EXPLICIT RECIPROCITY LAWS AND IWASAWA THEORY FOR MODULAR FORMS

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Abstract. We prove that the Mazur-Tate elements of an eigenform f sit inside the Fitting ideals of the corresponding dual Selmer groups along the cyclotomic $\mathbb{Z}_p$-extension (up to scaling by a single constant). Our method begins with a construction of local cohomology classes built via the p-adic local Langlands correspondence. From these classes, we build *algebraic* analogues of the Mazur-Tate elements which we directly verify sit in the appropriate Fitting ideals. Using Kato's Euler system and explicit reciprocity laws, we prove that these algebraic elements divide the corresponding Mazur-Tate elements, implying our theorem.

1. Introduction

Associated to a newform $f \in S_k(\Gamma_1(N), \psi, \overline{\mathbb{Q}}_p)$ is its collection of Mazur-Tate elements $\theta^{\mathrm{an}}_{n,r}(f) \in \mathcal{O}[\mathcal{G}_n]$ with $n \geq 0$, $\mathcal{O}/\mathbb{Z}_p$ finite, and $\mathcal{G}_n \cong \mathrm{Gal}(\mathbb{Q}(\mu_{p^n}/\mathbb{Q})$. Here the superscript 'an' is meant to remind us that these elements are analytically defined and are connected to L-values via[1]

$$\chi(\theta^{\mathrm{an}}_{n,r}(f)) = \chi(-1) \cdot (r-1)! \cdot p^{n(r-1)} \cdot \tau(\chi^{-1}) \cdot \frac{L(f, \chi, r)}{(-2\pi i)^{r-1}\Omega_f^{\pm}}$$

where χ is a primitive Dirichlet character of conductor p^n and $\Omega_f^{\pm}$ are the canonical periods attached to f as in [36].

When f corresponds to an elliptic curve E, Mazur and Tate in [25] conjectured that $\theta^{\mathrm{an}}_{n,r}(f)$ (defined with respect to an appropriate period, which should coincide, up to p-adic unit, with the canonical period when the Galois representation on $E[p]$ is irreducible) belongs to the Fitting ideal of the corresponding (dual) Selmer group. In fact, they formulated their conjecture more generally, looking at Selmer groups over any abelian extension of $\mathbb{Q}$. However, as the methods of this paper are rooted in Iwasawa theory, we restrict ourselves to the case of p-power cyclotomic fields and aim to prove an analogous conjecture for modular forms of arbitrary weight.

To state our main theorem, we first set some notation. Let ρ_f denote the (cohomological) p-adic Galois representation attached to f and let V_f denote the underlying vector space. Let V_f^* denote the linear dual of V_f and let T_f^* denote a Galois stable lattice of V_f^*. Set $A_f^* = V_f^*/T_f^*$ and let $H^1_g(\mathbb{Q}(\mu_{p^n}), A_f^*)$ denote the Block-Kato Selmer group attached to A_f^* over $\mathbb{Q}(\mu_{p^n})$.

The following hypotheses will be needed for our main result:

(Irred) $\overline{\rho}_f$ and $\rho_f|_{G_{\mathbb{Q}_p}}$ are irreducible;

(No pole at $s = r$) the local Euler factor of f at p does not have a pole at $s = r$;

[1] See Remarks 10.1.1 and 10.1.3 if you were expecting χ to be replaced by χ^{-1}.

(Euler) there is a basis of T_f^* such that if we identify $\mathrm{Aut}(T_f^*)$ with $\mathrm{GL}_2(\mathcal{O})$, then the image of $\rho_f(T_f^*)$ contains $\mathrm{SL}_2(\mathcal{O})$.

Theorem A. *If* (Irred)*,* (No pole at $s = r$)*, and* (Euler) *hold, then there exists a non-zero constant $C \in \mathcal{O}$ (independent of n and r) such for all $n \geq 0$*

$$C \cdot \theta_{n,r}^{\mathrm{an}}(f) \in \mathrm{Fitt}_{\Lambda_n}(H_g^1(\mathbb{Q}(\mu_{p^n}), A_f^*(1-r))^\vee)$$

where $\Lambda_n := \mathcal{O}[\mathcal{G}_n]$ and $^\vee$ denotes Pontryagin dual.

Theorem A should be viewed as a "finite-level" (weak) main conjecture for f. When f is ordinary at p, one can deduce this theorem directly from the main conjecture under relatively mild hypotheses simply by using standard control theorems (see for instance [22, Theorem 1.14]). We note though that (Irred) forces f to be non-ordinary at p. In particular, the standard control theorems no longer hold and there seems to be no direct connection between the main conjecture and Theorem A in this case. We further note that our theorem holds without any restriction on the power of p that occurs in the level of our form: we have no crystalline hypotheses nor a finite slope hypothesis.

We will discuss the constant C appearing in the statement of Theorem A later in the introduction, but for now let us just say that it arises because of our inability to compare two (potentially different) normalizations of the periods attached to f.

Our method of proof of Theorem A follows two steps. First we define the notion of an *algebraic* θ-element, $\theta_n^{\mathrm{alg}}(f) \in \Lambda_n$, which is meant to be the algebraic counterpart of the corresponding Mazur-Tate element. The construction of such elements dates back to Perrin-Riou's work [30] (see also [33, 32]). As these elements are defined purely algebraically, we are able to check directly that $\theta_n^{\mathrm{alg}}(f)$ belongs to the Fitting ideal appearing in Theorem A. The second step then is to prove that $\theta_n^{\mathrm{alg}}(f)$ divides $C \cdot \theta_{n,r}^{\mathrm{an}}(f)$ in Λ_n. Unsurprisingly, our proof of this fact comes via Kato's Euler system.

Our construction of algebraic θ-elements relies upon certain local cohomology classes $c_{n,j} \in H_e^1(\mathbb{Q}_p(\mu_{p^n}), T_f(1+j))$ which for $n \geq 1$ satisfy the three-term relation

$$\mathrm{cor}_n^{n+1}(c_{n+1,j}) = a_p(f) c_{n,j} - \psi(p) p^{k-1} \mathrm{res}_{n-1}^n(c_{n-1,j}).$$

With these classes in hand, we can directly define $\theta_n^{\mathrm{alg}}(f)$. Namely

$$\theta_{n,r}^{\mathrm{alg}}(f) = \sum_{\sigma \in \mathcal{G}_n} \left\langle c_{n,r-1}^\sigma, w_n \right\rangle_n \sigma^{-1} \cdot \mathrm{char}_\Lambda \mathbb{H}_P^2(T_f^*(1-r))^\iota.$$

Here $(w_n)_n$ is a generator of $\mathbb{H}^1(T_f^*(1-r))$ which is free of rank 1 over $\Lambda := \mathcal{O}[[\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})]]$ and $\langle \cdot, \cdot \rangle_n$ is the Tate local duality pairing at p. (See section 12.1 for the remaining notation.)

The relation of $\theta_n^{\mathrm{alg}}(f)$ to $\theta_{n,r}^{\mathrm{an}}(f)$ comes via Kato's Euler system as we prove that

$$C \cdot \theta_{n,r}^{\mathrm{an}}(f) = \sum_{\sigma \in \mathcal{G}_n} \left\langle c_{n,r-1}^\sigma, z_{\mathrm{K},n} \right\rangle_n \sigma^{-1}$$

where $(z_{\mathrm{K},n})_n \in \mathbb{H}^1(T_f^*(1-r))$ is Kato's Euler system and C is some non-zero constant (independent of n and r). The divisibility of $C \cdot \theta_{n,r}^{\mathrm{an}}(f)$ by $\theta_{n,r}^{\mathrm{alg}}(f)$ then follows immediately from Kato's proof of the main conjecture without p-adic L-functions [20, Theorem 12.5].

We note that in section 8.2, when we introduce algebraic θ-elements, we give a different definition from the one above. This alternative definition is more easily connected with Fitting ideals, while the above definition more easily relates to $\theta^{\mathrm{an}}_{n,r}(f)$ via Kato's Euler system. In section 11.2, we verify that these two definitions are the same.

We are now left to describe our construction of the local classes $c_{n,j}$. We note that these local classes have appeared in literature in several places before. For example, Kobayashi in [23] made use of such local points to define plus/minus Selmer groups of elliptic curves at supersingular primes. His construction proceeded via the formal group attached to the elliptic curve. As we are working with arbitrary weight modular forms, this geometric object is unavailable to us. We instead give a purely local construction of these classes via the p-adic local Langlands correspondence.

Namely, let V denote an irreducible de Rham representations of $G_{\mathbb{Q}_p}$ defined over a finite extension $L/\mathbb{Q}_p$; we will produce elements $c_{n,j} \in H^1_e(\mathbb{Q}_p(\mu_{p^n}), T(1+j))$ for T a $G_{\mathbb{Q}_p}$-stable lattice in V. For the remainder of the introduction, we will assume that $j = 0$ to ease the exposition and simply write c_n for the class $c_{n,j}$. To construct these classes, we first produce, for each $n \geq 0$, a functional on $H^1_{\mathrm{Iw}}(V^*) := \left(\varprojlim_n H^1(\mathbb{Q}_p(\mu_{p^n}), T^*)\right) \otimes \mathbb{Q}_p$ which:

(a) factors through $H^1(\mathbb{Q}_p(\mu_{p^n}), T^*) \otimes \mathbb{Q}_p$,
(b) takes integral values on $H^1(\mathbb{Q}_p(\mu_{p^n}), T^*)$, and
(c) kills $H^1_g(\mathbb{Q}_p(\mu_{p^n}), T^*)$.

By Tate local duality, such a functional corresponds to a class $c_n \in H^1_e(\mathbb{Q}_p(\mu_{p^n}), T(1))$.

To produce this functional, we identify $H^1_{\mathrm{Iw}}(V^*)$ with ψ-invariants of the (φ, Γ)-module $\mathbf{D}(V^*)$. The theory of p-adic local Langlands gives rise to a Banach space representation $\pi(V)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$, and moreover, an embedding

$$\mathbf{D}(V^*)^{\psi=1} \hookrightarrow \pi(V)^*$$

where the superscript $*$ on the right denotes the topological dual.

Thus, to define functionals on $\mathbf{D}(V^*)^{\psi=1}$, it suffices to simply give natural elements of $\pi(V)$. Such natural elements arise from the locally algebraic vectors in $\pi(V)$. Namely, there is an embedding

$$\pi_{\mathrm{sm}}(V) \otimes (\mathrm{Sym}^k(L^2))^* \hookrightarrow \pi(V)$$

where $\pi_{\mathrm{sm}}(V)$ is the smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representation associated to V via classical local Langlands. Let v_{new} denote a newvector of $\pi_{\mathrm{sm}}(V)$ and let v_{hw} denote a highest weight vector of $(\mathrm{Sym}^k(L^2))^*$. For $n \geq 0$, set

$$d_n := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right) (v_{\mathrm{new}} \otimes v_{\mathrm{hw}}) \in \pi(V)$$

which, as described above, gives rise to a functional on $H^1_{\mathrm{Iw}}(V^*)$.

Moreover, we have the following explicit reciprocity law, proven (essentially) by Colmez [10, Prop. VI.3.4]: if $z' \in H^1_{\mathrm{Iw}}(V^*) \cong \mathbf{D}(V^*)^{\psi=1}$, we have

$$\{d_n, z'\} = \langle \alpha_n, \exp^*(z'_n) \rangle$$

where z'_n is the projection of z' to $H^1(\mathbb{Q}_p(\mu_{p^n}), V^*)$ and α_n is some element of $\mathbf{D}_{\mathrm{dR},n}(V(1))$ which is described precisely in Corollary 7.3.2. For now, let us just mention that α_n arises by viewing $(1-\varphi)^{-1}(d_n)$ in a generalized Kirillov model of $\pi(V)$, and then evaluating at p^{-n}.

From this explicit formula for $\{d_n, z'\}$ we see immediately that the pairing factors through $H^1(\mathbb{Q}_p(\mu_{p^n}), V^*)$, as the dependence of the right hand of the formula on z' is only through z'_n. Moreover, the formula vanishes for $z'_n \in H^1_g(\mathbb{Q}_p(\mu_{p^n}), V^*)$

as this subspace is the kernel of $\exp^*$. In particular, there are unique classes $c_n \in H^1_e(\mathbb{Q}_p(\mu_{p^n}), V^*)$ such that

$$\{d_n, z'\} = \langle c_n, z'_n \rangle_n;$$

here the pairing on the right is the pairing on Galois cohomology arising from Tate local duality. The fact that the c_n satisfy a three-term relation follows from the Hecke properties of v_{new}. Further we mention that this reciprocity law is the key to our comparison of $\sum_{\sigma \in \mathcal{G}_n} \langle c_n^\sigma, z_{\mathrm{K},n} \rangle_n \sigma^{-1}$ with $\theta^{\mathrm{an}}_{n,1}(f)$, as one recovers L-values from $z_{\mathrm{K},n}$ via the dual exponential map.

We now discuss the constant C occurring in the statement of Theorem A. The above construction of the classes c_n only determines them up to a scalar as the class $v_{\mathrm{new}} \otimes v_{\mathrm{hw}}$ is only well-defined up to a scalar. Our normalization of the choice of this element is spelled out in section 7.7 and is well-defined up to a p-adic unit. For now, we just mention that our normalization guarantees that $v_{\mathrm{new}} \otimes v_{\mathrm{hw}} \in \pi(T)$ and thus the c_n take values in $T(1)$ (rather than just in $V(1)$). It appears difficult however to compare this normalization to the normalization one makes to arise at the periods $\Omega_f^\pm$ which are also well-defined up to a p-adic unit. Without a comparison of these two normalizations, we are obliged to state our theorem with the ambiguity of a single non-zero constant in $\mathcal{O}$ (independent of n and r).

Lastly, we note that Chan-Ho Kim in [21] has recently proven (under some assumptions) Theorem A in the crystalline case with $j = k/2$, and moreover, he can control the periods that appear when $2 \leq k \leq p-1$.

Regarding the format of the paper, we have divided the paper into two parts. The first part is entirely local and devoted to the construction of the classes $c_{n,j}$. The second half of the paper is more global in nature and is devoted to the construction of algebraic theta elements and the proof of Theorem A sketched above. The second part depends on the first part only through (a) the existence of the local classes $c_{n,j}$ satisfying a three-term relation, and (b) an explicit formula for $\sum_\sigma \langle c^\sigma_{n,j}, z' \rangle_n \sigma$ (Proposition 7.4.1) which is derived through the explicit reciprocity law. The reader only interested in the global methods from Iwasawa theory may attempt to read the second part with only a limited knowledge of the first part.

Part 1. **The local theory**

2. Rings

We introduce various rings of Fontaine using the notation introduced by Berger and Colmez.

2.1. **Rings in characteristic p.**

The basic ring $\widetilde{\mathbf{E}}^+$ is the perfection of the characteristic p ring $\mathcal{O}_{\mathbb{C}_p}/p$:

$$\widetilde{\mathbf{E}}^+ := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p.$$

This ring admits another description, namely

$$\widetilde{\mathbf{E}}^+ := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p};$$

with this latter description, the multiplication is given in the obvious way, but the addition involves taking an appropriate limit (since raising to the p-th power is a multiplicative homomorphism of $\mathcal{O}_{\mathbb{C}_p}$, but is not an additive homomorphism).

These two definitions give the same object as the map from the second projective limit to the first given by reducing each component modulo p is an isomorphism.

The second description has an advantage over the first, in that it puts into evidence the fact that $\widetilde{\mathbf{E}}^+$ is a complete valuation ring, with valuation given simply by projecting onto the initial component in the projective limit, and taking the p-adic valuation.

We let $\widetilde{\mathbf{E}}$ denote the field of fractions of the valuation ring $\widetilde{\mathbf{E}}^+$. Alternately, it can be described as

$$\widetilde{\mathbf{E}} := \varprojlim_{x \mapsto x^p} \mathbb{C}_p,$$

with multiplication being given componentwise in the projective limit, and addition being given by the same formula as before.

We choose now a generator of $\mathbb{Z}_p(1)$ and denote the corresponding element of $\widetilde{\mathbf{E}}^+$ as ε. The valuation of $\varepsilon - 1$ is equal to $p/(p-1)$, which is positive. Thus we also have the description $\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}^+[(\varepsilon - 1)^{-1}]$.

We define

$$\mathbf{E}^+_{\mathbb{Q}_p} := \mathbb{F}_p[[\varepsilon - 1]] \subset \widetilde{\mathbf{E}}^+,$$

and

$$\mathbf{E}_{\mathbb{Q}_p} := \mathbb{F}_p((\varepsilon - 1)) \subset \widetilde{\mathbf{E}}.$$

The valuation on $\widetilde{\mathbf{E}}$ induces the usual discrete valuation (with the slightly odd $p/(p-1)$ normalization mentioned above) on $\mathbf{E}_{\mathbb{Q}_p}$.

The field $\widetilde{\mathbf{E}}$ is algebraically closed, and is in fact the completion of the algebraic closure of $\mathbf{E}_{\mathbb{Q}_p}$, or, equivalently, of the separable closure of $\mathbf{E}_{\mathbb{Q}_p}$. We write $\mathbf{E} \subset \widetilde{\mathbf{E}}$ to denote the separable closure of the field $\mathbf{E}_{\mathbb{Q}_p}$ (it is a non-complete valuation ring), and also write $\mathbf{E}^+ := \mathbf{E} \cap \widetilde{\mathbf{E}}^+$ to denote the ring of integers in $\mathbf{E}$. As already noted, $\widetilde{\mathbf{E}}$ is the completion of $\mathbf{E}$. Finally, we let $\widetilde{\mathbf{E}}_{\mathbb{Q}_p}$ denote the completion of the radiciel (i.e. purely inseparable) closure of $\mathbf{E}_{\mathbb{Q}_p}$ in $\widetilde{\mathbf{E}}$, and write $\widetilde{\mathbf{E}}^+_{\mathbb{Q}_p} := \widetilde{\mathbf{E}}_{\mathbb{Q}_p} \cap \widetilde{\mathbf{E}}^+$.

An important point is that $\mathrm{Aut}(\widetilde{\mathbf{E}}/\mathbf{E}_{\mathbb{Q}_p}) = \mathrm{Gal}(\mathbf{E}/\mathbf{E}_{\mathbb{Q}_p}) = H$, where $H \subset G_{\mathbb{Q}_p}$ is the kernel of the cyclotomic character, and also that $\mathbf{E}_{\mathbb{Q}_p} = \mathbf{E}^H$, while $\widetilde{\mathbf{E}}_{\mathbb{Q}_p} = \widetilde{\mathbf{E}}^H$.

2.2. **General notational principles.** In addition to introducing a host of fields and domains, the preceding subsection illustrates some general notational conventions: (i) Objects in characteristic p are denoted via $\mathbf{E}$; (ii) The objects with tildes are in some sense complete, while the objects without the tildes are the purely algebraic, "decompleted" analogues; (iii) The objects with $+$ superscripts are integral with respect to the valuation; (iv) The objects with the $\mathbb{Q}_p$-subscript are the ones with trivial Galois action.

There will be some further notational conventions introduced below: (v) objects in characteristic zero obtained from the $\mathbf{E}$s by Witt vector or Cohen ring constructions will be denoted with an $\mathbf{A}$; (vi) objects obtained from the various $\mathbf{A}$s by inverting p will be denoted with a $\mathbf{B}$.

2.3. **Rings in characteristic zero.** The key ring is now

$$\widetilde{\mathbf{A}}^+ := W(\widetilde{\mathbf{E}}^+)$$

where W denotes Witt vectors. This gives rise to other $\mathbf{A}$-type rings related to (φ, Γ)-modules, and also underlies the definition of $\mathbf{B}_{\mathrm{crys}}$ and $\mathbf{B}_{\mathrm{dR}}$. We let φ denote the Frobenius on $\widetilde{\mathbf{A}}^+$. As well as its p-adic topology, the ring $\widetilde{\mathbf{A}}^+$ has another topology, which is more natural to consider, namely its so-called weak topology: we think of the the Witt vectors $W(\widetilde{\mathbf{E}}^+)$ as being isomorphic to the product $\prod_{i=0}^{\infty} \widetilde{\mathbf{E}}^+$, and give them the corresponding product topology, where $\widetilde{\mathbf{E}}^+$ is given its valuation topology.

An important element in $\widetilde{\mathbf{A}}^+$ is $T := [\varepsilon] - 1$. (Here, as usual, $[x]$ denotes the circular lift to the Witt vectors of an element $x \in \widetilde{\mathbf{E}}^+$.) Note that this lifts the positive valuation element $\varepsilon - 1$ of $\widetilde{\mathbf{E}}^+$. Thus the weak topology on $\widetilde{\mathbf{A}}^+$ is also the (p, T)-adic topology.

Since T lifts $\varepsilon - 1$, we find that

$$T \equiv [\varepsilon - 1] \bmod p\widetilde{\mathbf{A}}^+,$$

and hence that

$$\widehat{\widetilde{\mathbf{A}}^+[\frac{1}{T}]} = \widehat{\widetilde{\mathbf{A}}^+[\frac{1}{[\varepsilon - 1]}]}$$

(where $\widehat{\ }$ denotes the p-adic completion); we denote these (equal) rings by $\widetilde{\mathbf{A}}$. Since $\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}^+[(\varepsilon - 1)^{-1}]$, we have the more canonical description

$$\widetilde{\mathbf{A}} := W(\widetilde{\mathbf{E}}).$$

We now imitate all the other $\mathbf{E}$ constructions in this new $\mathbf{A}$ context. Firstly, we set

$$\mathbf{A}_{\mathbb{Q}_p} := \widehat{\mathbb{Z}_p((T))} \subset \widetilde{\mathbf{A}}$$

(this is a discretely valued Cohen ring with uniformizer p and residue field $\mathbf{E}_{\mathbb{Q}_p}$), and then we write

$$\mathbf{A}_{\mathbb{Q}_p}^+ := \mathbb{Z}_p[[T]] = \mathbf{A}_{\mathbb{Q}_p} \cap \widetilde{\mathbf{A}}^+.$$

We then let $\mathbf{A}$ denote the p-adic completion of the maximal unramified extension of $\mathbf{A}_{\mathbb{Q}_p}$ (in $\widetilde{\mathbf{A}}$), which is a Cohen ring with residue field $\mathbf{E}$, and let $\mathbf{A}^+ := \mathbf{A} \cap \widetilde{\mathbf{A}}^+$. Since $\mathbf{E}^+$ is dense in $\widetilde{\mathbf{E}}^+$, we find that $\mathbf{A}^+$ is weakly dense in $\widetilde{\mathbf{A}}^+$.

Finally, we write

$$\widetilde{\mathbf{A}}_{\mathbb{Q}_p} := W(\widetilde{\mathbf{E}}_{\mathbb{Q}_p}),$$

and

$$\widetilde{\mathbf{A}}_{\mathbb{Q}_p}^+ := W(\widetilde{\mathbf{E}}_{\mathbb{Q}_p}^+) = \widetilde{\mathbf{A}}_{\mathbb{Q}_p} \cap \widetilde{\mathbf{A}}^+.$$

The H-action on $\widetilde{\mathbf{E}}$ induces an H-action on $\widetilde{\mathbf{A}}$, and

$$\widetilde{\mathbf{A}}^H = \widetilde{\mathbf{A}}_{\mathbb{Q}_p}.$$

Since H fixes ε, we find that the H-action on $\widetilde{\mathbf{A}}$ fixes T, and hence fixes $\mathbf{A}_{\mathbb{Q}_p}$ elementwise. Thus it preserves $\mathbf{A}$, and we have

$$\mathbf{A}^H = \mathbf{A}_{\mathbb{Q}_p}.$$

2.4. **The B-rings.** We can replace $\mathbf{A}$ by $\mathbf{B}$ everywhere by inverting p. There is the famous surjection

$$\theta : \widetilde{\mathbf{A}}^+ \to \mathcal{O}_{\mathbb{C}_p},$$

which extends to a surjection

$$\theta : \widetilde{\mathbf{B}}^+ \to \mathbb{C}_p,$$

and as usual we write $\mathbf{B}_{\mathrm{dR}}^+$ to denote the completion of $\widetilde{\mathbf{B}}^+$ with respect to the kernel of θ.

