Adjoints, monads, and comonads

If $G : \mathcal{D} \to \mathcal{C}$ is a functor between categories, it sometimes admits a left-adjoint $F : \mathcal{C} \to \mathcal{D}$, characterized (up to natural equivalences) by natural isomorphism,

$$\mathrm{Mor}_{\mathcal{C}}(X, G(Y)) = \mathrm{Mor}_{\mathcal{D}}(F(X), Y),$$

$$\forall X \in \mathrm{Ob}(\mathcal{C}), \quad Y \in \mathrm{Ob}(\mathcal{D}).$$

If this holds, and if $Y = \lim_{i \in I} Y_i$ is a limit in $\mathcal{D}$, then

$$\mathrm{Mor}_{\mathcal{C}}(X, G(Y)) \underset{\text{adjunction}}{=} \mathrm{Mor}_{\mathcal{D}}(F(X), \lim_{i \in I} Y_i)$$

$$\underset{\text{def. of limit}}{=} \lim_{i \in I} \mathrm{Mor}_{\mathcal{D}}(F(X), Y_i) \overset{\text{adjunction}}{=} \lim_{i \in I} \mathrm{Mor}_{\mathcal{C}}(X, G(Y_i))$$

$$\underset{\text{def. of limit}}{=} \mathrm{Mor}_{\mathcal{C}}(X, \lim_{i \in I} G(Y_i))$$

Since X was arbitrary, we find that

$$G(Y) = \lim_{i \in I} G(Y_i)$$

i.e. that G preserves limits, or (by def'n) is continuous.

Basic examples of functors that preserve limits are representable functors, i.e. functors

$\mathrm{Mor}_{\mathcal{D}}(Y', -)$ for some $Y' \in \mathrm{Ob}(\mathcal{D})$.

(We used this above — it is essentially the def<u>n</u> of a limit.)

Now if G preserves limits, then the functor

$$\mathrm{Mor}_{\mathcal{C}}(X, G(-)) : \mathcal{D} \longrightarrow \mathrm{Sets}$$ preserves limits. If it is representable, i.e. if we can write it as $\mathrm{Mor}_{\mathcal{D}}(F(X), -)$ for some object

$F(X) \in \mathrm{Ob}(\mathcal{D})$, then F would define an adjoint functor to G.

(Given $G : \mathcal{D} \longrightarrow \mathcal{C}$ continuous, we are defining a putative adjoint

$$F : \mathcal{C}^{op} \longrightarrow \mathrm{Funct}_{cont}(\mathcal{D}, \mathrm{Sets})$$

~~via~~

as the composite

$$\mathcal{C}^{op} \xhookrightarrow{\text{Yoneda}} \mathrm{Funct}_{cont}(\mathcal{C}, \mathrm{Sets}) \xrightarrow{\circ G} \mathrm{Funct}_{cont}(\mathcal{D}, \mathrm{Sets}),$$

and asking if it factors through the subcategory

$$\mathcal{D}^{op} \xhookrightarrow{\text{Yoneda}} \mathrm{Funct}_{cont}(\mathcal{D}, \mathrm{Sets}) \, .)$$

Suppose that $\mathcal{D}$ admits all limits, i.e. is complete. Then we have the following

<u>Moral Lemma</u> If $\mathcal{D}$ is complete, the Yoneda embedding

$$\mathcal{D}^{op} \hookrightarrow \mathrm{Funct}_{cont}(\mathcal{D}, \mathrm{Sets})$$

is an equivalence.

<u>Pf</u>: ~~Let I be the cat consisting~~
The Yoneda embedding is fully faithful, so we just have to prove essential surjectivity.

Let $F : \mathcal{D} \to \mathrm{Sets}$ be continuous.
We regard the Yoneda embedding to think of $\mathcal{D}^{op}$ as a subset of the functor category,

so eg. $F(X) = \mathrm{Mor}(F, X)$, if $X \in \mathrm{Ob}(\mathcal{D})$. Then the assumption that F is continuous, ie. that

$$F(\lim X_i) = \lim F(X_i),$$

can be rewritten as $\mathrm{Mor}(F, \lim X_i)$

$$= \lim \mathrm{Mor}(F, X_i).$$

(In other words, $\lim X_i$ is a limit not just in $\mathcal{D}$, but in $\mathrm{Funct}_{\mathrm{cont}}(\mathcal{D}, \mathrm{Sets})$.)

