analogue of ψ; it is equivariant w.r.t.

$$\psi : A/I \longrightarrow A/pA \otimes D^\cdot/pD^\cdot.$$

$\therefore$ The map $R \longrightarrow (\varphi_A)_* (D^\cdot/pD^\cdot)$

induces $R^{(1)} := R \underset{A/I, \varphi}{\otimes} A/pA \longrightarrow D^\cdot/pD^\cdot$

Now the diagram

$$\begin{tikzcd} & D^\cdot \arrow[d] \\ R^{(1)} \arrow[r] & D^\cdot/pD^\cdot \end{tikzcd}$$

realize $D^\cdot$ as a co-simplicial object in $\Delta_{R^{(1)}/A}$.

~~As B ranges over all possible choices~~

$\therefore$ get $\Delta_{R^{(1)}/A} \longrightarrow D^\cdot = R\Gamma_{crys}(R/A)$.

As B ranges over all possible choices, these maps are compatible. Since the category of $\{B \twoheadrightarrow R\}$ is sifted, we get a well-defined map

$$\Delta_{R^{(1)}/A} \longrightarrow R\Gamma_{crys}(R/A)$$

ind. of choice of B.

To check it's an iso., choose a lift $\hat{R}$ of R as before, and choose B s.t.

$B \twoheadrightarrow \hat{R} \twoheadrightarrow R$.

p-completion of free δ-A-alg.

Then $B^\bullet \to \hat{R}$, with kernel $J^\bullet$, and our results on prismatic envelopes show that

$$C^\bullet := B^\bullet\left\{\frac{J^\bullet}{p}\right\}^\wedge \quad \text{is base-change compatible}.$$

(Again, there is an "ind" argument to be made here to pass from smooth to ind-smooth.)

(See Cor. 3.14 & Construction 4.16 of Bhatt–Scholze for examples of these sorts of arguments.)

Base-change compatibility shows that $\varphi_A^* C^\bullet$ compute Čech nerve of the $\Delta_{R^{(1)}/A}$. (B/c $\varphi_A^* C^\bullet$ is the prismatic envelope of the p-completely free δ-A-alg. $\varphi_A^* B$, and this prismatic envelope is weakly final, since $\varphi_A^* B$ is p-completely free over A.)

p-complete freeness of $B^\bullet$ over A shows that $A \to B^\bullet$ is a homotopy equivalence, $\therefore$

$$\varphi_A^* C^\bullet \xrightarrow{\sim} \varphi_{B^\bullet}^* C^\bullet = \varphi_{B^\bullet}^*\left(B^\bullet\left\{\frac{J^\bullet}{p}\right\}^\wedge\right)$$

$$= B^\bullet\left\{\frac{\varphi(J^\bullet)}{p}\right\}^\wedge$$

$$= D_{B^\bullet}(J^\bullet)^\wedge = D_{B^\bullet}(K^\bullet)^\wedge$$

$= \ker(B^\bullet \to R) = (J^\bullet, I)$

This again uses our results about d.p. envelopes and δ-structures, with some Ind-argument thrown in.

b/c I has d.powers

As we know, the last expression computes $RP_{cryp}(R/A)$,

$\therefore$ we get the required isomorphism

$$\Delta_{R^{(1)}/A} \xrightarrow{\sim} RP_{cryp}(R/A).$$

$\square$