Note that the kernel of θ in $\widetilde{\mathbf{A}}^+$, and hence also in $\widetilde{\mathbf{B}}^+$, is principal, generated by $T/\varphi^{-1}(T)$. This latter element is traditionally denoted ω. Note also that the famous element t (see just below) does not belong to $\widetilde{\mathbf{B}}^+$, but only to the completion $\mathbf{B}_{\mathrm{dR}}$; there is thus some advantage to being aware of the element ω, which generates the kernel of θ *in* $\widetilde{\mathbf{B}}^+$. Indeed, we will have use for ω later.

Recall that $\mathbf{B}_{\mathrm{dR}}^+$ is a DVR with uniformizer $t := \log(1+T)$ and with residue field $\mathbb{C}_p$ (given by the extension of θ, which we again denote by $\theta : \mathbf{B}_{\mathrm{dR}}^+ \to \mathbb{C}_p$). We set

$$\mathbf{B}_{\mathrm{dR}} := \mathbf{B}_{\mathrm{dR}}^+[1/t] = \mathbf{B}_{\mathrm{dR}}^+[1/T].$$

(To see the asserted equality, note that

$$t = \log(1+T) = T\left(\sum_{n=0}^{\infty}(-1)^n\frac{T^n}{n+1}\right),$$

where the second factor in the right-hand expression is a unit in $\mathbf{B}_{\mathrm{dR}}^+$, as $T \in \ker\ \theta$.)

There are further observations regarding T, t, and $\mathbf{B}_{\mathrm{dR}}^+$ that are useful. For example, $\varphi(T)/T = \varphi(\omega)$ is a unit in $\mathbf{B}_{\mathrm{dR}}^+$ (since its image under θ is p, which is non-zero). Thus in $\mathbf{B}_{\mathrm{dR}}^+$, the elements t and $\varphi^n(T)$ differ by a unit for all $n \geq 0$. On the other hand, if $n < 0$ then $\varphi^n(T)$ is a unit in $\mathbf{B}_{\mathrm{dR}}^+$ (since its image under θ is non-zero).

There are two more rings that we have to introduce, namely $\mathbf{B}_{\max}$ and $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+$. From the point of view of p-adic Hodge theory, these rings play the same role as $\mathbf{B}_{\mathrm{crys}}^+$ (i.e. they compute the crystalline Dieudonné module), but they have the advantage of being better behaved, and relating more directly to the rings of (φ,Γ)-module theory. The definitions are as follows: we first let $\mathbf{A}_{\max}$ denote the p-adic completion of $\widetilde{\mathbf{A}}^+[\omega/p]$, and then set $\mathbf{B}_{\max} := \mathbf{A}_{\max}[1/p]$; these are both subrings of $\mathbf{B}_{\mathrm{dR}}$. Note that $\varphi(\omega) \equiv \omega^p \bmod p\widetilde{A}^+$ (by definition of the action of φ on a ring of Witt vectors), so that $\varphi(\omega)/p \in \widetilde{A}^+[\omega/p]$. Thus φ extends by continuity to $\mathbf{A}_{\max}$ and $\mathbf{B}_{\max}$. However φ is not surjective on either of these rings, and we set

$$\widetilde{\mathbf{B}}_{\mathrm{rig}}^+ := \bigcap_{n\geq 0}\varphi^n(\mathbf{B}_{\max});$$

we then have, by construction, that φ is bijective on $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+$. The sequence of inclusions

$$\widetilde{\mathbf{B}}^+ \subset \widetilde{\mathbf{B}}_{\mathrm{rig}}^+ \subset \mathbf{B}_{\mathrm{dR}}^+$$

will be of fundamental importance to us.

The following proposition relating divisibilities in the rings involved in these inclusions may seem technical, but is crucial.

Proposition 2.4.1.

(1) *If $x \in \widetilde{\mathbf{A}}^+$ (resp. $\widetilde{\mathbf{B}}^+$) is such that $\varphi^n(x) \in \ker\theta$ for all $n \geq 0$ (i.e. if t divides $\varphi^n(x)$ in $\mathbf{B}_{\mathrm{dR}}^+$ for all $n \geq 0$), then $x \in T\widetilde{\mathbf{A}}^+$ (resp. $T\widetilde{\mathbf{B}}^+$).*

(2) *If $x \in \widetilde{\mathbf{B}}_{\mathrm{rig}}^+$ is such that $\varphi^n(x) \in \ker\theta$ for all $n \in \mathbb{Z}$ (i.e. if t divides $\varphi^n(x)$ in $\mathbf{B}_{\mathrm{dR}}^+$ for all $n \in \mathbb{Z}$), then $x \in t\widetilde{\mathbf{B}}_{\mathrm{rig}}^+$.*

Proof. The first part is [6, Lemma III.3.7]. For the second part, in [6, Lemma III.3.4], the analogous statement for $\mathbf{B}_{\max}^+$ is proven which immediately implies our statement for $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+$. ∎

Corollary 2.4.2. *For any $m \geq 0$, the inclusion $\varphi^m(T)\widetilde{\mathbf{A}}^+ \subset \widetilde{\mathbf{A}}^+ \cap \varphi^m(T)\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ is an equality (and similarly with $\widetilde{\mathbf{B}}^+$ in place of $\widetilde{\mathbf{A}}^+$.)*

Proof. Let $x \in \widetilde{\mathbf{A}}^+ \cap \varphi^m(T)\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$. Then $\varphi^{-m}(x) \in \widetilde{\mathbf{A}}^+ \cap T\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$. Then for any $n \geq 0$, we have $\varphi^{n-m}(x) \in \varphi^n(T)\widetilde{\mathbf{B}}^+_{\mathrm{rig}} \subset \ker\theta$, and so part (1) of the preceding proposition shows that $\varphi^{-m}(x) \in T\widetilde{\mathbf{A}}^+$. Applying φ^m, we find that $x \in \varphi^m(T)\widetilde{\mathbf{A}}^+$, as required. $\square$

3. (φ, Γ)-modules and p-adic Hodge theory

We will recall the constructions of (φ, Γ)-modules attached to representations of $G_{\mathbb{Q}_p}$, as well as various constructions in p-adic Hodge theory, and the relations between them. This material is all standard, but the systematic use of the rings with tildes is perhaps less familiar than it might be, and is very important for us, in part because it gives a very simple approach to relating (φ, Γ)-module theory to p-adic Hodge theory.

3.1. (φ, Γ)-modules.

If T is a continuous $G_{\mathbb{Q}_p}$-representation on a finite type $\mathbb{Z}_p$-module, then we define

$$\mathbf{D}(T) := (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^H,$$

which is an étale (φ, Γ)-module over $\mathbf{A}_{\mathbb{Q}_p}$. We can equally well define

$$\widetilde{\mathbf{D}}(T) := (\widetilde{\mathbf{A}} \otimes_{\mathbb{Z}_p} T)^H,$$

which is an étale (φ, Γ)-module over $\widetilde{\mathbf{A}}_{\mathbb{Q}_p}$. (Note that in this second context, the notion of étale is particularly simple: the map φ should simply be bijective.) If V is a continuous $G_{\mathbb{Q}_p}$-representation on a finite dimensional $\mathbb{Q}_p$-vector space (a *p-adic representation*, for short), then we define $\mathbf{D}(V)$ and $\widetilde{\mathbf{D}}(V)$ by the same formulas (or, perhaps a little more naturally, we could use $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ instead), and obtain étale (φ, Γ)-modules over $\mathbf{B}_{\mathbb{Q}_p}$ or $\widetilde{\mathbf{B}}_{\mathbb{Q}_p}$ instead. Either one of these functors, $\mathbf{D}$ or $\widetilde{\mathbf{D}}$, gives an equivalence of categories between $G_{\mathbb{Q}_p}$-representations and étale (φ, Γ)-modules. (This is probably most familiar in the $\mathbf{D}$ context.)

Now one can similarly define $\mathbf{D}^+(T)$ and $\widetilde{\mathbf{D}}^+(T)$ using $\mathbf{A}^+$ and $\widetilde{\mathbf{A}}^+$ instead. If T is *torsion* over $\mathbf{Z}_p$, then $\mathbf{D}^+(T)$ is a finite type $\mathbf{A}^+_{\mathbb{Q}_p}$-submodule of $\mathbf{D}(T)$, which generates $\mathbf{D}(T)$ over $\mathbf{A}_{\mathbb{Q}_p}$. However, if T is not torsion, then $\mathbf{D}^+(T)$ vanishes in general. Similarly, in the case of a p-adic representation V, it is generally the case that $D^+(V)$ vanishes. One says that V is of *finite height* if $\mathbf{D}^+(V)$ generates $\mathbf{D}(V)$ over $\mathbf{A}_{\mathbb{Q}_p}$, and this is a very restrictive condition.[2]

On the other hand, the module $\widetilde{\mathbf{D}}^+(V)$ *always* generates $\widetilde{\mathbf{D}}(V)$ over $\widetilde{\mathbf{A}}_{\mathbb{Q}_p}$. Thus there is an asymmetry between the $\mathbf{D}^+$ context and the $\widetilde{\mathbf{D}}^+$ context, which is not there when we remove the $+$.

For later use, we note that there is a kind of trace map $\mathrm{Tr} : \widetilde{\mathbf{D}}(V) \to \mathbf{D}(V)$, which restricts to the identity on $\mathbf{D}(V)$ (or equivalently, it is a projection onto $\mathbf{D}(V)$, i.e. it is surjective, and $\mathrm{Tr} \circ \mathrm{Tr} = \mathrm{Tr}$), and with the additional properties that it is continuous (with respect to the weak topologies on source and target),

[2] Related to this, Colmez [7] has proved a conjecture of Fontaine, stating that all crystalline representations are of finite height. The theory of Wach modules [3] is a strong witnessing of this fact, since the Wach module of a crystalline representation V is pretty close to $\mathbf{D}^+(V)$.

it is $\mathbf{A}_{\mathbb{Q}_p}$-linear, and $\mathrm{Tr}\circ\varphi^{-1} = \psi\circ\mathrm{Tr}$. (Here $\psi : \mathbf{D}(V) \to \mathbf{D}(V)$ is the usual left inverse of φ on étale (φ,Γ)-modules.)

Using Tr, we may define a map

$$\widetilde{\mathbf{D}}(V) \to \varprojlim_{\psi} \mathbf{D}(V) =: \mathbf{D}(V) \boxtimes \mathbb{Q}_p,$$

via

$$\tilde{z} \mapsto \big(\mathrm{Tr}\,\varphi^n(\tilde{z})\big)_{n\geq 0}.$$

This map is injective (but not surjective in general), and the image of $\widetilde{\mathbf{D}}^+(V)$ is equal to the intersection of the image of $\widetilde{\mathbf{D}}(V)$ with $\mathbf{D}^{\natural}(V)\boxtimes\mathbb{Q}_p$. Here $\mathbf{D}^{\natural}(V)$ is as defined in [10, I.3.2].

There are a slew of other species of (φ,Γ)-module — see, for instance, [10, pg. 116–117]. For instance, we can define $\widetilde{\mathbf{D}}^+_{\mathrm{rig}}(V) := (\widetilde{\mathbf{B}}^+_{\mathrm{rig}}\otimes V)^H$ and this module will play a large role in what follows. In section 5, we will also make use of several other kinds of (φ,Γ)-modules. Namely, if $\mathscr{R}$ is the Robba ring over $\mathbb{Q}_p$ — that is, the collection of $\mathbb{Q}_p$-power series which converge on an annulus defined by $0 < v_p(\cdot) \leq r$ for *some* r — we have a (φ,Γ)-module $\mathbf{D}_{\mathrm{rig}}(V)$ which is a module over $\mathscr{R}$.

We will also need (φ,Γ)-modules over even larger rings. Namely, the ring $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ is the largest of all the period rings we will use in this paper (see [1, §2]). With this ring in hand, we define $\widetilde{\mathbf{D}}^{\dagger}_{\mathrm{rig}}(V) := (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}\otimes V)^H$ which is a module over $\widetilde{\mathscr{R}} := (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}})^H$.

3.2. **Sen's theory, and Fontaine's generalization.** The relation between $\widetilde{\mathbf{D}}$ and $\mathbf{D}$ (and in particular, the fact that no information is lost when one passes from the former to the latter), is an instance of a more general principle of "decompletion", first studied by Sen.

The basic point in the $\widetilde{\mathbf{D}}/\mathbf{D}$-context is that the ring $\widetilde{\mathbf{A}}$ used to compute $\widetilde{\mathbf{D}}$ is the completion (in the weak topology) of the ring $\mathbf{A}$ used to compute $\mathbf{D}$. Since the resulting coefficient ring $\widetilde{\mathbf{A}}_{\mathbb{Q}_p}$ is *not* the completion of $\mathbf{A}_{\mathbb{Q}_p}$, this situation does not compare perfectly with Sen's, but the analogy is still meaningful.

Let us now recall Sen's context. He works with the period ring $\mathbb{C}_p$. Recall that $\mathbb{C}_p^H = \widehat{K}_{\infty,p}$ where $K_{\infty,p} := \mathbb{Q}_p(\mu_{p^\infty})$. Thus if V is a continuous $G_{\mathbb{Q}_p}$-representation over $\mathbb{Q}_p$, we can consider $\widetilde{\mathbf{D}}_{\mathrm{Sen}}(V) := (\mathbb{C}_p\otimes_{\mathbb{Q}_p} V)^H$, which is a semi-linear Γ-representation over $\widehat{K}_{\infty,p}$, of dimension (over $\widehat{K}_{\infty,p}$) equal to the dimension of V over $\mathbb{Q}_p$. Now Sen's theorem says that we can descend this canonically to a semi-linear Γ-representation over $K_{\infty,p}$ itself, denoted $\mathbf{D}_{\mathrm{Sen}}(V)$, where "descend" has the meaning that the natural map

$$\widehat{K}_{\infty,p}\otimes_{K_{\infty,p}}\mathbf{D}_{\mathrm{Sen}}(V) \to \widetilde{\mathbf{D}}_{\mathrm{Sen}}(V)$$

is an isomorphism.

What is the benefit of this? Well, any element of $K_{\infty,p}$ is actually invariant under a sufficiently small open subgroup of Γ, and so the Lie algebra $\mathrm{Lie}\,\Gamma$ (which is canonically identified with $\mathbb{Q}_p$ via the derivative of the cyclotomic character) acts *linearly* on the $K_{\infty,p}$-vector space $\mathbf{D}_{\mathrm{Sen}}(V)$. In particular, the basis vector $1\in\mathbb{Q}_p$ gives rise to a linear operator Δ_V on $\mathbf{D}_{\mathrm{Sen}}(V)$, and it is the eigenvalues of this operator that are the Hodge–Sen–Tate weights of V. (So the decompletion allows us to pass from a semi-linear context to a linear one, where we can then use linear algebra to define invariants.)

Fontaine [11] generalized Sen's theory as follows: instead of $\mathbb{C}_p$, the period ring is now $\mathbf{B}_{\mathrm{dR}}$. Remember that this a twisted version of $\mathbb{C}_p((t))$. Since it contains the algebraic closure $\overline{\mathbb{Q}}_p$, it does contain $\overline{\mathbb{Q}}_p((t))$ as a dense subfield. Noting that H fixes t, we find that $\mathbf{B}_{\mathrm{dR}}^H$ contains $K_{\infty,p}((t))$ as a dense subfield.

Now if V is as before, we define

$$\widetilde{\mathbf{D}}_{\mathrm{dif}}(V) := (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^H,$$

which is a vector space over $\mathbf{B}_{\mathrm{dR}}^H$ of dimension equal to the dimension of V over $\mathbb{Q}_p$, equipped with a semi-linear Γ-action. What Fontaine shows is that we may canonically descend $\widetilde{\mathbf{D}}_{\mathrm{dif}}(V)$, together with its Γ-action, to a vector space $\mathbf{D}_{\mathrm{dif}}(V)$ over $K_{\infty,p}((t))$, where again "descend" means that

$$\mathbf{B}_{\mathrm{dR}}^H \otimes_{K_{\infty,p}((t))} \mathbf{D}_{\mathrm{dif}}(V) \to \widetilde{\mathbf{D}}_{\mathrm{dif}}(V)$$

is an isomorphism. On $\mathbf{D}_{\mathrm{dif}}(V)$, the basis vector $1 \in \mathbb{Q}_p = \mathrm{Lie}\,\Gamma$ acts by a $K_{\infty,p}$-linear differential operator Δ_V (satisfying the Leibnitz rule $\Delta_V(tx) = tx + t\Delta_V(x)$).

We can proceed similarly with $\mathbf{B}_{\mathrm{dR}}^+$ in place of $\mathbf{B}_{\mathrm{dR}}$, to define $\widetilde{\mathbf{D}}_{\mathrm{dif}}^+(V)$ and $\mathbf{D}_{\mathrm{dif}}^+(V)$, which are modules over $(\mathbf{B}_{\mathrm{dR}}^+)^H$ and $K_{\infty,p}[[t]]$ respectively, and both of which are invariant under Γ. The latter is thus also preserved by the differential operator Δ_V. Note that reducing mod t returns us to Sen's situation:

$$\mathbf{D}_{\mathrm{dif}}^+(V)/t\mathbf{D}_{\mathrm{dif}}^+(V) \xrightarrow{\sim} \mathbf{D}_{\mathrm{Sen}}(V)$$

(compatibly with the operators Δ_V).

3.3. **So what?** The inclusions $\widetilde{\mathbf{B}}^+ \subset \widetilde{\mathbf{B}}_{\mathrm{rig}}^+ \subset \mathbf{B}_{\mathrm{dR}}^+$ gives rise to canonical inclusions

$$\widetilde{\mathbf{D}}^+(V) \subset \widetilde{\mathbf{D}}_{\mathrm{rig}}^+(V) \subset \widetilde{\mathbf{D}}_{\mathrm{dif}}^+(V) \tag{1}$$

(the first being (φ,Γ)-equivariant, and the second being Γ-equivariant). These inclusions set up a relation between the world of (φ,Γ)-modules and the world of p-adic Hodge theory which is at the basis of certain explicit reciprocity laws, and also at the heart of Colmez's approach to p-adic local Langlands.

To begin our discussion, we first note that the inclusion $\widetilde{\mathbf{D}}^+(V) \subset \widetilde{\mathbf{D}}_{\mathrm{dif}}^+(V)$ does not extend to a map with the $+$'s removed, because $\widetilde{\mathbf{D}}(V)$ is obtained by inverting T and then p-adically completing, while $\widetilde{\mathbf{D}}_{\mathrm{dif}}(V)$ is obtained by inverting t (which certainly allows us to invert T, since T divides t), but without any subsequent p-adic completion.

Nevertheless, each of the elements $\varphi^n(T)$ divides t in $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+$, and so also in $\mathbf{B}_{\mathrm{dR}}^+$, and so we do have inclusions

$$\widetilde{\mathbf{D}}^+(V)[1/\varphi^n(T)] \subset \widetilde{\mathbf{D}}_{\mathrm{rig}}^+(V)[1/t] \subset \widetilde{\mathbf{D}}_{\mathrm{dif}}^+(V)[1/t] = \widetilde{\mathbf{D}}_{\mathrm{dif}}(V),$$

for each $n \geq 0$. Thus, if we write

$$\widetilde{\mathbf{D}}^+(V)[\big(1/\varphi^n(T)\big)_{n\geq 0}] := \bigcup_{n\geq 0} \widetilde{\mathbf{D}}^+(V)[1/\varphi^n(T)],$$

then we obtain the inclusions

$$\widetilde{\mathbf{D}}^+(V)[\big(1/\varphi^n(T)\big)_{n\geq 0}] \subset \widetilde{\mathbf{D}}_{\mathrm{rig}}^+(V)[1/t] \subset \widetilde{\mathbf{D}}_{\mathrm{dif}}(V), \tag{2}$$

which in turn induces maps

$$\widetilde{\mathbf{D}}^+(V)[\big(1/\varphi^n(T)\big)_{n\geq 0}]/\widetilde{\mathbf{D}}^+(V) \to \widetilde{\mathbf{D}}_{\mathrm{rig}}^+(V)[1/t]/\widetilde{\mathbf{D}}_{\mathrm{rig}}^+(V) \to \widetilde{\mathbf{D}}_{\mathrm{dif}}(V)/\widetilde{\mathbf{D}}_{\mathrm{dif}}^+(V).$$

The first of these maps is an injection, by Corollary 2.4.2, and we regard it as an inclusion. The second map, which is not injective (unless $V = 0$), will be crucial for us, and so we give it a name, namely $\imath_0^-$. (The minus sign is to remind us that we are quotienting out by the plus objects; the reason for the subscript 0 will become apparent soon.)

A fundamental fact is that we have φ actions on our (φ, Γ)-modules, but not in p-adic Hodge theory. In other words, $\imath_0^-$ is a Γ-equivariant map from a module with a φ-action to a module without such an action. General principles of algebra then suggest that we induce $\widetilde{\mathbf{D}}_{\mathrm{dif}}(V)/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V)$ to obtain the module

$$\prod_{i\in\mathbb{Z}} \widetilde{\mathbf{D}}_{\mathrm{dif}}(V)/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V),$$

which we equip with the φ-action given by translation, namely

$$\varphi\big((x_i)\big) := (x_{i-1}).$$

(The reason for the negative, rather than positive, shift, is to accord with existing conventions in the literature, in particular in [10].) We make the product $\prod_{i\in\mathbb{Z}} \widetilde{\mathbf{D}}_{\mathrm{dif}}(V)/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V)$ a module over $\widetilde{\mathbf{B}}^+_{\mathbb{Q}_p}$ by defining the action of an element b of this ring on an element $(x_i)_{i\in\mathbb{Z}}$ of the product via

$$b \cdot (x_i)_{i\in\mathbb{Z}} := (\varphi^{-i}(b)x_i).$$

In this way the product $\prod_{i\in\mathbb{Z}} \widetilde{\mathbf{D}}_{\mathrm{dif}}(V)/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V)$, becomes an étale (φ, Γ)-module over $\widetilde{\mathbf{B}}^+_{\mathbb{Q}_p}$.

We then define a map

$$\imath^- : \widetilde{\mathbf{D}}^+_{\mathrm{rig}}(V)[1/t]/\widetilde{\mathbf{D}}^+_{\mathrm{rig}}(V) \to \prod_{i\in\mathbb{Z}} \widetilde{\mathbf{D}}_{\mathrm{dif}}(V)/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V)$$

of (φ, Γ)-modules over $\widetilde{\mathbf{B}}^+_{\mathbb{Q}_p}$ via the formula

$$\imath^-(x) = \Big(\imath_0^-\big(\varphi^{-i}(x)\big)\Big)_{i\in\mathbb{Z}}.$$

For any $i \in \mathbb{Z}$, we define

$$\imath_i^- : \widetilde{\mathbf{D}}^+_{\mathrm{rig}}(V)[1/t]/\widetilde{\mathbf{D}}^+_{\mathrm{rig}}(V) \to \widetilde{\mathbf{D}}_{\mathrm{dif}}(V)/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V)$$

via $\imath_i^-(x) := \imath_0^-\big(\varphi^{-i}(x)\big)$, so that $\imath^-$ also admits the description $\imath^-(x) := \big(\imath_i^-(x)\big)_{i\in\mathbb{Z}}$.