Now let I be the category whose objects are pairs $Y \in \mathcal{D}$ and morphisms $g : F \to Y$, and where morphism $h : (Y, g) \mapsto (Y', g')$ are $h : Y \to Y'$ in $\mathcal{D}$ s.t. $g' = h \circ g$.

Then we have ~~es~~ maps

$F \xrightarrow[g]{} Y$ for objects (Y, g) of I

that are compatible with all the morphisms h,

$\therefore$ get $F \xrightarrow{f} \lim_{I} Y =: X$

If $g \in F(Y)$ for some Y, then the tautological projection $\pi : X \to Y$ gives a commutative diagram

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[dr, "g"'] & X \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

i.e. $g = \pi \circ f$

Thus $\mathrm{Mor}(X, -) \xrightarrow{\circ f} \mathrm{Mor}(F, -)$

is an epi. of sets, $\therefore$ f is monic.

We will construct a subobject X'

of X s.t. $F \xrightarrow{\cong} X' \hookrightarrow X$.

Namely, let X' = simultaneous equalizer for all (Y, g)

of all $a : X \to Y$

s.t. $a \circ f = g$.

Then X' exists (equalizers are limits),

and by construction (and again remembering that

F is continuous)

$f : F \to X$ factors through X', say via

$f' : F \to X'$.

Let $e : X' \hookrightarrow X$ be the canonical mono; then $f = e \circ f'$

There exists $h: X \to X'$ s.t. $f' = h \circ f$. Then $f = e \circ f' = e \circ h \circ f$,

$\therefore$ id_X and $e \circ h$ are equalized by X' (by its construction), i.e. $e = e \circ h \circ e$,

$\therefore h \circ e = id_{X'}$, since e is monic

Now $Mor(X', -) \xrightarrow{\circ f'} Mor(F, -)$ is again surjective

(since $Mor(X, -) \xrightarrow{\circ e} Mor(X', -) \xrightarrow{\circ f'} Mor(F, -)$ is.)

We claim it is bijective.

Indeed, if $a, b: X' \to Y$ s.t.

$a \circ f' = b \circ f'$,

i.e. s.t. $a \circ h \circ f = b \circ h \circ f$,

then $a \circ h$ and $b \circ h$ are equalized by X', i.e. $a \circ h \circ e = b \circ h \circ e$, i.e. $a = b$.

Then F is represented by X'. $\square$

The moral lemma then implies

<u>Moral adjoint functor theorem</u>

If $G : \mathcal{D} \to \mathcal{C}$ preserves limits and $\mathcal{D}$ is complete then G admits a left adjoint $F : \mathcal{C} \to \mathcal{D}$

<u>Pf</u>: As in the above discussion, define F via

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, "\text{Yoneda}"] & \mathrm{Funct}_{cont}(\mathcal{C}, Sets) \arrow[r, "\circ G"] & \mathrm{Funct}_{cont}(\mathcal{D}, Sets) \\
& & \mathcal{D}^{op} \arrow[u, "\text{Yoneda}"', "\wr"]
\end{tikzcd}$$

where the moral lemma gives the indicated equivalence. □

<u>Why "moral"?</u> B/c there are set-theoretic issues we're ignoring: the limits should be taken over small categories, but if $\mathcal{D}$ is complete then $\mathrm{Ob}(\mathcal{D})$ is probably not small, and so things like $\lim_{(Y,g)}$ that appear in the proof of the moral lemma aren't actually valid expressions.

One has to put some size conditions into the statement of the moral lemma, or the moral adjoint functor theorem, to get genuinely correct statements.

See e.g. [Stacks Project, Tag 0AHM]

<u>Basic example</u> $\mathcal{D}$ = cat. of some kind of algebraic structure

$\mathcal{C}$ = cat. of sets

$G : \mathcal{D} \to \mathcal{C}$ the forgetful functor.

Then G preserves limits, and so has an adjoint. The adjoint $F : \mathcal{C} \to \mathcal{D}$ has the concrete interpretation

$X \longmapsto$ free structure generated by X.

There are many variants: eg. if $A \to B$ is a morphism of (commutative) rings (with unit), then the forgetful functor ~~[illegible]~~ $B\text{-Mod} \to A\text{-Mod}$ has the left adjoint $B \otimes_A -$.

Now a group (or ring, module, etc.) is a set "with extra structure".

If $A \to B$ is a morphism of rings, then ~~a~~ B-module is an A-module with extra structure.

The theory of <u>monads</u> abstracts this.

<u>Eg</u>. If M is an A-module, then giving M a B-module structure is the same as giving a morphism of A-modules $B \otimes_A M \xrightarrow{\alpha} M$ s.t. the diagram

$$\begin{tikzcd}
B \otimes_A B \otimes_A M \arrow[r, "id_B \otimes \alpha"] \arrow[d, "mult_B \otimes id_M"'] & B \otimes_A M \arrow[d, "\alpha"] \\
B \otimes_A M \arrow[r, "\alpha"'] & M
\end{tikzcd}$$

and
$$\begin{tikzcd} M \arrow[r, "m \mapsto 1 \otimes m"] \arrow[dr, equal] & B \otimes_A M \arrow[d, "\alpha"] \\ & M \end{tikzcd}$$
both commute.

Note: If we put ourselves in the set-up $G : \mathcal{D} \longrightarrow \mathcal{C}$ with adjoint F

(in our example, $F = B \otimes_A -$, adjoint to the forgetful functor from B-mod to A-mod)

then $B \otimes_A -$ (as an A-module)

is just the composite $G \circ F : \mathcal{C} \longrightarrow \mathcal{C}$

If we denote this composite by

$T : \mathcal{C} \longrightarrow \mathcal{C}$,

then T is an example of a monad.

To explain, recall that the adjunction b/w F & G can be expressed by the existence of natural transformations

$$\eta : id_{\mathcal{C}} \longrightarrow G \circ F \quad (\text{the unit})$$

and $\psi : F \circ G \longrightarrow id_{\mathcal{D}}$ (the counit)

satisfying the so-called "triangle identities"

$$\psi_F \circ F(\eta) = id_F$$

and $G(\psi) \circ \eta_G = id_G$.

If we set $T = G \circ F$,

then $\eta : id_{\mathcal{C}} \to T$,

while ψ gives $\mu : T \circ T = G \circ (F \circ G) \circ F \xrightarrow{G(\psi_F)} G \circ F = T$

The functor T, with the data of η (the unit), and μ (the composition) is called a monad.

The triangle identities imply that $T, \eta, \varepsilon, \mu$ ~~[illegible]~~ fit into commutative diagram

$$
\begin{tikzcd}
T = \mathrm{id}_{\mathcal{C}} \circ T \arrow[r, "\eta_T"] \arrow[dr, equal] & T \circ T \arrow[d, "\mu"] \\
& T
\end{tikzcd}
$$

unit axiom

$$
\begin{tikzcd}
T = T \circ \mathrm{id}_{\mathcal{C}} \arrow[r, "T(\eta)"] \arrow[dr, equal] & T \circ T \arrow[d, "\mu"] \\
& T
\end{tikzcd}
$$

and

$$\begin{tikzcd} T\circ T\circ T \arrow[r, "T(\mu)"] \arrow[d, "\mu_T"] & T\circ T \arrow[d, "\mu"] \\ T\circ T \arrow[r, "\mu"] & T \end{tikzcd}$$

associative axiom.

In our A- & B-modules example,

T is the functor $M \longmapsto B \otimes_A M$ (thought of as an A-module)

μ is the natural transformation

$$B \otimes_A (B \otimes_A M) = (B \otimes_A B) \otimes_A M \xrightarrow{\text{mult}_B} B \otimes_A M$$

and η is the natural transformation

$$M \xrightarrow{m \mapsto 1 \otimes m} B \otimes_A M .$$

A monad is a functor $T : \mathcal{C} \to \mathcal{C}$

with natural transformation

$\eta : id_{\mathcal{C}} \to T$

and $\mu : T \circ T \to T$,

satisfying the above unit and associativity axioms.

An algebra over the monad T

is an object $X \in Ob(\mathcal{C})$ with a morphism $h : TX \to X$, s.t.

the diagrams

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & TX \arrow[d, "h"] \\ & X \end{tikzcd}$$

and

$$\begin{tikzcd} T(TX) \arrow[r, "T(h)"] \arrow[d, "\mu_X"] & TX \arrow[d, "h"] \\ TX \arrow[r, "h"'] & X \end{tikzcd}$$

commute.

A morphism of algebras $(X, h) \to (X', h')$ is $f : X \to X'$ in $\mathcal{C}$ s.t.

$$\begin{tikzcd} TX \arrow[r, "h"] \arrow[d, "T(f)"] & X \arrow[d, "f"] \\ TX' \arrow[r, "h'"] & X' \end{tikzcd}$$

commutes.

We let $\mathcal{C}^T$ denote the category of T-algebras.