Proposition 3.3.1. *We have:*

1. *The map $\imath^-$ is injective.*
2. *For any $n \geq 0$, the restriction of $\imath_i^-$ to $\widetilde{\mathbf{D}}^+(V)[1/\varphi^n(T)]/\widetilde{\mathbf{D}}^+(V)$ vanishes if $i > n$.*

Proof. Fix $x = b/\varphi^n(T)^k \in \widetilde{\mathbf{D}}^+(V)[1/\varphi^n(T)]$. For any $i \in \mathbf{Z}$ we see that $\varphi^{-i}(x) = \varphi^{-i}(b)/(\varphi^{n-i}(T))^k$. Since $\varphi^{n-i}(T)$ is a unit in $\mathbf{B}^+_{\mathrm{dR}}$ when $i > n$, we see that $\varphi^{-i}(x) \in \widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V)$ when $i > n$, proving (2). On the other hand, if $\varphi^{-i}(x) \in \widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V)$ for some $x \in \widetilde{\mathbf{D}}^+_{\mathrm{rig}}(V)[1/t]/\widetilde{\mathbf{D}}^+_{\mathrm{rig}}(V)$ and all i, then we conclude from part (2) of Proposition 2.4.1 that $x \in \widetilde{\mathbf{D}}^+_{\mathrm{rig}}(V)$, proving part (1) of the present proposition. $\square$

From part (2) of the preceding result we see in particular that, when restricted to $\widetilde{\mathbf{D}}^+(V)[\big(1/\varphi^n(T)\big)_{n\geq 0}]/\widetilde{\mathbf{D}}^+(V)$, the map $\imath^-$ in fact takes values in

$$\varinjlim_n \prod_{i\leq n} \widetilde{\mathbf{D}}_{\mathrm{dif}}(V)/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V).$$

The next result involves a slight refinement of the argument used to prove part (1) of the preceding proposition.

Proposition 3.3.2. *We have:*

(1) *For $n \neq i$ we have that $\imath_i^-$ vanishes on*

$$\widetilde{\mathbf{D}}^+(V)[1/\varphi^n(\omega)]/\widetilde{\mathbf{D}}^+(V),$$

while $\imath_n^-$ induces an isomorphism

$$\widetilde{\mathbf{D}}^+(V)[1/\varphi^n(\omega)]/\widetilde{\mathbf{D}}^+(V) \xrightarrow{\sim} \widetilde{\mathbf{D}}_{\mathrm{dif}}(V)/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V).$$

(2) *The evident map*

$$\bigoplus_{n\in\mathbb{Z}} \widetilde{\mathbf{D}}^+(V)[1/\varphi^n(\omega)]/\widetilde{\mathbf{D}}^+(V) \to \widetilde{\mathbf{D}}^+(V)[\big(1/\varphi^n(\omega)\big)_{n\geq 0}]/\widetilde{\mathbf{D}}^+(V)$$

is an isomorphism, and $\imath^-$ induces an isomorphism

$$\widetilde{\mathbf{D}}^+(V)[\big(1/\varphi^n(\omega)\big)_{n\geq 0}]/\widetilde{\mathbf{D}}^+(V) \xrightarrow{\sim} \bigoplus_{i\in\mathbb{Z}} \widetilde{\mathbf{D}}_{\mathrm{dif}}(V)/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V).$$

Proof. The key point is that if $n \neq 0$, then $\varphi^n(\omega)$ is a unit in $\mathbf{B}^+_{\mathrm{dR}}$, which in turn follows from the facts that if $a \geq b \geq 0$, then $\varphi^a(T)/\varphi^b(T)$ is a unit in $\mathbf{B}^+_{\mathrm{dR}}$, while if $n < 0$, then $\varphi^n(T)$ is a unit in $\mathbf{B}^+_{\mathrm{dR}}$. $\square$

4. Representations of the mirabolic

Let $P := \begin{pmatrix} \mathbb{Q}_p^\times & \mathbb{Q}_p \\ 0 & 1 \end{pmatrix}$ denote the mirabolic subgroup of GL_2. We also write $P^+ := \begin{pmatrix} \mathbb{Z}_p \setminus \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$; this is a submonoid of P. Note that P is generated by P^+ together with $\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, and that giving a representation of P is the same as giving a representation of P^+ on which the action of $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ is invertible. Also, set $N := \begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & 1 \end{pmatrix}$, the unipotent subgroup of GL_2.

4.1. P-representations from (φ,Γ)-modues.

A (φ,Γ)-module $\mathbf{D}$ with coefficients in $\mathbf{A}^+_{\mathbb{Q}_p}$ may be equipped with the structure of a P^+ representation in the following manner: the matrix $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ acts via φ, the matrices $\begin{pmatrix} \Gamma & 0 \\ 0 & 1 \end{pmatrix}$ act through the given action of Γ on the (φ,Γ)-module, and the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ acts via $1+T$. This construction applies in particular if $\widetilde{\mathbf{D}}$ is an étale (φ,Γ)-module over $\widetilde{\mathbf{A}}^+_{\mathbb{Q}_p}$. Since in this case the action of φ on $\widetilde{\mathbf{D}}$ is invertible (this is the definition of étale in this context), we see that the P^+-action on $\widetilde{\mathbf{D}}$ extends to an action of P, so that $\widetilde{\mathbf{D}}$ is naturally a P-representation.

In particular, if V is any p-adic $G_{\mathbb{Q}_p}$-representation, we obtain a natural P-representation on the quotients

$$\widetilde{\mathbf{D}}\big(V(1)\big)/\widetilde{\mathbf{D}}^+\big(V(1)\big),$$

$$\widetilde{\mathbf{D}}^+\big(V(1)\big)\big[\big(1/\varphi^n(T)\big)_{n\geq 0}\big]/\widetilde{\mathbf{D}}^+\big(V(1)\big),$$

and

$$\widetilde{\mathbf{D}}^+_{\mathrm{rig}}\big(V(1)\big)[1/t]/\widetilde{\mathbf{D}}^+_{\mathrm{rig}}\big(V(1)\big),$$

as well as on the product

$$\prod_{i\in\mathbb{Z}}\widetilde{\mathbf{D}}_{\mathrm{dif}}\big(V(1)\big)/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}\big(V(1)\big),$$

and the map $\imath^-$ from either of the latter two quotients to the product is P-equivariant.

In terms of this P-action, the subspace $\widetilde{\mathbf{D}}^+\big(V(1)\big)\big[\big(1/\varphi^n(T)\big)_{n\geq 0}\big]/\widetilde{\mathbf{D}}^+\big(V(1)\big)$ of $\widetilde{\mathbf{D}}\big(V(1)\big)/\widetilde{\mathbf{D}}^+\big(V(1)\big)$ is identified as its subspace of N-locally algebraic vectors. In particular, the N-smooth vectors in $\widetilde{\mathbf{D}}\big(V(1)\big)/\widetilde{\mathbf{D}}^+\big(V(1)\big)$ are equal to the subspace $\bigcup_{n\geq 0}\dfrac{1}{\varphi^n(T)}\widetilde{\mathbf{D}}^+\big(V(1)\big)/\widetilde{\mathbf{D}}^+\big(V(1)\big)$. Their image under the map $\imath^-$ lies in the subspace of N-smooth vectors in $\prod_{i\in\mathbb{Z}}\widetilde{\mathbf{D}}_{\mathrm{dif}}\big(V(1)\big)/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}\big(V(1)\big)$, which is equal to

$$\prod_{i\in\mathbb{Z}}\frac{1}{t}\widetilde{\mathbf{D}}^+_{\mathrm{dif}}\big(V(1)\big)/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}\big(V(1)\big)=\prod_{i\in\mathbb{Z}}\frac{1}{t}\widetilde{\mathbf{D}}_{\mathrm{Sen}}\big(V(1)\big)=\prod_{i\in\mathbb{Z}}\widetilde{\mathbf{D}}_{\mathrm{Sen}}(V).$$

The P-smooth vectors in $\widetilde{\mathbf{D}}\big(V(1)\big)/\widetilde{\mathbf{D}}^+\big(V(1)\big)$ may be identified with the Γ-smooth vectors of the N-smooth vectors. Thus, they map under $\imath^-$ to the Γ-smooth vectors in $\prod_{i\in\mathbb{Z}}\widetilde{\mathbf{D}}_{\mathrm{Sen}}(V)$, which is contained in the product of the Γ-smooth vectors in each individual factor. Sen theory shows that the Γ-smooth vectors in $\widetilde{\mathbf{D}}_{\mathrm{Sen}}(V)$ are non-zero if and only if V admits zero as a Hodge–Sen–Tate weight, and, more precisely, that their dimension as $K_{\infty,p}$-vector space is equal to the multiplicity of zero as a Hodge–Sen–Tate weight of V. In particular, if V does not admit zero as a Hodge–Sen–Tate weight, then the subspace of P-smooth vectors in $\widetilde{\mathbf{D}}\big(V(1)\big)/\widetilde{\mathbf{D}}^+\big(V(1)\big)$ vanishes.

4.2. **The relationship with $\mathbf{D}_{\mathrm{crys}}$.** For any p-adic representation V of $G_{\mathbb{Q}_p}$, we have

$$\mathbf{D}_{\mathrm{crys}}(V):=\big(\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t]\otimes_{\mathbb{Q}_p}V\big)^{G_{\mathbb{Q}_p}}=\big(\widetilde{\mathbf{D}}^+_{\mathrm{rig}}(V)[1/t]\big)^{\Gamma}.$$

(The fact that we can do this with $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t]$, rather then with $\mathbf{B}_{\mathrm{crys}}$, is one of the basic results of the theory; see e.g. [6, §III.2].) Thus, replacing V by $V(1)$, we get a φ-equivariant map

$$\mathbf{D}_{\mathrm{crys}}\big(V(1)\big)\to\Big(\widetilde{\mathbf{D}}^+_{\mathrm{rig}}\big(V(1)\big)[1/t]/\widetilde{\mathbf{D}}^+_{\mathrm{rig}}\big(V(1)\big)\Big)^{\Gamma}. \tag{3}$$

If we compose this with $\imath_0^-$, we just get the canonical map

$$\mathbf{D}_{\mathrm{crys}}\big(V(1)\big)\to\mathbf{D}_{\mathrm{dR}}\big(V(1)\big)/\mathbf{D}^+_{\mathrm{dR}}\big(V(1)\big).$$

Remark 4.2.1. The image of $\mathbf{D}_{\mathrm{crys}}\big(V(1)\big)$ under $\imath^-$ is finite-dimensional and stable under the torus (since it is fixed by Γ, and stable under φ). Thus, this image has trivial intersection with $\imath^-\Big(\widetilde{\mathbf{D}}^+\big(V(1)\big)[\big(1/\varphi^n(T)\big)_{n\geq 0}]/\widetilde{\mathbf{D}}^+\big(V(1)\big)\Big)$, since elements of $\imath^-\Big(\widetilde{\mathbf{D}}^+\big(V(1)\big)[\big(1/\varphi^n(T)\big)_{n\geq 0}]/\widetilde{\mathbf{D}}^+\big(V(1)\big)\Big)$ have support bounded away from infinity, whereas cleary a non-zero φ-eigenvector doesn't have this property.

5. Iwasawa cohomology and explicit reciprocity laws

Let L denote a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$. For the remainder of the paper we will consider p-adic representations which are L-vector spaces. Note that if V is such a representation and $T \subseteq V$ is a Galois stable $\mathcal{O}_L$-lattice, then $D(T)$ is a module over $\mathbf{A}_{\mathbb{Q}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_L$.

We also set some notation that will be in place for the remainder of the paper. For a vector space V over L, set $V^* = \mathrm{Hom}(V, L)$. For an $\mathcal{O}_K$-lattice T, set $T^* = \mathrm{Hom}(T, \mathcal{O}_L)$. For an $\mathcal{O}_L$-module A, set $A^\vee = \mathrm{Hom}(A, L/\mathcal{O}_L)$.

5.1. (φ,Γ)**-modules and Iwasawa cohomology.** The fundamental fact (see [5]) relating (φ,Γ)-modules to Iwasawa theory is the following: if T is a $G_{\mathbb{Q}_p}$-invariant $\mathcal{O}_L$-lattice in a finite-dimensional representation V of $G_{\mathbb{Q}_p}$ over L, then

$$\mathbf{D}(T)^{\psi=1} \xrightarrow{\sim} H^1_{\mathrm{Iw}}(T). \tag{4}$$

Inverting p then yields an isomorphism

$$\mathbf{D}(V)^{\psi=1} \xrightarrow{\sim} H^1_{\mathrm{Iw}}(V). \tag{5}$$

We let $\mathcal{R}^+(\Gamma)$ denote the ring of locally analytic distributions on Γ, or equivalently, the ring of rigid analytic functions on weight space $\mathcal{W} := \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \overline{\mathbb{Q}}_p^\times)$. (This notation is due to Colmez, and is inspired by the analogous notation in the theory of the Robba ring.) The isomorphism (4) induces an isomorphism

$$\mathcal{R}^+(\Gamma) \otimes_{\mathcal{O}_L[[\Gamma]]} \mathbf{D}(T)^{\psi=1} \xrightarrow{\sim} \mathcal{R}^+(\Gamma) \otimes_{\mathcal{O}_L[[\Gamma]]} H^1_{\mathrm{Iw}}(T).$$

There is also a natural isomorphism

$$\mathcal{R}^+(\Gamma) \otimes_{\mathcal{O}_L[[\Gamma]]} \mathbf{D}(V)^{\psi=1} \xrightarrow{\sim} \mathbf{D}_{\mathrm{rig}}(V)^{\psi=1}, \tag{6}$$

and hence we obtain an isomorphism

$$\mathbf{D}_{\mathrm{rig}}(V)^{\psi=1} \xrightarrow{\sim} \mathcal{R}^+(\Gamma) \otimes_{\mathcal{O}_L[[\Gamma]]} H^1_{\mathrm{Iw}}(T). \tag{7}$$

5.2. **Pairings:** (φ,Γ)**-modules.** Let $\mathbf{D} := \mathbf{D}(T)$ be an étale (φ,Γ)-module over $\mathbf{A}_{\mathbb{Q}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_L$, associated to a $G_{\mathbb{Q}_p}$-invariant $\mathcal{O}_L$-lattice T in a p-adic representation V. As in [9, I.2.2], we define the Tate dual of $\mathbf{D}$ as $\check{\mathbf{D}} := \mathrm{Hom}_{\mathbf{A}_{\mathbb{Q}_p}}(\mathbf{D}, \mathbf{A}_{\mathbb{Q}_p}\frac{dT}{1+T} \otimes_{\mathbb{Z}_p} \mathcal{O}_L)$, its (φ,Γ)-module structure on $\check{\mathbf{D}}$ being determined by the formulas $\sigma_a(\frac{dT}{1+T}) = a\frac{dT}{1+T}$ for $\sigma_a \in \Gamma$ corresponding to an element $a \in \mathbb{Z}_p^\times$, and $\varphi(\frac{dT}{1+T}) = \frac{dT}{1+T}$. Note that these formulas identify $\mathbf{A}_{\mathbb{Q}_p}\frac{dT}{1+T}$ with $\mathbf{D}\big(\mathbb{Z}_p(1)\big)$, and so Fontaine's equivalence of categories yields an identification $\check{\mathbf{D}} = \mathbf{D}\big(T^*(1)\big)$. We denote the corresponding tautological pairing by $\langle\cdot,\cdot\rangle : \mathbf{D} \times \check{\mathbf{D}} \to \mathbf{A}_{\mathbb{Q}_p}\frac{dT}{1+T} \otimes_{\mathbb{Z}_p} \mathcal{O}_L$.

Define the residue map $\mathrm{res}_0 : \mathbf{A}_{\mathbb{Q}_p} dT = \mathbf{A}_{\mathbb{Q}_p}\frac{dT}{1+T} \to \mathbb{Z}_p$ by $\mathrm{res}_0(f\,dT) = a_{-1}$ where $f = \sum_k a_k T^k$. Extending this map $\mathcal{O}_L$-linearly, and then passing to residues in $\langle\cdot,\cdot\rangle$, yields a perfect pairing

$$\{\cdot,\cdot\}:\mathbf{D}\times\check{\mathbf{D}}\to\mathcal{O}_L$$
$$(x,y)\mapsto \mathrm{res}_0(\langle x,y\rangle)$$

which identifies $\check{\mathbf{D}}$ with the topological $\mathcal{O}_L$-dual of $\mathbf{D}$.

The preceding discussion has analogues for other flavors of (φ,Γ)-modules. The most evident comes from working over L rather than $\mathcal{O}_L$, so that $\mathbf{D}$ is the étale (φ,Γ)-module associated to the p-adic representation V itself (rather than its lattice T). Then $\check{\mathbf{D}}$ is associated to $V^*(1)$, and the pairing $\{\cdot,\cdot\}$ identifies $\check{\mathbf{D}}$ with the topological L-dual of $\mathbf{D}$. Similarly, for étale (φ,Γ)-modules defined over the Robba ring $\mathscr{R}\otimes_{\mathbb{Q}_p}L$, we have analogous perfect pairings, still denoted by $\langle\cdot,\cdot\rangle$ and $\{\cdot,\cdot\}$, again valued in L. See [10, pg. 128] for details.

Somewhat less obviously, the story carries over to étale (φ,Γ)-modules over $\widetilde{\mathbf{A}}_{\mathbb{Q}_p}$ and $\widetilde{\mathscr{R}}$. Indeed, using the trace maps $\widetilde{\mathbf{A}}_{\mathbb{Q}_p}\to\mathbf{A}_{\mathbb{Q}_p}$ and $\widetilde{\mathscr{R}}\to\mathscr{R}$ (see [8, Chapter 8] and [9, §4.3]), one can define res_0 for these rings, and thus deduce pairings $\{\cdot,\cdot\}$ from the tautological pairing $\langle\cdot,\cdot\rangle$, just as in the cases already considered above.

5.3. **Pairings: p-adic Hodge theory.** Recall that $\widetilde{\mathbf{D}}_{\mathrm{dif}}(V)=(V\otimes_{\mathbb{Q}_p}\mathbf{B}_{\mathrm{dR}})^H$ and that we can descend $\widetilde{\mathbf{D}}_{\mathrm{dif}}(V)$ to an $L_\infty((t))$-module $\mathbf{D}_{\mathrm{dif}}(V)$ where $L_\infty:=K_{\infty,p}\otimes_{\mathbb{Q}_p}L$. Further, for each $n\gg 0$, we can define a vector space $\mathbf{D}_{\mathrm{dif},n}(V)$ over $L_n((t))$, where $L_n:=K_{n,p}\otimes_{\mathbb{Q}_p}L$ and $K_{n,p}=\mathbb{Q}_p(\mu_{p^n})$. Namely, following the notation from [10, §VI.3.1], on $\mathbf{D}^{]0,r_n]}\subseteq\mathbf{D}_{\mathrm{rig}}(V)$, there is a map $\imath_n:\mathbf{D}^{]0,r_n]}\to\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V)$. We denote the image of this map by $\mathbf{D}^+_{\mathrm{dif},n}(V)$ and set $\mathbf{D}_{\mathrm{dif},n}(V):=\mathbf{D}^+_{\mathrm{dif},n}(V)[1/t]$. (Then $\mathbf{D}_{\mathrm{dif}}(V)$ is recovered by extending the scalars of $\mathbf{D}_{\mathrm{dif},n}(V)$ from $L_n((t))$ to $L_\infty((t))$.

Then, as in [10, pg. 151], there is a perfect Γ-equivariant pairing

$$\langle\cdot,\cdot\rangle_{\mathrm{dif}}:\mathbf{D}_{\mathrm{dif},n}(V(1))\times\mathbf{D}_{\mathrm{dif},n}(V^*)\longrightarrow L$$

which is independent of $n\gg 0$ (in the sense that, if $n'\geq n\gg 0$, then the pairing on the $\mathbf{D}_{\mathrm{dif},n'}$ restricts to the pairing on the $\mathbf{D}_{\mathrm{dif},n}$), and under which $\mathbf{D}^+_{\mathrm{dif},n}(V(1))$ and $\mathbf{D}^+_{\mathrm{dif},n}(V^*)$ are orthogonal complements.

We set $\mathbf{D}_{\mathrm{dR},n}(V)=\widetilde{\mathbf{D}}_{\mathrm{dif}}(V)^{\Gamma_n}$ equal to the de Rham Dieudonné modules at cyclotomic level n where Γ_n is the subgroup of Γ of index $p^{n-1}(p-1)$. By [10, pg. 231], we have

$$\mathbf{D}_{\mathrm{dR},n}(V)\cong\mathbf{D}_{\mathrm{dR},K_{n,p}}(V)\cong\mathbf{D}_{\mathrm{dR},\mathbb{Q}_p}(V)\otimes_{\mathbb{Q}_p}K_{n,p}$$

where $\mathbf{D}_{\mathrm{dR},K}(V):=(V\otimes\mathbf{B}_{\mathrm{dR}})^{G_K}$ for any finite extension $K/\mathbb{Q}_p$.

There is a L_n-valued natural pairing

$$\langle\cdot,\cdot\rangle'_{\mathrm{dR},n}:\mathbf{D}_{\mathrm{dR},K_{n,p}}(V(1))\times\mathbf{D}_{\mathrm{dR},K_{n,p}}(V^*)\to\mathbf{D}_{\mathrm{dR},K_{n,p}}(L(1))\cong L_n$$

where the final isomorphism is defined by taking t^{-1} as a basis of $\mathbf{D}_{\mathrm{dR},L_n}(L(1))$. We then have an induced the pairing

$$\langle\cdot,\cdot\rangle_{\mathrm{dR},n}:\mathbf{D}_{\mathrm{dR},L_n}(V(1))\times\mathbf{D}_{\mathrm{dR},L_n}(V^*)\to L$$

defined by $\langle\cdot,\cdot\rangle_{\mathrm{dR},n}=\mathrm{Tr}_L^{L_n}(\langle\cdot,\cdot\rangle'_{\mathrm{dR},n})$.

We note that $\mathbf{D}_{\mathrm{dR},n}(V)\subseteq\mathbf{D}_{\mathrm{dif},n}(V)$ if $n\gg 0$, and under this inclusion we have the relation $\frac{1}{p^n}\langle\cdot,\cdot\rangle_{\mathrm{dR}.n}=\langle\cdot,\cdot\rangle_{\mathrm{dif}}$ as the pairing on $\mathbf{D}_{\mathrm{dif},n}$ objects is defined via a normalized trace while on $\mathbf{D}_{\mathrm{dR},n}$ objects, the pairing is defined via an absolute trace.

We close this subsection with a relation between $\imath_n$ and the dual exponential map

$$\exp^* : H^1(\mathbb{Q}(\mu_{p^n}), V) \to \mathbf{D}_{\mathrm{dR},n}(V).$$

Lemma 5.3.1. *For $z \in \mathbf{D}(V)^{\psi=1} \cong H^1_{\mathrm{Iw}}(V)$, write z_n for the projection of z to $H^1(\mathbb{Q}_p(\mu_{p^n}), V)$. We then have*

$$p^n \exp^*(z_n) = \imath_m(z)$$

for $m \gg 0$.

Proof. This lemma is essentially [10, Lemme VIII.2.1], but we need to explain the factor of p^n appearing. The reason this factors appears is that the dual exponential used in [10] is normalized in a way that does not match the standard normalization as in [19].

Let's write $\widetilde{\exp}^*$ for the dual exponential map normalized as in [10]. Then for $x \in \mathbf{D}_{\mathrm{dR},n}(V^*(1))$ and $y \in H^1(K_{n,p}, V)$, by [10, Théorème VIII.2.2], we have

$$\langle x, \widetilde{\exp}^*(y)\rangle_{\mathrm{dif}} = \langle \exp(x), y\rangle_n$$

where $\langle \cdot, \cdot\rangle_n$ is the pairing of Tate local duality. Meanwhile, we also have that

$$\langle x, \exp^*(y)\rangle_{\mathrm{dR},n} = \langle \exp(x), y\rangle_n$$

by [19, Theorem 1.4.1(4)]. Since $\frac{1}{p^n}\langle \cdot, \cdot\rangle_{\mathrm{dR},n} = \langle \cdot, \cdot\rangle_{\mathrm{dif}}$, we deduce that $p^n \exp^* = \widetilde{\exp}^*$ and the lemma follows from [10, Lemme VIII.2.1]. $\square$

5.4. **An explicit reciprocity law.** Colmez has proven the following explicit reciprocity law [10, Prop. VI.3.4]. (Recall that $\mathbf{D}_{\mathrm{rig}}$ is the union of the $\mathbf{D}^{]0,r_n]}$, and so for any element of $\mathbf{D}_{\mathrm{rig}}$, it is in the domain of the map $\imath_n : \mathbf{D}^{]0,r_n]} \to \widetilde{\mathbf{D}}^+_{\mathrm{dif}}$ for $n \gg 0$.)