Eg. In our running example, where $T(M) = B \otimes_A M$ (as an A-module), a T-algebra structure on the A-module M is a morphism $B \otimes_A M \xrightarrow{h} M$

s.t. $M \xrightarrow{m \mapsto 1 \otimes m} B \otimes_A M \xrightarrow{h} M$ is equal to id_M,

and s.t.

$$\begin{tikzcd}
B \otimes_A B \otimes_A M \arrow[r, "id_B \otimes h"] \arrow[d, "mult_B \otimes id_M"] & M \arrow[d, "h"] \\
B \otimes_A M \arrow[r, "h"] & M
\end{tikzcd}$$

commutes.

As we saw above, this structure is the same as making M into a B-module.

In other words, in this particular set-up of adjoint functors $G : \mathcal{D} \to \mathcal{C}$, $F : \mathcal{C} \to \mathcal{D}$, and $T = G \circ F : \mathcal{C} \to \mathcal{C}$,

we have an equivalence of categories

$$\mathcal{D} \xrightarrow{\simeq} \mathcal{C}^T .$$

—

In general, if F, G are adjoint functors, we get a functor

$$\mathcal{D} \longrightarrow \mathcal{C}^T$$

defined via

$$Y \longmapsto \left(G(Y), \; \begin{array}{c} TG(Y) = G\circ F \circ G(Y) \\ \downarrow{\scriptstyle G(\psi_Y)} \\ G(Y) \end{array} \right)$$

We say that G is _monadic_ if this functor is an equivalence

What we have seen is that the forgetful functor $B\text{-Mod} \to A\text{-Mod}$ is monadic.

These forgetful functors have a special property (in addition to being right adjoints): they preserve "reflexive co-equalizers."

A coequalizer is the colimit of a diagram

$$Y \rightrightarrows Y'$$

The pair of morphisms is called "reflexive" if they have a common section $Y \leftarrow Y'$.

Now if $Y \underset{g}{\overset{f}{\rightrightarrows}} Y'$ are morphisms of sets, the coequalizer of f & g is the quotient of Y' by the equivalence relation arising from "zig zags"

$$y'_1 = h_1(y_1) \sim h_2(y_1) = y'_2 = h_3(y_2) \sim h_4(y_2) = y'_3 \quad \ldots$$

where the $y'_i \in Y'$, the $y_i \in Y$, and each h_i is either f or g.

If Y and Y' are algebraic structure, and f, g are homomorphisms, then this equivalence rel'n may not be a "congruence" (in the sense of universal algebra)

i.e. may not be respected by the algebraic operations.

(Eg f could be the inclusion of a ~~[illegible]~~ subgroup Y into a group Y', and g could be the composite $Y \twoheadrightarrow \{1\} \hookrightarrow Y'$.

Then in sets, we coequalize f & g just by crushing Y to a point and leaving all other points alone.

But in groups we have to crush all cosets of Y to points. And also replace Y by its normal closure, if Y is not normal.)

But suppose now $Y \overset{f}{\underset{g}{\rightrightarrows}} Y'$

is reflexive, with common section

$s : Y' \to Y$.

Then if we have a zig zag, that begins

$y' = g(y) \sim \dots$,

we can add $y' = f(s(y')) \sim g(s(y')) = y' = g(y) \dots$
so that it begins with "f".
Similarly, we can make it end with "g".

Suppose somewhere in the middle we have

$$\cdots = g(y) \sim f(y) = y' = g(\bar{y}) \sim f(\bar{y}) = \cdots$$

we can pad out this zig-zag to look like

$$\cdots = g(y) \sim f(y) = y' = f(s(y')) \sim g(s(y')) = y' = g(\bar{y}) \sim f(\bar{y})$$
$$= \cdots$$

(and similarly if the roles of f & g are switched)

and thus we can assume that the h_i are in the pattern

$$f, g, g, f, f, g, g, f, f, \ldots, g, g, f, f, g$$

Using s in a similar way, we can always pad out a zig-zag to be longer (adding any pattern of the form $g, f, f, g, g, \ldots, f, f, g$ at the end).

Now its easy to see that if Y and Y' are equipped with same collection of n-ary operations (for various choices of n) and if f, g ~~maps~~ are homomorphisms, then the equiv. relation that coequalizes f and g is a congruence.

(The more-or-less equivalent fact is that when f & g admit a common section s, the coequalizer (in sets) of the pair

$$Y^n \underset{g^n}{\overset{f^n}{\rightrightarrows}} (Y')^n$$

is the n-fold product of the coequalizer (in sets) of f & g.)