Theorem 5.4.1. *Let $z \in \mathbf{D}^+_{\mathrm{rig}}\big(V(1)\big)[1/t]$ and let $z' \in \mathbf{D}_{\mathrm{rig}}(V^*)^{\psi=1}$. If $(1-\varphi)z$ lies in $\mathbf{D}^+_{\mathrm{rig}}\big(V(1)\big)$, then the sequence $\langle \imath_n^-(z), \imath_n(z')\rangle_{\mathrm{dif}}$ of elements of L becomes constant for $n \gg 0$, and this constant value is equal to $\{(1-\varphi)z, \sigma_{-1}\cdot z'\}$.*

We need the following variant of Theorem 5.4.1.

Theorem 5.4.2. *Let $z \in \widetilde{\mathbf{D}}^+_{\mathrm{rig}}\big(V(1)\big)[1/t]$, $z' \in \mathbf{D}_{\mathrm{rig}}(V^*)^{\psi=1}$. Assume*

1. *$(1-\varphi)z \in \widetilde{\mathbf{D}}^+(V(1))\,[1/\varphi^r(T)]$ for some $r \in \mathbb{Z}^{\geq 0}$,*
2. *z is fixed by Γ_n for some n.*

Then

$$\{(1-\varphi)z, \sigma_{-1}\cdot z'\} = \langle \imath_s^-(z), \imath_m(z')\rangle_{\mathrm{dif}}$$

for $s \geq r$ and m large enough.

Remark 5.4.3. We explain the pairings in the statement of Theorem 5.4.2. First note that we can view $(1-\varphi)z \in \widetilde{\mathbf{D}}^+(V(1))\,[1/\varphi^r(T)] \subseteq \widetilde{\mathbf{D}}^\dagger_{\mathrm{rig}}(V(1))[1/t]$ and $z' \in \mathbf{D}_{\mathrm{rig}}(V^*) \subseteq \widetilde{\mathbf{D}}^\dagger_{\mathrm{rig}}(V^*)$. The pairing $\{(1-\varphi)z, \sigma_{-1}\cdot z'\}$ is thus the pairing between $\widetilde{\mathbf{D}}^\dagger_{\mathrm{rig}}(V(1))$ and $\widetilde{\mathbf{D}}^\dagger_{\mathrm{rig}}(V^*)$ noted at the end of section 5.2, as these are (φ,Γ)-modules defined over $\widetilde{\mathscr{R}} \otimes_{\mathbb{Q}_p} L$, extended to a pairing between $\widetilde{\mathbf{D}}^\dagger_{\mathrm{rig}}(V(1))[1/t]$ and $\widetilde{\mathbf{D}}^\dagger_{\mathrm{rig}}(V^*)[1/t]$ in the evident way. (The pairing $\langle\ ,\ \rangle$ now takes values in $\widetilde{\mathscr{R}}[1/t] \otimes_{\mathbb{Q}_p}$

L, the normalized traces map this to $\mathscr{R}[1/t] \otimes_{\mathbb{Q}_p} L$, and the pairing $(f, g) \mapsto \operatorname{res}_0(fg\frac{dT}{1+T}) = \operatorname{res}_0(fgdt)$ still makes sense.)

To make sense of $\langle \imath_s^-(z), \imath_m(z')\rangle_{\mathrm{dif}}$ for $z \in \widetilde{\mathbf{D}}_{\mathrm{rig}}^+\big(V(1)\big)[1/t]$ and for m large enough, note that since z is fixed by Γ_n, we have

$$\imath_s(z) \in \widetilde{\mathbf{D}}_{\mathrm{dif}}(V(1))^{\Gamma_n} \subseteq \mathbf{D}_{\mathrm{pdR},n}(V(1)) \subseteq \mathbf{D}_{\mathrm{dif},n}(V(1)) \subseteq \mathbf{D}_{\mathrm{dif},m}(V(1))$$

as long as $m \geq n$ (see [10, VI.3.3]). As $\imath_m(z') \in \mathbf{D}_{\mathrm{dif},m}(V^*)$, the pairing is defined as in section 5.3.

Proof of Theorem 5.4.2. Following the proof of [10, Prop. VI.3.4], mutatis mutandis, we find that

$$\{(1-\varphi)z, \sigma_{-1}\cdot z'\} = \langle \imath_m^-(z), \imath_m(z')\rangle_{\mathrm{dif}}$$

for m large enough. Further, since $(1-\varphi)z$ lies in $\widetilde{\mathbf{D}}^+\big(V(1)\big)[1/\varphi^r(T)]$, by Proposition 3.3.1, we have $\imath_s^-\big((1-\varphi)z\big) = 0$ for $s > r$. Thus,

$$\imath_s^-(z) = \imath_s^-(\varphi z) = \imath_{s-1}^-(z) = \cdots = \imath_{r+1}^-(z) = \imath_{r+1}^-(\varphi z) = \imath_r^-(z)$$

as desired. $\square$

6. Kirillov models and local Mellin transforms

This section is an interlude, in which we recall some basic facts regarding Kirillov models and local Mellin transform formulas for local Euler factors. We begin with the case of $\mathrm{GL}_2(\mathbb{Q}_p)$-representations with complex coefficients before explaining the modifications necessary for treating the case of representations with p-adic coefficients.

6.1. **Kirillov models of complex smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representations.** If π is an infinite-dimensional irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\mathbb{C}$, its Kirillov model identifies it with a space of locally constant functions on $\mathbb{Q}_p^\times$, each of which is compactly supported as a function on $\mathbb{Q}_p$. (More precisely, the closure in $\mathbb{Q}_p$ of the support of each function in the Kirillov model is compact, or, equivalently, the support in $\mathbb{Q}_p^\times$ of each such function is bounded away from infinity.) The entire $\mathrm{GL}_2(\mathbb{Q}_p)$-action on this space of functions is a bit tricky to describe, but the action of B is quite explicit. Firstly, the centre acts by the central character of π. Secondly, the subgroup $\begin{pmatrix} \mathbb{Q}_p^\times & 0 \\ 0 & 1 \end{pmatrix}$ acts by scaling the variable. Finally, if $b \in \mathbb{Q}_p$, then $\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\phi\right)(x) = e^{2\pi i b x}\phi(x)$. (Note that $e^{2\pi i y}$ makes sense for $y \in \mathbb{Q}_p$, via the natural isomorphism $\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\sim} (\mathbb{Q}/\mathbb{Z})[p^\infty]$.) This can be summarized by the following formula:

$$\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\phi\right)(x) = \delta(d)e^{2\pi i(b/d)x}\phi\big((a/d)x\big),$$

where δ is the central character of π.

An important aspect of the theory is that the realization of π in the Kirillov model is unique (up to rescaling by a non-zero constant). One way to phrase and prove this is as follows: if ϕ is an element of π, thought of in the Kirillov model, the map $\phi \mapsto \phi(1)$ gives a non-zero linear functional $\ell : \pi \to \mathbb{C}$, with the property that $\ell\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} v\right) = e^{2\pi i b x}\ell(v)$ for all $v \in \pi$. The uniqueness of Kirillov models can then be stated as saying that (when π is irreducible and infinite-dimensional) the space of such ℓ is one-dimensional, hence ℓ is uniquely determined up to a non-zero scalar. Note that we can then recover the Kirillov function ϕ_v attached to an element $v \in \pi$ via the formula

$$\phi_v(a) = \ell\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v\Big).$$

As an example of Kirillov models, suppose that π is the principal series

$$\begin{aligned} &\mathrm{Ind}_{\overline{B}}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2 \\ &= \{f : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathbb{C} \mid f(\overline{n}tg) = (\chi_1 \otimes \chi_2)(t) f(g) \text{ for all } g \in \mathrm{GL}_2(\mathbb{Q}_p), \overline{n} \in \overline{\mathrm{N}}, t \in T\} \end{aligned}$$

(where $\overline{\mathrm{N}}$ denotes the lower triangular unipotent subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$, T denotes the diagonal torus, and $\overline{B} = \overline{N}T$ denotes the lower triangular Borel), equipped with the right regular $\mathrm{GL}_2(\mathbb{Q}_p)$-action. We may regard π as the space of locally constant functions on $\mathbb{Q}_p$ which are proportional to χ_1/χ_2 in a neighbourhood of infinity, via

$$f(x) := f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right);$$

the $\mathrm{GL}_2(\mathbb{Q}_p)$-action on π is then given by the formula

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} f\right)(x) = \frac{\chi_1}{\chi_2}(cx+a)\chi_2(ad-bc) f\left(\frac{dx+b}{cx+a}\right).$$

Forming the Kirillov model of π in this case amounts to taking the Fourier transform: we map a function f in the principal series to (a rescaling of) its Fourier transform, namely to

$$\phi(x) := |x|_p \chi_1(x) \int_{\mathbb{Q}_p} f(y) e^{-2\pi i x y} dy.$$

(For definiteness, we normalize Haar measure on $\mathbb{Q}_p$ so that $\mathbb{Z}_p$ has measure 1.)

6.2. **Local newvectors.** If π is an irreducible infinite-dimensional smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, then the newvector in π (which is well-defined up to a scalar) is the vector fixed by $\begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ which is of smallest conductor, i.e. which is also fixed under $\begin{pmatrix} 1 & 0 \\ p^n\mathbb{Z}_p & 1 \end{pmatrix}$ for the smallest possible choice of n.

We will also consider the p-deprived newvector of π (which is again well-defined up to a scalar); this is any element of π which is fixed by $\begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ and which is annihilated by the operator

$$U_p := \sum_{i=0}^{p-1} \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix}.$$

Note that newvector and p-deprived newvector of π coincide if and only if the the local Euler factor attached to π is trivial.

As already noted, both the Kirillov model of π and the newvector (or the p-deprived newvector) of π are only determined up to a non-zero scalar. However, given a choice of Kirillov model, we can fix the newvector precisely by requiring that the corresponding function in the Kirillov model takes the value 1 at the point

$1 \in \mathbb{Q}_p^\times$. With this normalization, the p-deprived newvector corresponds in the Kirillov model to the characteristic function $\mathbb{1}_{\mathbb{Z}_p^\times}$.

If π is the unramified principal series $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_\alpha \otimes \chi_{\beta/p}$, where χ_λ is the unramified character taking p to λ (and π is assumed to be irreducible; equivalently, we assume that $\alpha/\beta \neq p^{\pm 1}$), then the newvector v_{new}, as a function on $\mathbb{Q}_p$, coincides (up to scaling) with the function

$$v_{\mathrm{new}} = (1 - \beta/(\alpha p))^{-1}\big(\mathbb{1}_{\mathbb{Z}_p} + (\chi_{\alpha p/\beta})\mathbb{1}_{\mathbb{Q}_p \setminus \mathbb{Z}_p}\big).$$

Converting to the Kirillov model (an exercise in taking Fourier transforms) one finds that in the Kirillov model (so now as a function on $\mathbb{Q}_p^\times$), one has

$$v_{\mathrm{new}} = \sum_{n=0}^{\infty} \frac{1}{p^n} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \mathbb{1}_{p^n \mathbb{Z}_p^\times} = \sum_{n=0}^{\infty} \frac{(\alpha/p)^{n+1} - (\beta/p)^{n+1}}{(\alpha/p) - (\beta/p)} \mathbb{1}_{p^n \mathbb{Z}_p^\times}.$$

(This is the $\mathrm{GL}_2(\mathbb{Q}_p)$ case of a formula of Casselman and Shalika.)

6.3. **The local Birch Lemma.** The goal of this subsection is to explain how to describe the p-deprived newvector of a ramified twist of a smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representation π in terms of the newvector of π.

If $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ is a character, then $f \mapsto f\chi$ gives a bijection between the Kirillov model of π and the Kirillov model of the twist $\pi_\chi := (\chi \circ \det) \otimes \pi$. In particular, if ϕ in Kirillov model corresponds to the newvector of π (normalized as above so that $\phi(1) = 1$), then $\phi\chi$ is a function in the Kirillov model of π_χ (satisfying $(\phi\chi)(1) = 1$).

Let p^n be the conductor of $\chi|_{\mathbb{Z}_p^\times}$. If $n = 0$, i.e. if χ is unramified, then $\phi\chi$ is the newvector of π_χ. However, if $n > 0$, i.e. if χ is ramified, then $\phi\chi$, although it is invariant under $\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$, is not invariant under $\begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix}$, and so cannot be the newvector. We remedy this by forming a Gauss sum: one easily derives the formula

$$\sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(a) \begin{pmatrix} 1 & a/p^n \\ 0 & 1 \end{pmatrix} (\phi\chi) = \tau(\chi) \mathbb{1}_{\mathbb{Z}_p^\times},$$

where $\tau(\chi)$ is the usual Gauss sum attached to χ. Thus the expression on the left is equal to $\tau(\chi)$ times the p-deprived newform of π_χ. (Note that, since χ is ramified, typically the p-deprived newform of π_χ coincides with the newform itself; the only exceptions occur for very particular choices of χ when π is a ramified principal series or a ramified twist of Steinberg.) This formula is a local version of the formula known as the Birch Lemma in the theory of modular symbols.

6.4. **Local Mellin transforms.** Let π be an infinite dimensional irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\mathbb{C}$, and let $\phi \in \mathcal{C}^{\mathrm{sm}}(\mathbb{Q}_p^\times, \mathbb{C})$ be an element of π, thought of in the Kirillov model. The local Mellin transform of ϕ is the integral

$$Z(\phi, s) := \int_{\mathbb{Q}_p^\times} |x|^{s-1} \phi(x) d^\times x.$$

(We normalize the Haar measure $d^\times x$ on $\mathbb{Q}_p^\times$ so that $\mathbb{Z}_p^\times$ has unit measure.) *A priori*, unless ϕ is compactly supported on $\mathbb{Q}_p^\times$, this integral only converges when the real part of s is sufficiently large. However, for a fixed choice of ϕ, the convergent values of the integral are a rational function of p^{-s}, and so the integral can be defined for any value of s in terms of this rational function.

These integrals come up in Jacquet–Langlands [17] in the description of local Euler factors and epsilon factors; in particular, taking ϕ to be the newvector gives the local Euler factor. For example, in the case of a principal series, taking ϕ to be the newvector, we find that this integral is a sum of geometric series, which we may formally evaluate to obtain the value

$$(1-\alpha p^{-s})^{-1}(1-\beta p^{-s})^{-1}.$$

Note also that the local Euler factor is the "largest" denominator that can occur in any $Z(\phi,s)$; more precisely, it is divisible by the denominator of $Z(\phi,s)$ for any function ϕ in the Kirillov model.

We now describe an alternative approach to evaluating the Mellin transform integral. Since $Z(\phi,s)=Z(|\ \ |^{s-1}\phi,1)$ (thinking of $|\ \ |^{s-1}\phi$ as a function in the Kirillov model of the twist $\pi_{|\det|^{s-1}}$), we focus on the value at $s=1$.

Proposition 6.4.1. *Suppose that the local Euler factor attached to π does not have a pole at $s=1$. Then given any function ϕ in the Kirillov model of π, we may find a uniquely determined function ξ in $\mathcal{C}^{\mathrm{sm}}(\mathbb{Q}_p^\times,\mathbb{C})$ such that:*

(1) *The restriction of ξ to (the intersection with $\mathbb{Q}_p^\times$ with) any compact neighbourhood of 0 in $\mathbb{Q}_p$ lies in the Kirillov model of π.*
(2) $\left(1-\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}\right)\xi=\phi.$

If ϕ is Γ-invariant, then so is ξ. Furthermore, one then has that

$$Z(\phi,1)=\int_{p^{-i}\mathbb{Z}_p^\times}\xi(x)d^\times x=\int_{\mathbb{Z}_p^\times}\xi(p^{-i}x)d^\times x,$$

for any i chosen so large that ϕ vanishes outside of $p^{-i}\mathbb{Z}_p$.

Proof. We first consider the case where ϕ has support bounded away from 0. In this case, we define $\xi=\sum_{n\geq 0}\left(\begin{smallmatrix} p^n & 0\\ 0 & 1\end{smallmatrix}\right)\phi$. Note that for a fixed $x\in\mathbb{Q}_p^\times$, $\xi(x)$ is a finite sum. Since ϕ has support bounded away from 0, the same is true of ξ, proving (1). Further (2) follows immediately from the definition of ξ, as does the fact that ξ is Γ-invariant if ϕ is. Lastly, for i large enough so that ϕ vanishes outside of $p^{-i}\mathbb{Z}_p$, we have

$$\int_{\mathbb{Z}_p^\times}\xi(p^{-i}x)d^\times x=\sum_{n\geq 0}\int_{\mathbb{Z}_p^\times}\phi(p^{n-i}x)d^\times x=\int_{p^{-i}\mathbb{Z}_p\setminus\{0\}}\phi(x)d^\times x=\int_{\mathbb{Q}_p^\times}\phi(x)d^\times x$$

as desired.

For the general case, we may write $\phi=\phi^-+\phi^+$ where ϕ^+ is bounded away from 0, and ϕ^- is a linear combination of functions supported on $\mathbb{Z}_p$ of the form $x\mapsto\gamma^{\mathrm{ord}_p(x)}$ for some $\gamma\in\mathbb{C}$ (see e.g. [12, Chapter 6]). Note further that our assumption on the local Euler factor forces that $\gamma\neq 1$. To complete the proof, we are thus reduced to the case where $\phi(x)=\begin{cases}\gamma^{\mathrm{ord}_p(x)} & x\in\mathbb{Z}_p,\\ 0 & \text{otherwise.}\end{cases}$

In this case, we define $\xi(x)=\begin{cases}\frac{\gamma^{\mathrm{ord}_p(x)}}{1-\gamma} & x\in\mathbb{Z}_p,\\ \frac{1}{1-\gamma} & \text{otherwise.}\end{cases}$ Clearly, ξ satisfies (1) as near zero it is of the form $x\mapsto D\gamma^{\mathrm{ord}_p(x)}$ for some constant D. To check (2), for

$x \in \mathbb{Z}_p$, we have

$$\left(1 - \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \xi(x) = \xi(x) - \xi(px) = \frac{\gamma^{\mathrm{ord}_p(x)}}{1-\gamma} - \frac{\gamma^{\mathrm{ord}_p(x)+1}}{1-\gamma} = \gamma^{\mathrm{ord}_p(x)} = \phi(x)$$

while for $x \notin \mathbb{Z}_p$, we have

$$\left(1 - \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \xi(x) = \xi(x) - \xi(px) = \frac{1}{\gamma - 1} - \frac{1}{\gamma - 1} = 0 = \phi(x)$$

as desired.

Further, we note that from our explicit construction of ξ, it is clear that if ϕ is invariant under Γ, then so is ξ.

Lastly, we compute that

$$Z(\phi, s) = \int_{\mathbb{Q}_p^\times} |x|^{s-1} \phi(x) d^\times x = \sum_{n=0}^{\infty} p^{-n(s-1)} \gamma^n = \frac{1}{1 - \frac{\gamma}{p^{s-1}}}$$

for $s \gg 0$. Thus, $Z(\phi, 1) = \frac{1}{1-\gamma}$. Finally, for $i > 0$,

$$\int_{\mathbb{Z}_p^\times} \xi(p^{-i}x) d^\times x = \int_{\mathbb{Z}_p^\times} \frac{1}{1-\gamma} d^\times x = \frac{1}{1-\gamma} = Z(\phi, 1),$$

completing the proof. □

6.5. **Kirillov models of smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ with p-adic coefficients.** We now consider the theory of Kirillov models for infinite dimensional irreducible smooth representations π of $\mathrm{GL}_2(\mathbb{Q}_p)$ having coefficients in L, a finite extension of $\mathbb{Q}_p$. The main difference between this case and the case of complex coefficients is one of rationality: the p-power roots of unity are not all in L.

Recall $L_\infty := K_{\infty,p} \otimes_{\mathbb{Q}_p} L$. If we let $\mathcal{C}^{\mathrm{sm}}(\mathbb{Q}_p^\times, L)$ (resp. $\mathcal{C}^{\mathrm{sm}}(\mathbb{Q}_p^\times, L_\infty)$) denote the space of locally constant functions on $\mathbb{Q}_p^\times$ with values in L (resp. L_∞), then the Kirillov model will be an embedding

$$\pi \hookrightarrow K_{\infty,p} \otimes_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{sm}}(\mathbb{Q}_p^\times, L) \subseteq \mathcal{C}^{\mathrm{sm}}(\mathbb{Q}_p^\times, L_\infty),$$

which is B-equivariant for a certain B-action on the target.

To describe this B-action, we have to replace our complex valued additive character on $\mathbb{Q}_p$ by a $K_{\infty,p}$-valued one. Recall that we have chosen a generator ε of $\mathbb{Z}_p(1)$; write $\varepsilon = (\varepsilon^{(n)})$, where $\varepsilon^{(n)}$ is a primitive p^nth root of unity, and $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$. We will replace the complex valued character $e^{2\pi i x}$ of $\mathbb{Q}_p$ by the $\overline{\mathbb{Q}}_p$-valued character

$$\varepsilon(x) := \lim_{n \to \infty} (\varepsilon^{(n)})^{p^n x}.$$

(Note that the sequence $(\varepsilon^{(n)})^{p^n x}$ eventually stabilizes, so that the limit is just equal to this stable value.) We then define the B-action on $\mathcal{C}^{\mathrm{sm}}(\mathbb{Q}_p^\times, L_\infty)$ via the formula

$$\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \phi \right)(x) = \delta(d) \varepsilon\big((b/d)x\big) \phi\big((a/d)x\big),$$

where, as before, δ denotes the central character of π. We may also define a Galois-twisted action of Γ on $\mathcal{C}^{\mathrm{sm}}(\mathbb{Q}_p^\times, L_\infty)$ via the formula $(a \cdot \phi)(x) = \sigma_a\big(\phi(a^{-1}x)\big)$. (Here the Galois element σ_a acts on L_∞ via its action on the first factor in the tensor product which we normalize so that $\sigma_a(\zeta) = \zeta^a$ for $\zeta \in \mu_{p^\infty}$.) Using the fact that π is defined over L, together with the uniqueness of Kirillov models, one then shows that the image of the Kirillov model lies in $\big(K_{\infty,p} \otimes \mathcal{C}^{\mathrm{sm}}(\mathbb{Q}_p^\times, L)\big)^\Gamma$, where the invariants are computed with respect to the Galois-twisted action.

Note that the Kirillov functional given by evaluation at $1 \in \mathbb{Q}_p^\times$ is now a non-zero linear functional

$$\ell : \pi \to L_\infty,$$

such that

$$\ell\Big(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} v\Big) = \varepsilon(b)\ell(v),$$

and such that

$$\ell\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v\Big) = \sigma_a\big(\ell(v)\big),$$

for all $a \in \Gamma$. The second property of ℓ shows that (unlike in the complex case), in the p-adic setting the functions in the Kirillov model of π are determined by their values on the elements p^i ($i \in \mathbb{Z}$) of $\mathbb{Q}_p^\times$. Thus, in the following, we will frequently consider the functions in the Kirillov model just as functions on the powers p^i (and lose no information by so doing).

6.6. **Local Mellin transforms with p-adic coefficients.** Suppose that π is defined over L, and let

$$\phi \in \mathcal{C}^{\mathrm{sm}}(\mathbb{Q}_p^\times, L_\infty)^\Gamma$$

be an element of π, thought of via its Kirillov model. (Here Γ-invariance of ϕ refers to invariance under the Galois-twisted Γ-action, as discussed above.)

When s is an integer, we can write down the local Mellin transform integral, and express it as a rational function of p^{-s}. However, we can further simplify the integral in this context to be just a sum over powers of p (just as the Kirillov model can be regarded simply as functions on the set of powers of p, rather than on all of $\mathbb{Q}_p^\times$), as we now explain.

Let $\mathrm{Tr} : K_{\infty,p} \to \mathbb{Q}_p$ be the normalized trace, *i.e.* its restriction to $K_{n,p} := \mathbb{Q}_p(\mu_{p^n})$ is given by $\dfrac{1}{(p-1)p^{n-1}} \mathrm{Tr}_{\mathbb{Q}_p}^{K_{n,p}}$. We then have the following formula, easily proved using the Galois-twisted Γ-invariance of ϕ:

$$\int_{\mathbb{Z}_p^\times} \phi(x) d^\times x = (\mathrm{Tr} \otimes \mathrm{id}_L)\big(\phi(1)\big).$$

From this, we deduce the corresponding formula for the local Mellin transform:

$$\int_{\mathbb{Q}_p^\times} \phi(x) d^\times x = (\mathrm{Tr} \otimes \mathrm{id}_L)\Big(\sum_{n=-\infty}^{\infty} \phi(p^n)\Big).$$

In light of this formula, we regard the expression

$$\widetilde{Z}(\phi, 1) = \sum_{n=-\infty}^{\infty} \phi(p^n) \in L_\infty \tag{8}$$

(which takes values in $K_{n,p} \otimes_{\mathbb{Q}_p} L$ if ϕ is invariant under the (regular, not Galois-twisted) action of $1 + p^n\mathbb{Z}_p$) as being a refined Mellin transform.

We then have the following analogue of Proposition 6.4.1.

Proposition 6.6.1. *Suppose that the local Euler factor attached to π does not have a pole at $s = 1$. Then given any function ϕ in the Kirillov model of π, we may find a uniquely determined function ξ in $\mathcal{C}^{\mathrm{sm}}(\mathbb{Q}_p^\times, L_\infty)^\Gamma$ such that:*

(1) *The restriction of* ξ *to (the intersection with* $\mathbb{Q}_p^\times$ *with) any compact neighbourhood of* 0 *in* $\mathbb{Q}_p$ *lies in the Kirillov model of* π.
(2) $\left(1 - \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right) \xi = \phi$.

If ϕ *is invariant under the regular* Γ*-action, then so is* ξ. *Furthermore, one then has that* $\tilde{Z}(\phi, 1) = \xi(p^{-i})$, *for any* i *chosen so large that* ϕ *vanishes outside of* $p^{-i}\mathbb{Z}_p$.

Proof. The proof of this proposition proceeds along the same lines as that of Proposition 6.4.1. □

6.7. **The case of locally algebraic representations.** In what follows, we are closely following [10, VI.4]. Namely, we now consider Kirillov models of locally algebraic representations (*e.g.* representations of the form $\pi \otimes \mathrm{Sym}^k(L^2) \otimes \det^\ell$ with π a smooth representation). Here we view $\mathrm{Sym}^k(L^2)$ as homogenous polynomials in e_1 and e_2 of degree k where $\mathrm{GL}_2(\mathbb{Q}_p)$ acts by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} e_1 = ae_1 + ce_2 \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix} e_2 = be_1 + de_2.$$

In the smooth p-adic case (section 6.5), our Kirillov models were smooth functions taking values in L_∞. We now instead consider larger spaces of functions.

Define $\mathrm{LP}^{[\ell,\ell+k]}(\mathbb{Q}_p^\times, t^\ell L_\infty[t]/t^{k+1+\ell} L_\infty[t])$ to be the space of locally polynomial functions on $\mathbb{Q}_p^\times$ valued in $t^\ell L_\infty[t]/t^{k+1+\ell} L_\infty[t]$. We equip this space with an action of the mirabolic via

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \phi\right)(x) = \delta(d)[(1+T)^{bx/d}]\phi(ax/d)$$

where δ is the central character of our locally algebraic representation, and where $[(1+T)^z]$ is the character of $\mathbb{Q}_p$ discussed in [10, §I.1.6]. The simplest way to describe it in our present context is to use the formula $[(1+T)^z] = \varepsilon(z)\,(e^{tz} - 1)$ (see *loc. cit.*).

In [10, Prop VI.2.9], it is proven that $\pi \otimes \mathrm{Sym}^k(L^2) \otimes \det^\ell$ admits a locally algebraic Kirillov model; that is, it admits an equivariant map to

$$\begin{aligned} \pi \otimes \mathrm{Sym}^k(L^2) \otimes \det^\ell &\longrightarrow \mathrm{LP}^{[\ell,\ell+k]}(\mathbb{Q}_p^\times, t^\ell L_\infty[t]/t^{k+1+\ell} L_\infty[t]). \\ z &\mapsto \mathcal{K}_z \end{aligned}$$

Explicitly, fix a Kirillov model of π where we write ϕ_v for the function corresponding to $v \in \pi$. Then for

$$z = \sum_{i=0}^{k} z_i \otimes e_1^{k-i} e_2^i \in \pi \otimes \mathrm{Sym}^k(L^2) \otimes \det^\ell$$

we set

$$\mathcal{K}_z(x) = (tx)^\ell \sum_{i=0}^{k} i!\,\phi_{z_i}(x)(tx)^{k-i}. \tag{9}$$

It is easy to see that for $a \in \mathbb{Z}_p^\times$, we have $\sigma_a(\mathcal{K}_z(x)) = \mathcal{K}_z(ax)$ where $\sigma_a(t) = at$ as usual. Thus, as before, $\mathcal{K}_z$ is invariant under the Galois-twisted action of Γ. In particular, all of the values of $\mathcal{K}_z(x)$ are determined by its values on powers of p.

Proposition 6.7.1. *Suppose that the local Euler factor attached to π does not have a pole at $s = j + 1 - \ell - k$. Then given*

$$z = v \otimes e_1^{k-j} e_2^j \in \pi \otimes \mathrm{Sym}^k(L^2) \otimes \det^\ell,$$

we may find a uniquely determined function ξ in $\mathrm{LP}^{[\ell,\ell+k]}(\mathbb{Q}_p^\times, t^\ell L_\infty[t]/t^{k+1+\ell} L_\infty[t])^\Gamma$ such that:

(1) *The restriction of ξ to (the intersection with $\mathbb{Q}_p^\times$ with) any compact neighbourhood of 0 in $\mathbb{Q}_p$ lies in the Kirillov model of $\pi \otimes \mathrm{Sym}^k(L^2) \otimes \det^\ell$.*

(2) $\left(1 - \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right) \xi = \mathcal{K}_z.$

If ϕ_v is invariant under the regular Γ-action, then so is ξ. Furthermore, one then has that $\tilde{Z}(\mathcal{K}_z, 1) = \xi(p^{-i})$, for any i chosen so large that ϕ_v vanishes outside of $p^{-i}\mathbb{Z}_p$.

Proof. First note that we have

$$\mathcal{K}_z(x) = (tx)^\ell j!\, \phi_v(x)(tx)^{k-j} = j!(tx)^{k+\ell-j}\phi_v(x).$$

If ϕ_v is bounded away from 0, then we can argue exactly as in Proposition 6.4.1 and define $\xi = \sum_{n\geq 0} \left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right) \mathcal{K}_z$. Otherwise, again as in Proposition 6.4.1, we can reduce to the case where $\phi_v(x) = \begin{cases} \gamma^{\mathrm{ord}_p(x)} & x \in \mathbb{Z}_p, \\ 0 & \text{otherwise.} \end{cases}$ In this case, we define $\xi(x) = \begin{cases} \dfrac{(tx)^a \gamma^{\mathrm{ord}_p(x)}}{1 - p^a\gamma} & x \in \mathbb{Z}_p, \\ \dfrac{(tx|x|)^a}{1 - p^a\gamma} & x \in \mathbb{Z}_p, \end{cases}$ where $a = k + \ell - j$. A direct computation with this function verifies the remaining claims of the proposition. We also note that to define ξ, we need that $\gamma \neq p^{-a} = p^{j-k-\ell}$ which is implied by our condition on the local Euler factor of π. $\square$

7. p-adic local Langlands and Kirillov models

In this section we will turn to the consideration of Colmez's work on p-adic local Langlands. This work uses the constructions of the preceding sections in the particular case of two-dimensional p-adic representations of $G_{\mathbb{Q}_p}$ to associate a $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach space representation to any such two-dimensional representation. For us, one of the most important features of Colmez's work will be the very powerful analogue of the classical theory of Kirillov models for smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ that he develops using the maps $\imath_i^-$ and $\imath^-$ introduced above.

7.1. **Kirillov models via (φ, Γ)-modules.** If V is a continuous irreducible two-dimensional representation of $G_{\mathbb{Q}_p}$ over some finite extension L of $\mathbb{Q}_p$, then Colmez associates to V an L-Banach space representation $\pi(V)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$. (We use slightly different conventions from Colmez; he would call this $\pi(V(1))$.) We won't recall the construction of $\pi(V)$ as a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation here. Rather, we will focus on $\pi(V)$ just as a P-representation (recall P is the mirabolic subgroup), in which case $\pi(V)$ admits the following description: there is a canonical isomorphism of P-modules

$$\widetilde{\mathbf{D}}\big(V(1)\big)/\widetilde{\mathbf{D}}^+\big(V(1)\big) \xrightarrow{\sim} \pi(V),$$

where the source has the P-representation structure discussed in the section 4 (see [10, Corollaire II.2.9]).

Suppose now that V admits zero as a Hodge–Sen–Tate weight with multiplicity one. If we let $(\widetilde{\mathbf{D}}\big(V(1)\big)/\widetilde{\mathbf{D}}^+\big(V(1)\big))_{P-\mathrm{sm}}$ denote the subspace of P-smooth vectors of the P-representation $\widetilde{\mathbf{D}}\big(V(1)\big)/\widetilde{\mathbf{D}}^+\big(V(1)\big)$, and if we choose an L-basis $\mathbf{e}$ for the one-dimensional space of Γ-invariant vectors in $\widetilde{\mathbf{D}}_{\mathrm{Sen}}$, then we see that $\imath_0^-$ restricts to a map

$$\imath_0^- : \Big(\widetilde{\mathbf{D}}\big(V(1)\big)/\widetilde{\mathbf{D}}^+\big(V(1)\big)\Big)_{P-\mathrm{sm}} \to L_\infty \mathbf{e}, \tag{10}$$

with the following two properties: for any $n \geq 0$ and any $b \in \mathbb{Z}_p$,

$$\imath_0^-\left(\varphi^{-n}(1+T)^b x\right) = (\varepsilon^{(n)})^b \imath_0^-(x),$$

or, rewriting this in terms of the P-action,

$$\imath_0^-\left(\begin{pmatrix} 1 & b/p^n \\ 0 & 1 \end{pmatrix} x\right) = (\varepsilon^{(n)})^b \imath_0^-(x); \tag{11}$$

and for any $a \in \Gamma$,

$$\imath_0^-(a \cdot x) = \sigma_a\big(\imath_0^-(x)\big),$$

or, again rewriting this in terms of the P-action,

$$\imath_0^-\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} x\right) = \sigma_a\left(\imath_0^-(x)\right). \tag{12}$$

We suppose now (and for the remainder of this section) that V is furthermore de Rham, with Hodge–Tate weights equal to 0 and $1-k$, for some $k \geq 2$ (so that the jumps in the Hodge filtration of $\mathbf{D}_{\mathrm{dR}}(V)$ occur at 0 and $k-1$). The theory of p-adic local Langlands shows that $\pi(V)$ contains as a subrepresentation $\pi_{\mathrm{sm}}(V) \otimes_L (\mathrm{Sym}^{k-2} L^2)^*$, where $\pi_{\mathrm{sm}}(V)$ is the smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over L attached to the Weil–Deligne representation underlying the potentially semi-stable Dieudonné module of V (see [10, Théorème 0.20, 0.21]).

Let v_{hw} denote a highest weight vector of $(\mathrm{Sym}^{k-2} L^2)^*$ (well-defined up to scaling). Note that v_{hw} is not only fixed by N (by definition) but is actually fixed by P. Thus we have an inclusion

$$\pi_{\mathrm{sm}}(V) \xrightarrow{\sim} \pi_{\mathrm{sm}}(V) \otimes_L v_{\mathrm{hw}} \hookrightarrow \pi(V)_{P-\mathrm{sm}}.$$

In particular, we may restrict the map $\imath_0^-$ of (10) above to a map

$$\pi_{\mathrm{sm}}(V) \to L_\infty \mathbf{e}.$$

A consideration of the formulas (11) and (12) above then shows that this map is precisely of the form

$$v \mapsto \ell(v),$$

where ℓ is the (suitably scaled) Kirillov functional of $\pi_{\mathrm{sm}}(V)$. In summary, we have the following result, due to Colmez.

Theorem 7.1.1 (Colmez). *The Kirillov model of $\pi_{\mathrm{sm}}(V)$ (suitably scaled) satisfies the following: if $v_{\mathrm{sm}} \in \pi_{\mathrm{sm}}(V)$ corresponds to the function ϕ in the Kirillov model, then $\imath_0^-(v_{\mathrm{sm}} \otimes v_{\mathrm{hw}}) = \phi(1)\mathbf{e}$.*

Proof. See [10, Proposition VI.5.6]. □

Note that since V is assumed to be de Rham with Hodge–Tate weights 0 and $1-k$, the Γ-fixed points of $\widetilde{\mathbf{D}}_{\mathrm{Sen}}(V)$ are naturally identified with $\mathbf{D}_{\mathrm{dR}}(V(1))/\mathbf{D}^+_{\mathrm{dR}}(V(1))$, and so our choice of $\mathbf{e}$ is tantamount to a choice of basis of the latter quotient.

In light of the preceding theorem, we regard the map $\imath^-$ as providing generalized Kirillov models for the P-representations $\widetilde{\mathbf{D}}^+\big(V(1)\big)\big[\big(1/\varphi^n(T)\big)_{n\geq 0}\big]/\widetilde{\mathbf{D}}^+\big(V(1)\big)$ and $\widetilde{\mathbf{D}}^+_{\mathrm{rig}}\big(V(1)\big)[1/t]/\widetilde{\mathbf{D}}^+_{\mathrm{rig}}\big(V(1)\big)$. More precisely, we regard $\imath_i^-$ as evaluation at p^{-i}, and regard elements of these representations as functions on the set $\{p^{-i}\}_{i\in\mathbb{Z}}$, taking values in $\widetilde{\mathbf{D}}_{\mathrm{dif}}\big(V(1)\big)/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}\big(V(1)\big)$.

7.2. **The locally algebraic case.** In the previous subsection, we embedded $\pi_{\mathrm{sm}}(V)$ into $\pi(V)_{P-\mathrm{sm}}$ by utilizing the highest weight vector in $\mathrm{Sym}^{k-2}(L^2)^*$. However, using only the highest weight vector loses information and we want to be able to work with all of $\mathrm{Sym}^{k-2}(L^2)^*$. This consideration was exactly the reason we introducted Kirillov models of locally algebraic representations in section 6.7. We now explain how to see this locally algebraic Kirillov model in terms of the morphism $\imath^-$, i.e. in terms of Colmez's generalized Kirillov model.

To this end, following Colmez [10, VI.5.2], we first define

$$\pi(V)_{U-\mathrm{alg}} := \bigcup_{n\geq 0} \frac{1}{(\varphi^n(T))^{k-1}}\widetilde{\mathbf{D}}^+(V(1))/\widetilde{\mathbf{D}}^+(V(1)).$$

The map $\imath_0^-$ (of section 3.3) then induces a map

$$\imath_0^- : \pi(V)_{U-\mathrm{alg}} \longrightarrow t^{1-k}\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V(1))/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V(1)).$$

For $z \in \pi(U)_{U-\mathrm{alg}}$, we define a function

$$\begin{aligned}\widetilde{\mathcal{K}}_z : \mathbb{Q}_p^\times &\longrightarrow t^{1-k}\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V(1))/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V(1))\\ x &\mapsto \imath_0^-\left(\begin{pmatrix} x & 0\\ 0 & 1\end{pmatrix} z\right).\end{aligned}$$

In [10, Lemme VI.5.4], it is verified that $\sigma_a(\widetilde{\mathcal{K}}_z(x)) = \widetilde{\mathcal{K}}_z(ax)$ for $a \in \mathbb{Z}_p^\times$ and thus $\widetilde{\mathcal{K}}_z$ is uniquely determined by its values on powers of p. That is, $\widetilde{\mathcal{K}}$ is simply (a concrete reinterpretation of) the restriction of $\imath^-$ to $\pi(V)_{U-\mathrm{alg}}$.

We now state an analogue of Theorem 7.1.1, which will describe the precise relationship of this generalized Kirillov model to the Kirillov model of a locally algebraic representation of Section 6.7. To this end, recall that for each

$$z \in \pi_{\mathrm{sm}}(V) \otimes_L (\mathrm{Sym}^{k-2}L^2)^* \cong \pi_{\mathrm{sm}}(V) \otimes_L (\mathrm{Sym}^{k-2}L^2) \otimes \det{}^{2-k},$$

we have an associated locally polynomial map $\mathcal{K}_z : \mathbb{Q}_p^\times \to t^{2-k}L_\infty[t]/tL_\infty[t]$. Further, we have previously fixed a basis $\mathbf{e}$ of $\mathbf{D}_{\mathrm{dR}}(V(1))/\mathbf{D}^+_{\mathrm{dR}}(V(1))$. Since the Hodge-Tate weights of $V(1)$ are 1 and $2-k$, we have that $\mathbf{e} \in t^{-1}\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V(1))$. Thus

$$\begin{aligned}t^{2-k}L_\infty[t]/tL_\infty[t] &\to \widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V(1))[1/t]/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V(1))\\ f(t) &\mapsto f(t)\mathbf{e}\end{aligned}$$

is a well-defined Γ-equivariant map.

Write $\mathcal{K}_z\mathbf{e}$ to be the map with values in $\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V(1))[1/t]/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V(1))$ defined by sending x to $\mathcal{K}_z(x)\mathbf{e}$. Thus $\mathcal{K}_z\mathbf{e}$ and $\widetilde{\mathcal{K}}_z$ both take values in the same space. The following theorem of Colmez compares these two Kirillov models.

Theorem 7.2.1 (Colmez). *As functions of z on*

$$\pi_{\mathrm{sm}}(V) \otimes_L (\mathrm{Sym}^{k-2} L^2) \otimes \det{}^{2-k} \subseteq \pi(V)_{U-\mathrm{alg}},$$

$\mathcal{K}_z\mathbf{e}$ and $\widetilde{\mathcal{K}}_z$ agree up to multiplication by a scalar.

Proof. This theorem follows from [10, Prop. VI.5.6(iii)]. □

Note that by suitably scaling the Kirillov model of $\pi_{\mathrm{sm}}(V)$ we can (and will) force $\mathcal{K}_z\mathbf{e}$ and $\widetilde{\mathcal{K}}_z$ to exactly agree.

We now return to the problem of inverting $1 - \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in the setting of generalized Kirillov models. Recall that in Proposition 6.7.1, we solved the equation $\left(1 - \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \xi = \mathcal{K}_z$. However, the solution ξ was not literally in the Kirillov model of π, but instead ξ restricted to any compact neighborhood was in the Kirillov model of π. The following proposition realizes $\xi\mathbf{e}$ in a generalized Kirillov model.

Proposition 7.2.2. *Suppose that the local Euler factor of $\pi_{\mathrm{sm}}(V)$ does not have a pole at $s = 1 + j$. For $z = v \otimes e_1^{k-2-j} e_2^j \in \pi \otimes \mathrm{Sym}^{k-2}(L^2) \otimes \det^{2-k}$, let*

$$\xi \in \mathrm{LP}^{[2-k,0]}(\mathbb{Q}_p^\times, t^{2-k} L_\infty[t]/tL_\infty[t])^\Gamma$$

be the function of Proposition 6.7.1 such that $\left(1 - \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \xi = \mathcal{K}_z$. Then there exists an element $w \in \widetilde{\mathbf{D}}^+_{\mathrm{rig}}\big(V(1)\big)[1/t]/\widetilde{\mathbf{D}}^+_{\mathrm{rig}}\big(V(1)\big)$ such that

$$\imath_i^-(w) = \xi(p^{-i})\mathbf{e}$$

in $\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V(1))[1/t]/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V(1))$ for all i.

Proof. By the proof of Proposition 6.7.1, we can write the function ξ as $\xi_1 + \xi_2$ where ξ_1 lies in the Kirillov model locally near zero, and ξ_2 satisfies: $\xi_2(p^i) = 0$ for $i \geq 0$ and $\xi_2(p^i)$ is constant for $i < 0$. In light of Theorem 7.2.1, we thus simply need to show that there is some element w in $\widetilde{\mathbf{D}}^+_{\mathrm{rig}}\big(V(1)\big)[1/t]/\widetilde{\mathbf{D}}^+_{\mathrm{rig}}\big(V(1)\big)$ such that $\imath_i^-(w) = \xi_2(p^{-i})\mathbf{e}$ for $i \in \mathbb{Z}$. But the existence of such an element follows from [10, Lemme VI.4.11]. □

7.3. **Construction of local cohomology classes.** We now build the local cohomology classes $c_{n,j} \in H^1(K_{n,p}, V(1+j))$ which will play a key role in the global arguments in the second half of the paper.

For any $n \geq 0$, we define

$$d_{n,j} \in \pi_{\mathrm{sm}}(V) \otimes \mathrm{Sym}^{k-2}(L^2) \otimes \det{}^{2-k} \subseteq \pi(V) = \widetilde{\mathbf{D}}(V(1))/\widetilde{\mathbf{D}}^+(V(1))$$

by

$$d_{n,j} := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \otimes e_1^{k-2-j} e_2^j.$$

Recall that e_1 and e_2 are our basis of L^2 and the induced action on $\mathrm{Sym}^{k-2}(L^2)$ is described in Section 6.7.

Lemma 7.3.1. *We have*

(1) $d_{n,j} \in \dfrac{1}{\varphi^n(T)^{j+1}} \widetilde{\mathbf{D}}^+(V(1))/\widetilde{\mathbf{D}}^+(V(1))$;

(2) *for $a \in \Gamma_n = 1 + p^n\mathbb{Z}_p$, we have* $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) d_{n,j} = a^{-j} \cdot d_{n,j}$.

Proof. First note the simple matrix equation:

$$\left(\begin{smallmatrix} 1 & p^n \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right).$$

Thus,

$$\left(\begin{smallmatrix} 1 & p^n \\ 0 & 1 \end{smallmatrix}\right) d_{n,j} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \otimes e_1^{k-2-j}(p^n e_1 + e_2)^j$$

as $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ fixes v_{new}. Computing further, we see

$$\begin{aligned}\varphi^n(T)(d_{n,j}) &= \left(\begin{smallmatrix} 1 & p^n \\ 0 & 1 \end{smallmatrix}\right) d_{n,j} - d_{n,j} \\ &= \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \otimes (e_1^{k-2-j}(p^n e_1 + e_2)^j - e_1^{k-2-j} e_2^j);\end{aligned}$$

note that $e_1^{k-2-j}(p^n e_1 + e_2)^j - e_1^{k-2-j} e_2^j$ is a homogenous polynomial in e_1 and e_2 where e_2^{j-1} is the highest power of e_2 which occurs. The same computation as above then shows

$$(\varphi^n(T))^2(d_{n,j}) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \otimes g(e_1, e_2)$$

where $g(e_1, e_2)$ is again a homogenous polynomial in e_1 and e_2, but all powers of e_2 present are less than or equal to $j-2$. Iterating this argument yields that $(\varphi^n(T))^{j+1}(d_{n,j}) = 0$, proving the first part of the proposition.