Thus if $G : \mathcal{D} \to \mathcal{C}$ is a forgetful functor from one algebraic category to another (given by forgetting some part of the structure) then G preserves reflexive coequalizers. (And, by considering the case when $\mathcal{C} = \text{sets}$ and G forgets <u>all</u> the structure, we see that these coequalizers agree with those computed in sets.)

G also "reflects isomorphisms".
(Any functor G preserves isomorphisms, but in this case, if $G(g)$ is an iso., so is g itself. This again b/c a morphism of algebraic structures is an iso. iff it is a bijection, i.e. iff it induces an iso. on the underlying sets.)

We now have the following theorem, which places our discussion of A- and B-modules in a general context.

Thm. (Crude monadicity theorem)

If $G : \mathcal{D} \to \mathcal{C}$ admits a left adjoint $F : \mathcal{C} \to \mathcal{D}$, if $\mathcal{D}$ admits all reflexive coequalizers and if G preserves them, and if G reflects isomorphisms, then G is monadic, i.e. the induced functor

$$\mathcal{D} \to \mathcal{C}^T \qquad (\text{where } T = G \circ F)$$

is an equivalence.

This theorem is "crude" b/c it does not give a necessary condition for monadicity, just a sufficient one. There is a "precise monadicity theorem" too, but we don't discuss it here.

Applying this theorem to forgetful functors on categories of algebraic structures, we find that in general, "more complicated" algebraic structures can be understood as algebras over monads on "less complicated" structures.

Of course, there is a "dual" story given by reversing arrows, which leads to comonads.

A comonad $\mathcal{S}: \mathcal{D} \to \mathcal{D}$

has a counit $\psi: \mathcal{S} \to id_{\mathcal{D}}$

and a comultiplication $\Delta: \mathcal{S} \to \mathcal{S} \circ \mathcal{S}$,

satisfying some obvious axioms (counit axiom, coassociativity axiom).

If $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ are adjoint as before, then $\mathcal{S} = F \circ G: \mathcal{D} \to \mathcal{D}$ is a comonad.

Eg. If G is the forgetful functor

$$B\text{-Mod} \to A\text{-mod},$$

and F is $B \underset{A}{\otimes} -$,

then $\mathcal{S}: N \longmapsto B \underset{A}{\otimes} N$,

with the B-action on ~~the~~ $B \underset{A}{\otimes} N$ taking place on the left factor.

Comonads admit coalgebras,

i.e. $Y \in ob(\mathcal{D})$ equipped with a co-action

$$Y \xrightarrow{k} SY$$

satisfying the evident axioms (dual to those for algebras over a monad). These form a category $\mathcal{D}^S$

We then have

<u>Thm</u> (Crude comonadicity theorem)

Suppose $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ are adjoint as before, that $\mathcal{C}$ admits and F preserves equalizers,(*) and that F reflects isomorphisms. Then F is comonadic, i.e. the induced functor

$$\mathcal{C} \longrightarrow \mathcal{D}^S$$

is an equivalence.

(*) We need only consider "coreflexive" equalizers, but don't bother with this refinement.

Let's again consider our example

$$F = B \otimes_A - \quad , \quad G = \text{forgetful functor from } B\text{-Mod to } A\text{-Mod}.$$

When does B preserve ~~[illegible]~~ equalizers (i.e. preserve kernels) and reflect isomorphisms?

Exactly if B is faithfully flat over A!

$\therefore$ $A\text{-Mod} = S\text{-coalgebras in } B\text{-Mod}.$

Now what *is* an S coalgebra?

It is the data of $k: N \to B \otimes_A N$

a B-linear map (w/ the B-action on the target being given via the action on the first factor) s.t. $N \xrightarrow{k} B \otimes_A N \xrightarrow{B\text{-action on } N} N$ is id_N

and s.t.

$$\begin{tikzcd} N \arrow[r, "k"] \arrow[d, "k"'] & B \otimes_A N \arrow[d, "\mathrm{id}_B \otimes k"] \\ B \otimes_A N \arrow[r] & B \otimes_A B \otimes_A N \end{tikzcd}$$

$$b \otimes n \mapsto 1 \otimes b \otimes n$$

commutes.

So the comonadicity theorem in this context is faithfully flat descent.

The morphism $k : N \to B \otimes_A N$ induces a morphism $N \otimes_A B \to B \otimes_A N$ of $B \otimes_A B$-modules, which is the descent data. The commutative square gives the cocycle condition.