For the second part, for $a = 1 + p^n x$ with $x \in \mathbb{Z}_p$, we have

$$\begin{aligned}\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) d_{n,j} &= \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\Big(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \otimes e_1^{k-2-j} e_2^j\Big) \\ &= a^{-j}\left(\begin{smallmatrix} p^n a & a \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \otimes e_1^{k-2-j} e_2^j \\ &= a^{-j}\left(\begin{smallmatrix} p^n & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \otimes e_1^{k-2-j} e_2^j \\ &= a^{-j}\left(\begin{smallmatrix} p^n & 1 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \otimes e_1^{k-2-j} e_2^j\end{aligned}$$

proving the claim. We note that the factor of a^{-j} arises by combining the action of $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on $e_1^{k-2-j} e_2^j$ and the twist by $\det^{2-k}$. □

We now aim to build local cohomology classes from the elements $d_{n,j} \in \pi(V)$. To this end, note that as vector spaces we have $\pi(V) = \pi(V(j))$ though the underlying Γ-action differs. For $z \in \pi(V)$, we write $z^{(j)}$ for the element z, but viewed in $\pi(V(j))$. Let $z' \in \mathbf{D}(V^*(-j))^{\psi=1}$ which we identify with $H^1_{\mathrm{Iw}}(V^*(-j))$ and write z'_n for the image of z' to level n in $H^1(K_{n,p}, V^*(-j))$ where $K_{n,p} := \mathbb{Q}_p(\mu_{p^n})$. Lastly, recall the dual exponential map

$$\exp^* : H^1\big(K_{n,p}, V^*(-j)\big) \to \mathbf{D}^+_{\mathrm{dR}}\big(V^*(-j)\big) \otimes K_{n,p}.$$

We then have the following twisted version of Theorem 5.4.2.

Corollary 7.3.2. *Let $z \in \widetilde{\mathbf{D}}^+_{\mathrm{rig}}\big(V(1)\big)[1/t]$ and let $z' \in \mathbf{D}(V^*(-j))^{\psi=1}$. Suppose that*

1. *$(1-\varphi)z$ lies in $\widetilde{\mathbf{D}}^+\big(V(1)\big)[1/\varphi^r(T)]$ for some $r \geq 0$,*
2. *for some $n \geq 0$, we have $\gamma z = \gamma^{-j} \cdot z$ for all $\gamma \in 1 + p^n\mathbb{Z}_p = \Gamma_n \subseteq \Gamma$.*

Then

$$\{(1-\varphi)z^{(j)}, z'\} = p^n \langle \imath_r^-(z^{(j)}), \sigma_{-1} \exp^*(z'_n)\rangle_{\mathrm{dif}}.$$

Proof. By assumption (2), $z^{(j)}$ is fixed by Γ_n and Theorem 5.4.2 implies

$$\{(1-\varphi)z^{(j)}, z'\} = \langle \imath_r^-(z^{(j)}), \sigma_{-1} \cdot \imath_m(z')\rangle_{\mathrm{dif}}$$

for m large enough. This corollary then follows immediately from Lemma 5.3.1. □

As we intend to be pairing with $d_{n,j}$, to apply the above corollary, we need to show that $d_{n,j}$ is in the image of $1-\varphi$. To this end, suppose that the local Euler factor of $\pi_{\mathrm{sm}}(V)$ has no pole at $s=1+j$. Thus, by Propositions 6.7.1, we can solve the equation $\left(1-\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\xi_{n,j}=\mathcal{K}_{d_{n,j}}$ in the Kirillov model of $\pi_{\mathrm{sm}}(V)\otimes \mathrm{Sym}^{k-2}(L^2)\otimes \det^{2-k}$. By Proposition 7.2.2, we can then realize $\xi_{n,j}$ in a generalized Kirillov model for $\pi(V)$; that is, we can find $w_{n,j}\in \widetilde{\mathbf{D}}^{+}_{\mathrm{rig}}(V(1))[1/t]$ such that $\imath_i^-(w_{n,j})=\xi_{n,j}(p^{-i})\mathbf{e}$ for all i. Since $\imath^-$ is injective (Proposition 3.3.1), we deduce that $(1-\varphi)w_{n,j}\equiv d_{n,j} \bmod \widetilde{\mathbf{D}}^{+}_{\mathrm{rig}}\big(V(1)\big)$.

Remark 7.3.3. One can explicitly write down the classes $d_{n,j}$ and $w_{n,j}$ in the (locally algebraic) Kirillov model of $\pi_{\mathrm{sm}}(V)\otimes \mathrm{Sym}^{k-2}(L^2)\otimes \det^{2-k}$. For instance, when $\pi_{\mathrm{sm}}(V)$ is supercuspidal, then v_{new} corresponds to $\mathbf{1}_{\mathbb{Z}_p^\times}(x)$ in the (smooth) Kirillov model of $\pi_{\mathrm{sm}}(V)$ (when appropriately normalized) and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right)v_{\mathrm{new}}$ corresponds to $\varepsilon(x)\cdot \mathbf{1}_{\mathbb{Z}_p^\times}(p^n x)$. Thus, by (9),

$$\mathcal{K}_{d_{n,j}}(x)=j!\cdot (tx)^{-j}\cdot \varepsilon(x)\cdot \mathbf{1}_{\mathbb{Z}_p^\times}(p^n x);$$

that is,

$$\mathcal{K}_{d_{n,j}}(p^r)=\begin{cases} j!\cdot p^{nj}\cdot \varepsilon(p^{-n})\cdot t^{-j} & r=-n,\\ 0 & r\neq -n.\end{cases}$$

Further, tracing through the proof of Proposition 6.7.1, we see that

$$\mathcal{K}_{w_{n,j}}=\sum_{n\geq 0}\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right)\mathcal{K}_{d_{n,j}};$$

that is,

$$\mathcal{K}_{w_{n,j}}(p^r)=\begin{cases} 0 & r>-n,\\ j!\cdot p^{nj}\cdot \varepsilon(p^{-n})\cdot t^{-j} & r\leq -n.\end{cases}$$

Recall that when we view the element $d_{n,j}$ in $\pi(V(j))$ rather than $\pi(V)$, we write it as $d^{(j)}_{n,j}$. To ease notation a little, let's simply write $d^{(j)}_n$ for $d^{(j)}_{n,j}$ and likewise for $w^{(j)}_n$. Then given any element $z'\in \mathbf{D}(V^*(-j))^{\psi=1}$, Lemma 7.3.1 and Corollary 7.3.2 show that

$$\{d^{(j)}_n, z'\}=\{(1-\varphi)w^{(j)}_n, z'\}=p^n\left\langle \imath_n^-(w^{(j)}_n), \sigma_{-1}\exp^* z'_n\right\rangle_{\mathrm{dif}}. \tag{13}$$

Note that pairing with $d^{(j)}_n$ gives a functional on $\mathbf{D}(V^*(-j))^{\psi=1}\cong H^1_{\mathrm{Iw}}(V^*(-j))$, and by (13), we see that this pairing factors through $H^1(K_{n,p}, V^*(-j))$ (since z'_n is the projection of z' to level n). Thus, by Tate local duality, there is a unique $c_{n,j}\in H^1(K_{n,p}, V(1+j))$ such that

$$\langle c_{n,j}, z'_n\rangle_n=p^n\left\langle \imath_n^-(w^{(j)}_n), \sigma_{-1}\exp^* z'_n\right\rangle_{\mathrm{dif}}$$

where $\langle\cdot,\cdot\rangle_n$ is the perfect pairing on Galois cohomology induced by Tate local duality. Moreover, (13) implies that pairing with $c_{n,j}$ kills the kernel of $\exp^*$ which is $H^1_g(K_{n,p}, V^*)$, and thus $c_{n,j}\in H^1_e(K_{n,p}, V(1))$. We summarize these observations in the following proposition.

Proposition 7.3.4. *Suppose that the local Euler factor of $\pi_{\mathrm{sm}}(V)$ does not have a pole at $s = 1 + j$. Then there exist classes $c_{n,j} \in H^1_e(K_{n,p}, V(1+j))$ such that for $z' \in \mathbf{D}(V^*(-j))^{\psi=1} \cong H^1_{\mathrm{Iw}}(V^*(-j))$, we have*

$$\langle c_{n,j}, z'_n \rangle_n = p^n \left\langle \imath_n^-(w_n^{(j)}), \sigma_{-1} \exp^* z'_n \right\rangle_{\mathrm{dif}}$$

where $(1-\varphi) w_n^{(j)} = d_n^{(j)}$ in $\pi(V)$.

The local classes $c_{n,j}$ will play a key role in the global computations in the second half of the paper.

7.4. **Twisted pairing sums.** The formula of the following proposition will be used in the global half of the paper to relate our algebraic θ-elements to the Mazur-Tate elements.

In what follows, recall from section 5.3 that $\langle \cdot, \cdot \rangle'_{\mathrm{dR},n}$ denotes the pairing

$$\langle \cdot, \cdot \rangle'_{\mathrm{dR},n} : (\mathbf{D}_{\mathrm{dR}}(V(1+j)) \otimes K_{n,p}) \times (\mathbf{D}_{\mathrm{dR}}(V^*(-j)) \otimes K_{n,p}) \to L \otimes K_{n,p}$$

and $\langle \cdot, \cdot \rangle_{\mathrm{dR},n} = \mathrm{Tr}_L^{L_n} \langle \cdot, \cdot \rangle'_{\mathrm{dR},n}$.

Proposition 7.4.1. *Suppose that the local Euler factor of $\pi_{\mathrm{sm}}(V)$ does not have a pole at $s = 1 + j$. For $z'_n \in H^1(K_{n,p}, V^*(-j))$ and χ a Dirichlet character of conductor p^n, we have*

$$\sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi^{-1}(a) \left\langle c_{n,j}^{\sigma_a}, z'_n \right\rangle_n =$$
$$\begin{cases} j! \cdot p^{nj} \cdot \tau(\chi^{-1}) \left\langle t^{-j}\mathbf{e}, \displaystyle\sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(-a) \exp^*(z'_n)^{\sigma_a} \right\rangle'_{\mathrm{dR},n} & m \geq 1, \\ j! \cdot \widetilde{Z}(x^{-j}\phi_{v_{\mathrm{new}}}, 1) \left\langle t^{-j}\mathbf{e}, \exp^* z'_0 \right\rangle_{\mathrm{dR},0} & m = 0. \end{cases}$$

Proof. Using Proposition 7.3.4, we compute for $n \geq 1$

$$\begin{aligned} \langle \sigma_a c_{n,j}, z'_n \rangle_n &= p^n \left\langle \imath_n^-(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) w_n^{(j)}), \sigma_{-1} \exp^* z'_n \right\rangle_{\mathrm{dif}} \\ &= \left\langle \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \imath_n^-(w_n^{(j)}), \sigma_{-1} \exp^* z'_n \right\rangle_{\mathrm{dR},n} \\ &= \mathrm{Tr}_L^{L_n} \left\langle \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \imath_n^-(w_n^{(j)}), \sigma_{-1} \exp^* z'_n \right\rangle'_{\mathrm{dR},n} \\ &= \sum_b \sigma_b \left\langle \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \imath_n^-(w_n^{(j)}), \sigma_{-1} \exp^* z'_n \right\rangle'_{\mathrm{dR},n} \\ &= \sum_b \left\langle \left(\begin{smallmatrix} ab & 0 \\ 0 & 1 \end{smallmatrix}\right) \imath_n^-(w_n^{(j)}), \sigma_{-b} \exp^* z'_n \right\rangle'_{\mathrm{dR},n} \end{aligned}$$

where the sums above and below are over any systems of representatives for $(\mathbb{Z}/p^n\mathbb{Z})^\times$.

Thus

$$\begin{aligned}
&\sum_a \chi^{-1}(a)\left\langle \sigma_a c_{n,j}, z'_n\right\rangle_n \\
&\qquad = \sum_a \chi^{-1}(a) \sum_b \left\langle \left(\begin{smallmatrix} ab & 0 \\ 0 & 1\end{smallmatrix}\right) \imath_n^-(w_n^{(j)}), \sigma_{-b} \exp^* z'_n \right\rangle'_{\mathrm{dR},n} \\
&\qquad = \sum_b \left\langle \sum_a \chi^{-1}(a) \left(\begin{smallmatrix} ab & 0 \\ 0 & 1\end{smallmatrix}\right) \imath_n^-(w_n^{(j)}), \sigma_{-b} \exp^* z'_n \right\rangle'_{\mathrm{dR},n} \\
&\qquad = \sum_b \left\langle \sum_a \chi^{-1}(a) \left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right) \imath_n^-(w_n^{(j)}), \chi(b)\sigma_{-b} \exp^* z'_n \right\rangle'_{\mathrm{dR},n} \\
&\qquad = \left\langle \sum_a \chi^{-1}(a) \left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right) \imath_n^-(w_n^{(j)}), \sum_b \chi(b)\sigma_{-b} \exp^* z'_n \right\rangle'_{\mathrm{dR},n}.
\end{aligned}$$

Focusing on the first term in the above pairing and applying Propositions 6.7.1 and 7.2.2, we have

$$\begin{aligned}
\sum_a \chi^{-1}(a) \left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right) \imath_n^-(w_n^{(j)}) &= \sum_a \chi^{-1}(a) \left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right) \imath_n^-(w_{n,j}) a^j \\
&= \sum_a \chi^{-1}(a) \left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right) \widetilde{Z}\left(\mathcal{K}_{d_{n,j}}\mathbf{e}, 1\right) a^j = \widetilde{Z}\left(\mathcal{K}_\alpha \mathbf{e}, 1\right)
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= \sum_a \chi^{-1}(a) a^j \left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right) d_{n,j} \\
&= \sum_a \chi^{-1}(a) a^j \left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right) \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right) \left(\begin{smallmatrix} p^n & 0 \\ 0 & 1\end{smallmatrix}\right) v_{\mathrm{new}} \otimes e_1^{k-2-j} e_2^j \right) \\
&= \sum_a \chi^{-1}(a) \left(\begin{smallmatrix} p^n a & a \\ 0 & 1\end{smallmatrix}\right) v_{\mathrm{new}} \otimes e_1^{k-2-j} e_2^j \\
&= \sum_a \chi^{-1}(a) \left(\begin{smallmatrix} p^n & 0 \\ 0 & 1\end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & a/p^n \\ 0 & 1\end{smallmatrix}\right) \left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right) v_{\mathrm{new}} \otimes e_1^{k-2-j} e_2^j \\
&= \left(\begin{smallmatrix} p^n & 0 \\ 0 & 1\end{smallmatrix}\right) \left(\sum_a \chi^{-1}(a) \left(\begin{smallmatrix} 1 & a/p^n \\ 0 & 1\end{smallmatrix}\right) v_{\mathrm{new}} \right) \otimes e_1^{k-2-j} e_2^j.
\end{aligned}$$

By the local Birch lemma (section 6.3), we have $\sum_a \chi^{-1}(a) \left(\begin{smallmatrix} 1 & a/p^n \\ 0 & 1\end{smallmatrix}\right) v_{\mathrm{new}}$ corresponds simply to $\tau(\chi^{-1})\mathbf{1}_{\mathbb{Z}_p^\times}(x)$ in the smooth Kirillov model attached to $\pi_{\mathrm{sm}}(V)$. Thus, by (9), we have

$$\mathcal{K}_\alpha(x) = j! \cdot (tx)^{-j} \cdot \tau(\chi^{-1}) \cdot \mathbf{1}_{\mathbb{Z}_p^\times}(p^n x)$$

and

$$\widetilde{Z}(\mathcal{K}_\alpha \mathbf{e}, 1) = j! \cdot p^{nj} \cdot \tau(\chi^{-1}) \cdot t^{-j}\mathbf{e}.$$

Putting it all together gives

$$\sum_a \chi^{-1}(a)\left\langle \sigma_a c_{n,j}, z'_n\right\rangle_n = j! \cdot p^{nj} \cdot \tau(\chi^{-1}) \left\langle t^{-j}\mathbf{e}, \sum_b \chi(-b)\sigma_b \exp^* z'_n \right\rangle'_{\mathrm{dR},n}$$

as desired.

For $n = 0$, we have

$$\langle c_{0,j}, z_0' \rangle_0 = \left\langle \imath_0^-(w_0^{(j)}), \exp^* z_0' \right\rangle_{\mathrm{dif}}$$

and

$$\imath_0^-(w_0^{(j)}) = \imath_0^-(w_{0,j}) = \widetilde{Z}\left(\mathcal{K}_{d_{0,j}}, 1\right)\mathbf{e} = \widetilde{Z}\left(j!(tx)^{-j}\phi_{v_{\mathrm{new}}}, 1\right)\mathbf{e} = j!\widetilde{Z}\left(x^{-j}\phi_{v_{\mathrm{new}}}, 1\right)t^{-j}\mathbf{e}.$$

Thus

$$\langle c_{0,j}, z_0' \rangle_0 = j! \cdot \widetilde{Z}\left(x^{-j}\phi_{v_{\mathrm{new}}}, 1\right) \cdot \left\langle t^{-j}\mathbf{e}, \exp^* z_0' \right\rangle_{\mathrm{dif}}$$

as desired. □

7.5. **Three-term relation.** In this subsection, we describe some basic properties of the local cohomology classes $\{c_{n,j}\}$ including their key three-term relation.

We continue to assume that V is de Rham and let $\pi_{\mathrm{sm}}(V)$ be the associated smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ with central character χ_V. If T is the Hecke operator associated to the matrix $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$, write $Tv_{\mathrm{new}} = av_{\mathrm{new}}$ with $a \in \mathcal{O}_L$. Also, define $\delta = \chi_V(p)$ if $\pi_{\mathrm{sm}}(V)$ is unramified (i.e. if V is crystalline) and 0 otherwise. We then have the following lemma.

Lemma 7.5.1. *We have*

$$\sum_{i=0}^{p-1} \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} v_{\mathrm{new}} + \delta \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} v_{\mathrm{new}} = a \cdot v_{\mathrm{new}}.$$

Proof. We have

$$T = \begin{cases} \displaystyle\sum_{i=0}^{p-1} \left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) & \text{if } \pi_{\mathrm{sm}}(V) \text{ is unramified,} \\ \\ \displaystyle\sum_{i=0}^{p-1} \left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right) & \text{otherwise,} \end{cases}$$

which implies the lemma. □

For $m \geq n$, let $\mathrm{cor}_n^m : H^1(K_{n,p}, V(1+j)) \to H^1(K_{m,p}, V(1+j))$ and $\mathrm{res}_n^m : H^1(K_{m,p}, V(1+j)) \to H^1(K_{n,p}, V(1+j))$ denote the natural corestriction and restriction maps. We then have the following three-term relation for the local cohomology classes $\{c_n\}$.

Proposition 7.5.2. *For $n \geq 1$, we have*

$$\mathrm{cor}_n^{n+1}(c_{n+1,j}) = ac_n - \delta\,\mathrm{res}_{n-1}^n(c_{n-1,j}),$$

and

$$\mathrm{cor}_0^1(c_1) = (a - \delta - 1)c_0.$$

Proof. For $n \geq 1$, we compute

$$\begin{aligned}
\mathrm{cor}_n^{n+1}(d_{n+1}^{(j)}) &= \sum_{i=0}^{p-1} \left(\begin{smallmatrix} 1+ip^n & 0 \\ 0 & 1 \end{smallmatrix}\right) d_{n+1}^{(j)} = \sum_{i=0}^{p-1} \left(\begin{smallmatrix} 1+ip^n & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^{n+1} & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \otimes e_1^{k-2-j} e_2^j \\
&= \sum_{i=0}^{p-1} \left(\begin{smallmatrix} p^{n+1}(1+p^n i) & 1+ip^n \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \otimes e_1^{k-2-j} e_2^j \\
&= \sum_{i=0}^{p-1} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1+ip^n & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \otimes e_1^{k-2-j} e_2^j \\
&= \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\sum_{i=0}^{p-1} \left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \right) \otimes e_1^{k-2-j} e_2^j \\
&= \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(a \cdot v_{\mathrm{new}} - \delta \left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \right) \otimes e_1^{k-2-j} e_2^j \\
&= a \cdot \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \otimes e_1^{k-2-j} e_2^j - \delta \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^{n-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \otimes e_1^{k-2-j} e_2^j \\
&= a \cdot d_n^{(j)} - \delta \cdot d_{n-1}^{(j)}.
\end{aligned}$$

Since $\langle c_{n,j}, z'_n \rangle_n = \{d_n^{(j)}, z'\}$, the above relations for the $d_n^{(j)}$ imply the corresponding relations for the $c_{n,j}$ for $n \geq 1$.

For $n = 0$, we have

$$\begin{aligned}
\mathrm{cor}(d_1^{(j)}) &= \sum_{i=1}^{p-1} \left(\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix}\right) d_1^{(j)} = \sum_{i=1}^{p-1} \left(\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \otimes e_1^{k-2-j} e_2^j \\
&= \sum_{i=1}^{p-1} \left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \otimes e_1^{k-2-j} e_2^j \\
&= \left(a \cdot v_{\mathrm{new}} - \delta \left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} - \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathrm{new}} \right) \otimes e_1^{k-2-j} e_2^j \\
&= \left(a - \delta \left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \right) d_0^{(j)}.
\end{aligned}$$

The relation between $c_{1,j}$ and $c_{0,j}$ then follows as above combined with the fact that $\{z, z'\} = \left\{ \left(\begin{smallmatrix} p^{\pm 1} & 0 \\ 0 & 1 \end{smallmatrix}\right) z, z' \right\}$ for any $z \in \pi(V)$ and $z' \in D(V^*)^{\psi=1}$. □

7.6. **The crystalline case.** Suppose now that $V(1)$ is crystalline of dimension two, and that $D_{\mathrm{crys}}\big(V(1)\big)$ has distinct Frobenius eigenvalues α/p and β/p arising from eigenvectors

$$e_{\alpha/p}, e_{\beta/p} \in D_{\mathrm{crys}}\big(V(1)\big) \subseteq \widetilde{\mathbf{D}}^+_{\mathrm{rig}}\big(V(1)\big)[1/t].$$

Suppose that neither α nor β equals p so that the local Euler factor of $\pi_{\mathrm{sm}}(V)$ does not have a pole at $s = 1$. Further suppose that V is irreducible, so that $e_{\alpha/p}$ and $e_{\beta/p}$ both have non-zero image in $\mathbf{D}_{\mathrm{dR}}\big(V(1)\big)/\mathbf{D}^+_{\mathrm{dR}}\big(V(1)\big)$, and scale them so that their images coincide. We take this common image to be our basis $\mathbf{e}$ in the preceding discussion, and set $\omega = e_{\alpha/p} - e_{\beta/p} \in \mathbf{D}^+_{\mathrm{dR}}\big(V(1)\big)$.

In this case, we can give another (more direct) description of the classes $c_n := c_{n,0}$ as follows. (So we are fixing $j = 0$ for this subsection.). First, for $n \geq 0$, set

$$k_n = \varepsilon^{(n)} \otimes \varphi\omega + \cdots + \varepsilon^{(1)} \otimes \varphi^n \omega + (1 - \varphi)^{-1} \varphi^{n+1} \omega \in K_{n,p} \otimes_{\mathbb{Q}_p} D_{\mathrm{crys}}\big(V(1)\big)$$

where for $n = 0$ this formula simply means the final term. Note that $1 - \varphi$ is invertible as we have assumed that φ does not have 1 as an eigenvalue.

Recall the exponential map

$$\exp : K_{n,p} \otimes D_{\mathrm{crys}}(V(1)) \to H^1_f(K_{n,p}, V(1))$$

and consider the elements $\exp(k_n) \in H^1(K_{n,p}, V(1))$ for $n \geq 0$. A simple computation shows that these elements satisfy the three term relation given in Proposition 7.5.2. In fact, we will show the following:

Proposition 7.6.1. *For $n \geq 0$, we have $c_n = \left(\frac{\alpha}{p} - \frac{\beta}{p}\right)^{-1} \exp(k_n)$.*

Proof. Recall that c_n is defined by the formula

$$\langle c_n, z'_n \rangle_n = p^n \langle \imath_n^-(w_n), \sigma_{-1} \exp^* z'_n \rangle_{\mathrm{dif}}$$

where $w_n := w_{n,0}$. Further, by [10, Théorème VIII.2.2], we have

$$\langle \exp(k_n), \sigma_{-1} \cdot z'_n \rangle_n = p^n \langle k_n, \sigma_{-1} \exp^* z'_n \rangle_{\mathrm{dif}}.$$

Thus it suffices to see that the images of $\left(\frac{\alpha}{p} - \frac{\beta}{p}\right)^{-1} k_n$ and $\imath_n^-(w_n)$ agree in $D_{\mathrm{dR}}(V(1))/D^+_{\mathrm{dR}}(V(1))$.

To this end, we first explicitly describe the realization of $d_n := d_{n,0}$ and w_n in (generalized) Kirillov models. If ϕ denotes the realization of v_{new} in the Kirillov model of $\pi_{\mathrm{sm}}(V)$, then

$$\phi(p^n) = \begin{cases} \dfrac{(\alpha/p)^{n+1} - (\beta/p)^{n+1}}{(\alpha/p) - (\beta/p)} & n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

as in section 6.2. Thus,

$$\mathcal{K}_{d_n}(p^r)\mathbf{e} = \begin{cases} \phi(p^{n+r})\mathbf{e} & r \geq 0, \\ \varepsilon^{(-r)}\phi(p^{n+r})\mathbf{e} & -n \leq r \leq 0, \\ 0 & r < -n, \end{cases}$$

where $\varepsilon^{(m)} = \varepsilon(p^{-m})$ is a p^m-th root of unity.

Tracing through the proof of Proposition 6.6.1, we then see that

$$\xi_n(p^{-n}) =$$
$$\left(\frac{\alpha}{p} - \frac{\beta}{p}\right)^{-1} \left(\sum_{j=0}^{n-1} \varepsilon^{(n-j)} \left(\left(\frac{\alpha}{p}\right)^{j+1} - \left(\frac{\beta}{p}\right)^{j+1} \right) + \frac{(\alpha/p)^{n+1}}{1 - \alpha/p} - \frac{(\beta/p)^{n+1}}{1 - \beta/p} \right) \mathbf{e}.$$

Since $\omega = e_{\alpha/p} - e_{\beta/p}$, we have

$$\begin{aligned} k_n &= \sum_{j=0}^{n-1} \varepsilon^{(n-j)} \otimes \varphi^{j+1}\omega + (1 - \varphi)^{-1}\varphi^{n+1}\omega \\ &= \sum_{j=0}^{n-1} \varepsilon^{(n-j)} \otimes \left(\left(\frac{\alpha}{p}\right)^{j+1} e_{\alpha/p} - \left(\frac{\beta}{p}\right)^{j+1} e_{\beta/p} \right) + \frac{(\alpha/p)^{n+1}}{1 - \alpha/p} e_{\alpha/p} - \frac{(\beta/p)^{n+1}}{1 - \beta/p} e_{\beta/p} \end{aligned}$$

and thus

$$k_n \equiv \left(\frac{\alpha}{p} - \frac{\beta}{p}\right) \imath_n^-(w_n) \pmod{D^+_{\mathrm{dR}}(V(1))}$$

as desired. □

Remark 7.6.2. One can even go further and directly check that the classes $\exp(k_n)$ satisfy Proposition 7.4.1. Nonetheless, we note that the explicit construction in this section for the crystalline case do not supplant the main arguments given so far in this paper. Indeed, these explicit constructions with the exponential map do not give us any integral control of these cohomology classes. In the next section, we describe how the c_n can be normalized so that they are in $H^1(K_{n,p}, T(1))$ for all $n \geq 0$ for T some Galois-stable lattice in V. We cannot directly check this property for the classes $\exp(k_n)$.

7.7. **Normalizations.** If T is a $G_{\mathbb{Q}_p}$-invariant $\mathcal{O}_L$-lattice in the two-dimensional irreducible continuous $G_{\mathbb{Q}_p}$-representation V over L, then we can form $\widetilde{\mathbf{D}}^+\big(T(1)\big) \subset \widetilde{\mathbf{D}}^+\big(V(1)\big)$ and $\widetilde{\mathbf{D}}\big(T(1)\big) \subset \widetilde{\mathbf{D}}\big(V(1)\big)$. Note that $\widetilde{\mathbf{D}}^+\big(T(1)\big) = \widetilde{\mathbf{D}}\big(T(1)\big) \cap \widetilde{\mathbf{D}}^+\big(V(1)\big)$ (the intersection taking place in $\widetilde{\mathbf{D}}\big(V(1)\big)$), so that we also obtain an embedding

$$\pi(T) := \widetilde{\mathbf{D}}\big(T(1)\big)/\widetilde{\mathbf{D}}^+\big(T(1)\big) \hookrightarrow \widetilde{\mathbf{D}}\big(V(1)\big)/\widetilde{\mathbf{D}}^+\big(V(1)\big) = \pi(V),$$

which gives an integral structure on the target.

Suppose now, as before, that V is de Rham, with Hodge–Tate weights 0 and $1-k < 0$. Then we have the inclusion $\pi_{\mathrm{sm}}(V) \otimes_L (\mathrm{Sym}^{k-2} L^2)^* \hookrightarrow \pi(V)$, and we may normalize the vector $v_{\mathrm{new}} \in \pi_{\mathrm{sm}}(V)$ (up to an element of $\mathcal{O}_L^\times$) by asking that $v_{\mathrm{new}} \otimes_{\mathcal{O}_L} (\mathrm{Sym}^{k-2}\mathcal{O}_L^2)^*$ be contained in $\pi(T)$, but not in $\varpi\pi(T)$. (Here ϖ is a uniformizer of $\mathcal{O}_L$.) Further, if we apply $\imath_0^-$ to $v_{\mathrm{new}} \otimes v_{\mathrm{hw}}$ (where v_{hw} is a basis element for the highest weight space of $(\mathrm{Sym}^{k-2}\mathcal{O}_L^2)^*$), we obtain an element of $\mathbf{D}_{\mathrm{dR}}(V)/\mathbf{D}^+_{\mathrm{dR}}(V)$, well-determined up to multiplication by an element of $\mathcal{O}_L^\times$; in other words, we obtain an $\mathcal{O}_L$-structure on the one-dimensional L-vector space $\mathbf{D}_{\mathrm{dR}}(V)/\mathbf{D}^+_{\mathrm{dR}}(V)$. Setting $\mathbf{e}$ equal $\imath_0^-(v_{\mathrm{new}} \otimes v_{\mathrm{hw}})$ then pins down the elements $d_{n,j}$ (and thus the $c_{n,j}$) up to scaling by a single element in $\mathcal{O}_L^\times$. Further, we note that under this normalization for all $n \geq 0$, we have that $c_{n,j}$ is in $H^1(K_{n,p}, T(1+j))$ (as opposed to just being in $H^1(K_{n,p}, V(1+j))$).

Part 2. The global theory

8. Algebraic θ-elements

8.1. **Notations and assumptions.** Let f be a newform in $S_k(\Gamma_1(N), \psi, \overline{\mathbb{Q}}_p)$ for $k \geq 2$, and let $L/\mathbb{Q}_p$ denote the field generated by the Fourier coefficients of f. Let $\rho_f : G_{\mathbb{Q}} \to \mathrm{Aut}(V_f)$ denote the associated (cohomological) Galois representation; that is, the determinant of ρ_f is $\psi^{-1}\varepsilon^{1-k}$ where ε is the cyclotomic character. Let Σ denote the finite set of primes where ρ_f ramifies.

As f will be fixed for the remainder of the paper, we simply write $V := V_f$ and $\overline{V} := V_{\overline{f}}$ where $\overline{f}$ is the complex conjugate of f. Fix T a Galois stable lattice in V. Set $A = V/T$ and $A^* = V^*/T^*$. We similarly write $\overline{T}$, $\overline{A}$, and $\overline{A}^*$ for the analogous constructions using $\overline{V}$. Note that $A^* \cong \overline{A}(k-1)$. In what follows, to ease notation, we will simply write A^*_r for $A^*(1-r)$ for any $r \in \mathbb{Z}$ whose Tate dual is simply $T(r)$.

Further, set $K_n = \mathbb{Q}(\mu_{p^n})$, $K_\infty = \mathbb{Q}(\mu_{p^\infty})$, $\Gamma_n = \mathrm{Gal}(K_\infty/K_n)$, $\mathcal{G}_n = \mathrm{Gal}(K_n/\mathbb{Q})$, $\Lambda = \mathcal{O}_L[[\mathrm{Gal}(K_\infty/\mathbb{Q})]]$, and $\Lambda_n = \Lambda_{\Gamma_n} \cong \mathcal{O}_L[\mathrm{Gal}(K_n/\mathbb{Q})]$. We fix an isomorphism of $(\mathbb{Z}/p^n\mathbb{Z})^\times$ with $\mathcal{G}_n$ by mapping a to σ_a where $\sigma_a(\zeta) = \zeta^a$ for $\zeta \in \mu_{p^n}$.

Let $\pi_n^{n+1} : \Lambda_{n+1} \to \Lambda_n$ be the natural projection, and let $\mathrm{cor}_n^{n+1} : \Lambda_n \to \Lambda_{n+1}$ be the natural injection which sends a group like element $\sigma \in \mathcal{G}_n$ to $\sum \tau$ where τ runs over all elements in $\mathcal{G}_{n+1}$ whose image in $\mathcal{G}_n$ equals σ.

8.2. **Selmer groups.** In this section, we give a construction of algebraic θ-elements attached to f along the $\mathbb{Z}_p^\times$-extension $K_\infty/\mathbb{Q}$. We begin by recalling the definition of the relevant Selmer groups. We have

$$H^1_g(K_n, A_r^*) := \ker\left(H^1(K_n, A_r^*) \to \prod_{v \nmid p} H^1(I_v, A_r^*) \times \frac{H^1(K_{n,p}, A_r^*)}{H^1_g(K_{n,p}, A_r^*)} \right)$$

where v runs over places of K_n and I_v denotes an inertia group at v, and

$$H^1_g(K_\infty, A_r^*) := \ker\left(H^1(K_\infty, A_r^*) \to \prod_{w \nmid p} H^1(I_w, A_r^*) \right)$$

where w runs over places of K_∞. We note that no condition at p is imposed in this second Selmer group since $H^1_g(K_{\infty,p}, A_r^*) = H^1(K_{\infty,p}, A_r^*)$ by [4, Theorem A].

The key global input in what follows is the sequence in the following proposition which precisely describes the failure of the control theorem in this non-ordinary situation.

Proposition 8.2.1. *Assuming* (Irred) *there is a natural exact sequence*

$$0 \to H^1_g(K_n, A_r^*) \to H^1_g(K_\infty, A_r^*)^{\Gamma_n} \to \frac{H^1(K_{n,p}, A_r^*)}{H^1_g(K_{n,p}, A_r^*)} \oplus B_n \tag{14}$$

where B_n is a finite group with size bounded independent of n.

Proof. This sequence is derived via the Snake Lemma as in [14, Chapter 3] or as in [16, Theorem 3.1]. We note that the local ingredients needed to prove this are:

(1) $H^1(K_{\infty,w}/K_{n,v}, A_r^*(K_{\infty,w}))$ is finite with size bounded independent of n for $v \nmid p$.
(2) $H^1_g(K_{\infty,p}, A_r^*) = H^1(K_{\infty,p}, A_r^*)$.

The first of these facts can be proven as in [14, Lemma 3.3]. The second fact follows from [4, Theorem A] by Tate local duality and (Irred). □

Dualizing (14) and applying Tate local duality yields

$$H^1_e(K_{n,p}, T(r)) \oplus B_n^\vee \longrightarrow (H^1_g(K_\infty, A_r^*)^\vee)_{\Gamma_n} \longrightarrow H^1_g(K_n, A_r^*)^\vee \longrightarrow 0. \tag{15}$$

Let $\mathbb{Q}_\Sigma$ denote the maximal extension of $\mathbb{Q}$ unramified outside of Σ, and let

$$\mathbb{H}^i(T_r^*) := \varprojlim_n H^i(\mathbb{Q}_\Sigma/K_n, T_r^*).$$

The following is a deep theorem of Kato.

Theorem 8.2.2 (Kato)**.** *Assuming* (Irred) *we have*

(1) *$\mathbb{H}^1(T_r^*)$ is a free Λ-module of rank one.*
(2) *$H^1_g(K_\infty, A_r^*)^\vee$ is a rank one Λ-module.*

Proof. The first part is [20, Theorem 12.4.3]. For the second part, we have that the Λ-corank of $H^1(\mathbb{Q}_\Sigma/K_\infty, A_r^*)$ equals the Λ-rank of $\mathbb{H}^1(T_r^*)$ by [31, Proposition 1.3.2]. Further, consider the exact sequence

$$0 \to H^1_g(K_\infty, A_r^*) \to H^1(\mathbb{Q}_\Sigma/K_\infty, A_r^*) \to \bigoplus_{w \in \Sigma - \{p\}} H^1(K_{\infty,w}, A_r^*). \tag{16}$$

Since $H^1(K_{\infty,w}, A_r^*)$ is Λ-cotorsion if $w \nmid p$ (see [13, Proposition 2]), we have

$$\operatorname{corank}_\Lambda H^1_g(K_\infty, A_r^*) = \operatorname{corank}_\Lambda H^1(\mathbb{Q}_\Sigma/K_\infty, A_r^*) = 1$$

as desired. $\square$

To ease notation, let $X_r = H^1_g(K_\infty, A_r^*)^\vee$. Set $(X_r)_{\Lambda\text{-tor}}$ equal to the Λ-torsion submodule of X_r, and let $Z_r = X_r/(X_r)_{\Lambda\text{-tor}}$, a torsion-free Λ-module. Let $R_\Lambda(Z_r) := \operatorname{Hom}_\Lambda\big(\operatorname{Hom}_\Lambda(Z_r, \Lambda), \Lambda\big)$ denote the reflexive hull of Z_r. Choosing an isomorphism of $\alpha : R_\Lambda(Z_r) \cong \Lambda$ (via Theorem 8.2.2) yields maps for each $n \geq 0$

$$(X_r)_{\Gamma_n} \longrightarrow (Z_r)_{\Gamma_n} \longrightarrow \Lambda_n. \tag{17}$$

Combining (15) and (17), we get a map

$$\psi_n = \psi_{n,\alpha} : H^1_e(K_{n,p}, T(r)) \to H^1_e(K_{n,p}, T(r)) \oplus B_n^\vee \longrightarrow (X_r)_{\Gamma_n} \to (Z_r)_{\Gamma_n} \to \Lambda_n.$$

Lemma 8.2.3. *The diagram*

$$\begin{array}{ccc} H^1_e(K_{n,p}, T(r)) & \xrightarrow{\psi_n} & \Lambda_n \\ {\scriptstyle \mathrm{res}^n_m}\uparrow & & \uparrow{\scriptstyle \mathrm{cor}^n_m} \\ H^1_e(K_{m,p}, T(r)) & \xrightarrow{\psi_m} & \Lambda_m \end{array}$$

commutes for $n \leq n$.

Proof. This commutativity can easily be checked. See [16, Proposition 6.3] for a similar computation. $\square$

Recall our local cohomology classes $c_{n,r-1} \in H^1_e(K_{n,p}, T(r))$ defined in section 7.3. For $1 \leq r \leq k-1$, we set

$$\psi^{\mathrm{alg}}_{n,r}(f) = \psi_n(c_{n,r-1})$$

and

$$\theta^{\mathrm{alg}}_{n,r}(f) = \psi^{\mathrm{alg}}_{n,r}(f) \cdot \operatorname{char}_\Lambda((X_r)_{\Lambda\text{-tor}})$$

which is our n-th layer algebraic θ-element.

Further, let $I^{\mathrm{alg}}_{n,r}(f)$ (resp. $J^{\mathrm{alg}}_{n,r}(f)$) denote the ideal of Λ_n generated by the elements $\mathrm{cor}^n_m(\theta^{\mathrm{alg}}_{m,r}(f))$ (resp. $\mathrm{cor}^n_m(\psi^{\mathrm{alg}}_{m,r}(f))$) for $0 \leq m \leq n$. Note that $I^{\mathrm{alg}}_{n,r}(f) = J^{\mathrm{alg}}_{n,r}(f) \cdot \operatorname{char}_\Lambda(X_r)_{\Lambda\text{-tor}}$. Also, changing our fixed isomorphism α has the effect of scaling all of the $\psi^{\mathrm{alg}}_{n,r}(f)$ by a unit in Λ. In particular, the ideals $I^{\mathrm{alg}}_{n,r}(f)$ and $J^{\mathrm{alg}}_{n,r}(f)$ are independent of this choice.

Remark 8.2.4. This construction of algebraic θ-elements dates back to Perrin-Riou in her 1990 paper [30].

9. Algebraic results on Fitting ideals

9.1. **Main algebraic results.** The remainder of this section will be occupied with proving the following purely algebraic result.

Theorem 9.1.1. *If* (Irred) *holds, we have*

$$\psi_n(c) \cdot \mathrm{char}_\Lambda(X_r)_{\Lambda\text{-tor}} \in \mathrm{Fitt}_{\Lambda_n} H^1_g(K_n, A_r^*)^\vee$$

for any c in $H^1_e(K_{n,p}, T(r))$.

We note that as an immediate corollary we get.

Corollary 9.1.2. *If* (Irred) *holds,*

$$I^{\mathrm{alg}}_{n,r}(f) \subseteq \mathrm{Fitt}_{\Lambda_n} H^1_g(K_n, A_r^*)^\vee.$$

Proof. This corollary follows immediately from Theorem 9.1.1 by taking $c = \mathrm{cor}^n_m(c_{m,r})$ for $0 \leq m \leq n$. □

9.2. **Fitting ideal lemmas.**

Lemma 9.2.1. *If A_n, B_n, C_n are Λ_n-modules, and Y is a Λ-module then*

1. *If $A_n \to B_n$ is surjective, then $\mathrm{Fitt}_{\Lambda_n}(A_n) \subseteq \mathrm{Fitt}_{\Lambda_n}(B_n)$.*
2. *If $0 \to A_n \to B_n \to C_n \to 0$ is exact, then*
$$\mathrm{Fitt}_{\Lambda_n}(B_n) \supseteq \mathrm{Fitt}_{\Lambda_n}(A_n) \cdot \mathrm{Fitt}_{\Lambda_n}(C_n).$$
3. $\mathrm{Fitt}_{\Lambda_n}(Y_{\Gamma_n}) = \pi_n(\mathrm{Fitt}_\Lambda(Y))$ *where $\pi_n : \Lambda \to \Lambda_n$ is the natural map.*

Proof. See [27, Appendix: 1,9,4] □

Lemma 9.2.2. *If Y is a finitely generated torsion Λ-module with no non-zero finite submodules, then $\mathrm{Fitt}_\Lambda(Y) = \mathrm{char}_\Lambda(Y)$. In particular, $\mathrm{Fitt}_\Lambda(Y)$ is a principal ideal.*

Proof. See [35, Proposition 1.3.4]. □

We thank Cornelius Greither for the following argument.

Lemma 9.2.3. *Let $I \subseteq A$ be ideals of Λ with A finite-index in Λ. Then*

$$I \subseteq \mathrm{Fitt}_\Lambda(A/I).$$

Proof. We first note that it suffices to assume that I is a principal ideal. Indeed, assume that we have proven this lemma for all principal ideals, and let f be any element of I. Then we have

$$f\Lambda \subseteq \mathrm{Fitt}_\Lambda(A/f\Lambda) \subseteq \mathrm{Fitt}_\Lambda(A/I)$$

where the second inclusion follows from Lemma 9.2.1. Since this is true for all $f \in I$, we have $I \subseteq \mathrm{Fitt}_\Lambda(A/I)$.

Thus, we will now assume that $I = f\Lambda$. We then have an exact sequence

$$0 \to A/f\Lambda \to \Lambda/f\Lambda \to \Lambda/A \to 0.$$

Since A has finite index in Λ, for any height one prime $\mathfrak{p}$ in Λ, we have

$$(A/f\Lambda)_\mathfrak{p} \cong (\Lambda/f\Lambda)_\mathfrak{p}.$$

Thus,

$$\left(\mathrm{Fitt}_{\Lambda_\mathfrak{p}}(A/f\Lambda)\right)_\mathfrak{p} = \mathrm{Fitt}_{\Lambda_\mathfrak{p}}\left((A/f\Lambda)_\mathfrak{p}\right) = \mathrm{Fitt}_{\Lambda_\mathfrak{p}}\left((\Lambda/f\Lambda)_\mathfrak{p}\right) = (f\Lambda)_\mathfrak{p}.$$

By Lemma 9.2.2, we have $\mathrm{Fitt}_\Lambda(A/f\Lambda) = g\Lambda$ for some g in Λ. Thus,

$$(g\Lambda)_{\mathfrak{p}} = (f\Lambda)_{\mathfrak{p}}$$

for all height one primes $\mathfrak{p}$. Hence $f\Lambda = g\Lambda$ and $f\Lambda = \mathrm{Fitt}_\Lambda(A/f\Lambda)$ as desired. $\square$

Lemma 9.2.4. *Let A be some finite-index ideal of Λ, and let I_n be a Λ_n-submodule of A_{Γ_n}. Then*

$$i_n(I_n) \subseteq \mathrm{Fitt}_{\Lambda_n} A_{\Gamma_n}/I_n$$

where i_n is the natural map $A_{\Gamma_n} \to \Lambda_n$.

Proof. Let π_n denote the natural map from $\Lambda \to \Lambda_n$, and let p_n denote the natural map from A to A_{Γ_n} so that $i_n \circ p_n = \pi_n|_A$. The map p_n induces an isomorphism

$$\frac{A}{p_n^{-1}(I_n)} \cong \frac{A_{\Gamma_n}}{I_n},$$

and thus

$$\begin{aligned} \mathrm{Fitt}_{\Lambda_n}\left(\frac{A_{\Gamma_n}}{I_n}\right) &= \pi_n\left(\mathrm{Fitt}_\Lambda\left(\frac{A}{p_n^{-1}(I_n)}\right)\right) && \text{by Lemma 9.2.1.3} \\ &\supseteq \pi_n\left(p_n^{-1}(I_n)\right) && \text{by Lemma 9.2.3} \\ &= i_n(I_n) \end{aligned}$$

as desired. $\square$

We can now prove the main algebraic theorem of this section.

Proof of Theorem 9.1.1. Consider the sequence

$$0 \to (X_r)_{\Lambda\text{-tor}} \to X_r \to Z_r \to 0.$$

Taking Γ_n-coinvariants yields

$$0 \to ((X_r)_{\Lambda\text{-tor}})_{\Gamma_n} \to (X_r)_{\Gamma_n} \to (Z_r)_{\Gamma_n} \to 0$$

since the kernel of the first map equals $Z_r^{\Gamma_n} = 0$. Let E_n denote the image of $H^1_e(K_{n,p}, T(r)) \oplus \{0\}$ in $(X_r)_{\Gamma_n}$ under (15), and let F_n denote its image in $(Z_r)_{\Gamma_n}$. If i_n denotes the natural map $(Z_r)_{\Gamma_n} \to \Lambda_n$, we recall that by definition $i_n(F_n)$ equals the image of ψ_n.

We have

$$0 \to \frac{((X_r)_{\Lambda\text{-tor}})_{\Gamma_n}}{((X_r)_{\Lambda\text{-tor}})_{\Gamma_n} \cap E_n} \to \frac{(X_r)_{\Gamma_n}}{E_n} \to \frac{(Z_r)_{\Gamma_n}}{F_n} \to 0$$

Note that (15) equates a quotient of the middle term of this sequence with $H^1_g(K_n, A^*_r)^\vee$. Thus,

$$\begin{aligned}
&\mathrm{Fitt}_{\Lambda_n}(H^1_g(K_n, A^*_r)^\vee) \\
&\qquad \supseteq \mathrm{Fitt}_{\Lambda_n} \frac{(X_r)_{\Gamma_n}}{E_n} && \text{by Lemma 9.2.1.1} \\
&\qquad \supseteq \mathrm{Fitt}_{\Lambda_n}\left(\frac{((X_r)_{\Lambda\text{-tor}})_{\Gamma_n}}{((X_r)_{\Lambda\text{-tor}})_{\Gamma_n} \cap E_n}\right) \cdot \mathrm{Fitt}_{\Lambda_n}\left(\frac{(Z_r)_{\Gamma_n}}{F_n}\right) && \text{by Lemma 9.2.1.2} \\
&\qquad \supseteq \mathrm{Fitt}_{\Lambda_n}\left(((X_r)_{\Lambda\text{-tor}})_{\Gamma_n}\right) \cdot \mathrm{Fitt}_{\Lambda_n}\left(\frac{(Z_r)_{\Gamma_n}}{F_n}\right) && \text{by Lemma 9.2.1.1} \\
&\qquad = \pi_n(\mathrm{Fitt}_{\Lambda}\left((X_r)_{\Lambda\text{-tor}}\right)) \cdot \mathrm{Fitt}_{\Lambda_n}\left(\frac{(Z_r)_{\Gamma_n}}{F_n}\right) && \text{by Lemma 9.2.1.3} \\
&\qquad \supseteq \pi_n(\mathrm{char}_{\Lambda}\left((X_r)_{\Lambda\text{-tor}}\right)) \cdot \mathrm{Fitt}_{\Lambda_n}\left(\frac{(Z_r)_{\Gamma_n}}{F_n}\right) && \text{by Lemma 9.2.2} \\
&\qquad \supseteq \pi_n(\mathrm{char}_{\Lambda}\left((X_r)_{\Lambda\text{-tor}}\right)) \cdot i_n(F_n) && \text{by Lemma 9.2.4} \\
&\qquad \supseteq \pi_n(\mathrm{char}_{\Lambda}\left((X_r)_{\Lambda\text{-tor}}\right)) \cdot \psi_n(H^1_e(K_{n,p}, T(r)))
\end{aligned}$$

as desired.

We note that in applying Lemma 9.2.2, the fact that $(X_r)_{\Lambda\text{-tor}}$ has no non-zero finite submodules is a theorem of Greenberg (see [15, Proposition 4.1.1]). □

10. Mazur-Tate elements and Kato's Euler system

10.1. **Mazur-Tate elements and statement of main results.** Following [26], we define

$$\lambda(f, z^j, a, m) := 2\pi i \int_{\infty}^{-a/m} f(z)(mz + a)^j \, dz.$$

Setting $\lambda^{\pm}(f, z^j, a, m) := \lambda(f, z^j, a, m) \pm \lambda(f, z^j, a, m)$, we further define

$$\varphi(f, z^j, a, m) := \frac{\lambda^+(f, z^j, a, m)}{\Omega_f^+} + \frac{\lambda^-(f, z^j, a, m)}{\Omega_f^-}.$$

With this notation in hand, we define our Mazur-Tate elements of f over K_n:

$$\theta^{\mathrm{an}}_{n,r}(f) := \sum_{\sigma_a \in \mathcal{G}_n} \varphi(f, z^{r-1}, a, p^n)\sigma_a^{-1}.$$

For χ a primitive character of $\mathcal{G}_n$ we have

$$\chi(\theta^{\mathrm{an}}_{n,r}(f)) = \chi(-1) \cdot (r-1)! \cdot p^{n(r-1)} \cdot \tau(\chi^{-1}) \cdot \frac{L(f, \chi, r)}{(-2\pi i)^{r-1}\Omega_f^{\pm}}. \tag{18}$$

where the sign $\pm$ equals the sign of $(-1)^{r-1}\chi(-1)$.

Remark 10.1.1. Our definition here is a bit non-standard as Mazur and Tate's original definition would replace σ_a^{-1} above with σ_a. That is, if ι is the involution on Λ_n sending σ to σ^{-1}, then in our notation above $\theta^{\mathrm{an}}_{n,r}(f)^\iota$ is the more standard definition of Mazur-Tate elements. Nonetheless, the methods of this paper naturally show that our $\theta^{\mathrm{an}}_{n,r}(f)$ belong to a Fitting ideal of a Selmer group and so we chose to state our results in this more natural form.

The Mazur-Tate elements satisfy the relations:

$$\pi_n^{n+1}(\theta_{n+1,r}^{\mathrm{an}}(f)) = \begin{cases} a_p(f)\cdot\theta_{n,r}^{\mathrm{an}}(f) - \psi(p)p^{k-2}\cdot\mathrm{cor}_{n-1}^{n}(\theta_{n-1,r}^{\mathrm{an}}(f)) & n\geq 1, \\ \\ (a_p - 1 - \psi(p)p^{k-2})\cdot\theta_{0,r}^{\mathrm{an}}(f) & n=0, \end{cases} \tag{19}$$

where π_n^{n+1} is the natural projection from Λ_{n+1} to Λ_n. These formulas follow from [26, (4.2)]. See also [34, Proposition 2.5] for the case $r=1$.

We have the following conjecture relating these analytically defined Mazur-Tate elements to Selmer groups.

Conjecture 10.1.2. *(Mazur-Tate [25]) For each $n\geq 0$, we have*

$$\theta_{n,r}^{\mathrm{an}}(f)\in \mathrm{Fitt}_{\Lambda_n}(H_g^1(K_n,A_r^*)^\vee).$$

We note that Mazur and Tate only formulated this conjecture for elliptic curves, but in that formulation, the conjecture was much more general, covering Selmer groups over all abelian extensions of $\mathbb{Q}$.

Remark 10.1.3. The functional equation for L-values gives an equality

$$\theta_{n,r}^{\mathrm{an}}(f) = C\cdot\theta_{n,k-r}^{\mathrm{an}}(\overline{f})^\iota$$

where C is some constant. Thus, it may be equally reasonable to conjecture that $\theta_{n,k-r}^{\mathrm{an}}(\overline{f})^\iota$ belongs to the above Fitting ideal. Moreover, since $A_r^* = A^*(1-r)\cong \overline{A}(k-r)$, that second conjecture equivalently reads

$$\theta_{n,r}^{\mathrm{an}}(f)^\iota\in \mathrm{Fitt}_{\Lambda_n}(H_g^1(K_n,A(r))^\vee)$$

which might be a more familiar form of this conjecture for some (noting the presence of ι is caused by our non-standard definition of the Mazur-Tate element as in Remark 10.1.1).

However, as the above constant C need not be integral, neither conjecture is stronger nor weaker than the other. We do note that for n large enough, the constant C is a p-adic unit and thus the two conjectures (for n large) are equivalent. Also, showing that these Fitting ideals were stable under the ι-involution (which seems very plausible), would give a neat explanation of the equivalence of the two conjectures.

We set $I_n^{\mathrm{an}}(f)$ to be the ideal of Λ_n generated by the elements $\mathrm{cor}_m^n(\theta_{m,r}^{\mathrm{an}}(f))$ for $0\leq m\leq n$. The remainder of the paper will be devoted to proving the following divisibility.

Theorem 10.1.4. *If* (Irred)*,* (No pole at $s=r$)*, and* (Euler) *hold, then there exists a non-zero constant $C\in\mathcal{O}_L$ (independent of n and r) such that*

$$C\cdot I_{n,r}^{\mathrm{an}}(f)\subseteq I_{n,r}^{\mathrm{alg}}(f)$$

for all $n\geq 0$.

From Corollary 9.1.2 and Theorem 10.1.4, we get our main result.

Corollary 10.1.5. *If* (Irred)*,* (No pole at $s=r$)*, and* (Euler) *hold, then there exists a non-zero constant $C\in\mathcal{O}_L$ (independent of n and r) such that*

$$C\cdot I_{n,r}^{\mathrm{an}}(f)\subseteq \mathrm{Fitt}_{\Lambda_n}(H_g^1(K_n,A_r^*)^\vee).$$

for all $n\geq 0$. In particular, $C\cdot\theta_{n,r}^{\mathrm{an}}(f)\in\mathrm{Fitt}_{\Lambda_n}(H_g^1(K_n,A_r^*)^\vee)$ for all $n\geq 0$.

10.2. **Kato's Euler system.** In this section, we describe how to recover the Mazur-Tate elements from Kato's Euler system and the local classes $c_{n,j}$ constructed in section 7.3.

Let $z_{\mathrm{K}} = (z_{\mathrm{K},n})_n \in \mathbb{H}^1(T_r^*) = \mathbb{H}^1(\overline{T}(k-r))$ denote Kato's Euler system where we recall that $\overline{T}$ is a lattice in the Galois representation attached to $\overline{f}$, the complex conjugate of f. We now state Kato's theorem relating z_{K} to L-values. To this end, let $\mathbf{e}$ continue to denote our fixed basis from section 7.1 of $\mathbf{D}_{\mathrm{dR}}(V(1))/\mathbf{D}^+_{\mathrm{dR}}(V(1))$, and let ω_f denote the element of $\mathbf{D}^+_{\mathrm{dR}}(V^*)$ canonically identified with $\overline{f}$, via Eichler–Shimura theory and the comparison theorems.

Theorem 10.2.1 (Kato)**.** *Let χ denote a primitive character of $\mathcal{G}_n$. Then*

$$\sum_{\sigma \in \mathcal{G}_n} \chi(\sigma) \exp^*(\sigma(z_{\mathrm{K},n})) = \frac{L_{\{p\}}(f,\chi,r)}{(-2\pi i)^{r-1}\Omega_f^{\pm}} \cdot t^{r-1}\omega_f$$

where the sign $\pm$ equals the sign of $\chi(-1)$ and $L_{\{p\}}$ denotes the L-value with the Euler factor at p removed.

Proof. See [20, Theorem 12.5]. □

The following proposition shows how one can recover Mazur-Tate elements from Kato's Euler system and the local classes $c_{n,j}$.

Proposition 10.2.2. *If* (No pole at $s = r$) *holds, then there exists a non-zero constant $C \in \mathcal{O}_L$ (independent of n and r) such that*

$$C \cdot \theta^{\mathrm{an}}_{n,r}(f) = \sum_{\sigma \in \mathcal{G}_n} \left\langle c^{\sigma}_{n,r-1}, \mathrm{res}_p(z_{\mathrm{K},n}) \right\rangle_n \sigma^{-1}.$$

Proof. Set

$$\theta_{n,r} = \sum_{\sigma \in \mathcal{G}_n} \left\langle c^{\sigma}_{n,r-1}, \mathrm{res}_p(z_{\mathrm{K},n}) \right\rangle_n \sigma^{-1}.$$

To prove this proposition it suffices to check that the $\theta_{n,r}$ satisfy the same interpolation property (up to a constant) as the Mazur-Tate elements (e.g. (18)) for primitive characters, and that they satisfy the same three-term relation (e.g. (19)).

To this end, for $n > 0$, we have

$$\begin{aligned}
\chi(\theta_{n,r}) &= \sum_{\sigma \in \mathcal{G}_n} \chi(\sigma)^{-1} \left\langle c^{\sigma}_{n,r-1}, \mathrm{res}_p(z_{\mathrm{K},n}) \right\rangle_n \\
&= (r-1)! \cdot p^{m(r-1)} \cdot \tau(\chi^{-1}) \cdot \sum_{\sigma \in \mathcal{G}_n} \chi(\sigma) \left\langle t^{1-r}\mathbf{e}, \exp^*(z^{\sigma}_{\mathrm{K},n}) \right\rangle_{\mathrm{dR},n} && \text{[Proposition 7.4.1]} \\
&= \chi(-1) \cdot (r-1)! \cdot p^{m(r-1)} \cdot \tau(\chi^{-1}) \cdot \frac{L_{\{p\}}(f,\chi,r)}{(2\pi i)^{r-1}\Omega_f^{\pm}} \langle \mathbf{e}, \omega_f \rangle_{\mathrm{dR}} && \text{[Proposition 10.2.1]} \\
&= \chi(\theta^{\mathrm{an}}_{n,r}(f)) \cdot \langle \mathbf{e}, \omega_f \rangle_{\mathrm{dR}}.
\end{aligned}$$

since the local Euler factor at p is trivial for $L(f,\chi,s)$. Note that the application of Proposition 7.4.1 requires the hypothesis (No pole at $s = r$).

For $n=0$, we have

$$\begin{aligned}
\mathbf{1}(\theta_{0,r}) &= \langle c_{0,r}, \mathrm{res}_p(z_{\mathrm{K},0})\rangle_0 \\
&= (r-1)! \cdot \widetilde{Z}(x^{1-r}\phi_{v_{\mathrm{new}}}, 1) \left\langle t^{1-r}\mathbf{e}, \exp^* z_0' \right\rangle_{\mathrm{dR},0} && \text{[Proposition 7.4.1]} \\
&= (r-1)! \cdot \widetilde{Z}(x^{1-r}\phi_{v_{\mathrm{new}}}, 1) \cdot \frac{L_{\{p\}}(f,r)}{\Omega_f^+} \langle \mathbf{e}, \omega_f\rangle_{\mathrm{dR}} && \text{[Proposition 10.2.1]} \\
&= (r-1)! \cdot \frac{L(f,r)}{\Omega_f^+} \langle \mathbf{e}, \omega_f\rangle_{\mathrm{dR}} \\
&= \mathbf{1}(\theta_{0,r}^{\mathrm{an}}(f)) \cdot \langle \mathbf{e}, \omega_f\rangle_{\mathrm{dR}}\,.
\end{aligned}$$

Here $\phi_{v_{\mathrm{new}}}$ is the newvector associated to f and $\widetilde{Z}(x^{1-r}\phi_{v_{\mathrm{new}}}, 1)$ is the simply the local Euler factor at p for $L(f,r)$. Thus, $\theta_{n,r}$ and $\theta_{n,r}^{\mathrm{an}}(f)$ agree at primitive characters (up to a constant).

To finish the proof, we compute

$$\begin{aligned}
&\pi_n^{n+1}(\theta_{n,r}) = \pi_n^{n+1}\left(\sum_{\tau\in\mathcal{G}_{n+1}} \left\langle c_{n+1,r-1}^{\tau}, \mathrm{res}_p(z_{\mathrm{K},n+1})\right\rangle_{n+1} \tau^{-1}\right) \\
&= \sum_{\sigma\in\mathcal{G}_n}\left(\sum_{\substack{\tau\in\mathcal{G}_{n+1}\\ \tau\to\sigma}} \left\langle c_{n+1,r-1}^{\tau}, \mathrm{res}_p(z_{\mathrm{K},n+1})\right\rangle_{n+1}\right)\sigma^{-1} \\
&= \sum_{\sigma\in\mathcal{G}_n} \left\langle \mathrm{cor}_n^{n+1}(c_{n+1,r-1}), \mathrm{cor}_n^{n+1}(\mathrm{res}_p(z_{\mathrm{K},n+1}))\right\rangle_n \sigma^{-1} \\
&= \sum_{\sigma\in\mathcal{G}_n} \left\langle a_p(f) c_{n,r-1}^{\sigma} - \psi(p)p^{k-2}\,\mathrm{res}_{n-1}^n(c_{n-1,r-1}^{\sigma}), \mathrm{res}_p(z_{\mathrm{K},n})\right\rangle_n \sigma^{-1} \\
&= a_p(f)\cdot\sum_{\sigma\in\mathcal{G}_n} \left\langle c_{n,r-1}^{\sigma}, \mathrm{res}_p(z_{\mathrm{K},n})\right\rangle_n \sigma^{-1} - \psi(p)p^{k-2}\sum_{\sigma\in\mathcal{G}_n}\left\langle \mathrm{res}_{n-1}^n(c_{n-1,r-1}^{\sigma}), \mathrm{res}_p(z_{\mathrm{K},n})\right\rangle_n \sigma^{-1} \\
&= a_p(f)\cdot\sum_{\sigma\in\mathcal{G}_n} \left\langle c_{n,r-1}^{\sigma}, \mathrm{res}_p(z_{\mathrm{K},n})\right\rangle_n \sigma^{-1} - \psi(p)p^{k-2}\sum_{\sigma\in\mathcal{G}_n}\left\langle c_{n-1,r-1}^{\sigma}, \mathrm{res}_p(z_{\mathrm{K},n-1})\right\rangle_{n-1} \sigma^{-1} \\
&= a_p(f)\cdot\theta_{n,r} - \psi(p)p^{k-2}\cdot\mathrm{cor}_{n-1}^n(\theta_{n-1,r})
\end{aligned}$$

as desired. Here we used Proposition 7.5.2. A similar computation handles the case $n=0$ as well. ☐

11. Algebraic θ-elements, take II

Throughout this section we assume that (Irred) holds.

11.1. **An alternative construction of θ-elements.** In section 8, we gave a construction of algebraic θ-elements working with the Galois cohomology of the discrete module A_r^* which allowed us to relate these elements to the Fitting ideals of its Selmer group. We now give a simpler and more direct construction of these elements using the cohomology of T_r^*. The advantage of this second construction is that it will be easier to relate to Mazur-Tate elements (via Kato's results) and thus to prove Theorem 10.1.4. However, showing that these two constructions match then takes additional work, and will be carried out in this subsection.

By Theorem 8.2.2, let $w = (w_n) \in \mathbb{H}^1(T_r^*)$ denote some generator of this free Λ-module of rank 1. For $c \in H^1_e(K_{n,p}, T(r))$, set

$$\widetilde{\psi}_n(c) := \widetilde{\psi}_{n,w}(c) := \sum_{\sigma \in \mathcal{G}_n} \langle c^\sigma, \mathrm{res}_p(w_n)\rangle_n \, \sigma \in \Lambda_n$$

where res_p denotes restriction to the unique place over p. The pairing $\langle \cdot, \cdot \rangle_n$ is given by Tate local duality

$$H^1(K_{n,p}, T(r)) \times H^1(K_{n,p}, T_r^*) \to \mathcal{O}_L.$$

11.2. **Comparing the two constructions.** Let ι be the involution of Λ_n which sends a group-like element σ to σ^{-1}. We will see that the above simple and direct definition of $\widetilde{\psi}_n$ yields a function which, up to ι, equals ψ_n (which was the key input in defining algebraic θ-elements). However, to make this claim precise, recall that the definition of $\psi_n = \psi_{n,\alpha}$ depended on an isomorphism $\alpha : R_\Lambda(X_r) \cong \Lambda$ while the definition of $\widetilde{\psi}_n = \widetilde{\psi}_{n,w}$ depended on a choice of a generator w of $\mathbb{H}^1(T_r^*)$. Thus, to relate ψ_n to $\widetilde{\psi}_n$, we must first relate these choices.

We now show how to use our fixed $w = (w_n) \in \mathbb{H}^1(T_r^*)$ to give an explicit identification of $R_\Lambda(X_r)$ with Λ. Set $H_r = H^1(\mathbb{Q}_\Sigma/K_\infty, A_r^*)$ and $X_r^\Sigma := H_r^\vee$. Dualizing (16) yields

$$W \to X_r^\Sigma \to X_r \to 0$$

where W is some finitely generated torsion Λ-module. Thus, applying $\mathrm{Hom}_\Lambda(\cdot, \Lambda)$ gives a canonical isomorphism

$$\mathrm{Hom}_\Lambda(X_r, \Lambda) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(X_r^\Sigma, \Lambda)$$

and hence a canonical isomorphism $R_\Lambda(X_r) \cong R_\Lambda(X_r^\Sigma)$. It therefore suffices to make an identification of $R_\Lambda(X_r^\Sigma)$ with Λ which we now do.

From (Irred), we deduce that $H^0(\mathbb{Q}_p(\mu_p), \overline{\rho}_f) = 0$, and thus the control theorem holds for H_r; that is, the restriction map

$$r_n : H^1(\mathbb{Q}_\Sigma/K_n, A_r^*) \to H_r^{\Gamma_n}$$

is an isomorphism for any n. Let

$$\pi_n^m : H^1(\mathbb{Q}_\Sigma/K_n, T_r^*) \to H^1(\mathbb{Q}_\Sigma/K_n, T_r^*/p^m T_r^*) \cong H^1(\mathbb{Q}_\Sigma/K_n, A_r^*[p^m]) \xrightarrow{r_n} H_r^{\Gamma_n}[p^m]$$

denote the canonical map. Define a Λ-homomorphism

$$\xi_w : X_r^\Sigma \to \Lambda = \varprojlim_n \Lambda_n$$

given, for $\eta : H_r \to \mathbb{Q}_p/\mathbb{Z}_p$, by $\xi_w(\eta) = \left(\sum_{\sigma \in \mathcal{G}_n} a_\sigma \sigma\right)_n$ where

$$a_\sigma = \varprojlim_{m \geq n} p^m \eta\big(\pi_n^m(w_n^\sigma)\big) \in \mathbb{Z}_p.$$

Note that $\eta(\pi_n^m(w_n^\sigma))$ lies in $\frac{1}{p^m}\mathbb{Z}_p/\mathbb{Z}_p$, so that the m^{th} term in this limit lies in $\mathbb{Z}/p^m\mathbb{Z}$.

Lemma 11.2.1. $\mathrm{Hom}_\Lambda(X_r^\Sigma, \Lambda)$ *is free of rank one over* Λ *with generator* ξ_w.

Proof. We have the following Λ-module homomorphisms:

$$\begin{aligned}\mathrm{Hom}_\Lambda(X_r^\Sigma,\Lambda) &\cong \mathrm{Hom}_\Lambda(\Lambda^\vee,(X_r^\Sigma)^\vee) \cong \mathrm{Hom}_\Lambda(\Lambda^\vee,H_r) \cong \mathrm{Hom}_\Lambda\big(\varinjlim_n (\Lambda^\vee)^{\Gamma_n}[p^n],H_r\big)\\ &\cong \varprojlim_n \mathrm{Hom}_\Lambda\big((\Lambda^\vee)^{\Gamma_n}[p^n],H_r\big) \cong \varprojlim_n \mathrm{Hom}_\Lambda\big((\Lambda_n/p^n)^\vee,H_r\big)\\ &\cong \varprojlim_n \mathrm{Hom}_\Lambda(\Lambda_n/p^n,H_r) \cong \varprojlim_n H_r[p^n]^{\Gamma_n} \cong \mathbb{H}^1(T_r^*).\end{aligned}$$

Above we used the self-duality

$$\Lambda_n/p^n \times \Lambda_n/p^n \to \mathbb{Z}/p^n$$
$$\left\langle \sum a_\sigma\sigma, \sum b_\sigma\sigma\right\rangle \mapsto \sum a_\sigma b_{\sigma^{-1}}$$

as this is the duality which induces a Λ-module isomorphism: $(\Lambda_n/p^n)^\vee \cong \Lambda_n/p^n$. To prove this lemma, one then chases through the above maps to see that ξ_w in $\mathrm{Hom}_\Lambda(X_r^\Sigma,\Lambda)$ maps to the generator w in $\mathbb{H}^1(T_r^*)$. $\square$

Corollary 11.2.2. *The map*

$$\Xi_w : R_\Lambda(X_r) \xrightarrow{\sim} R_\Lambda(X_r^\Sigma) \to \Lambda$$

given by evaluation at ξ_w is an isomorphism.

We have thus built an explicit identification of $R_\Lambda(X_r)$ with Λ given a generator of $\mathbb{H}^1(T_r^*)$.

Theorem 11.2.3. *For $c \in H^1_e(K_{n,p},T(r))$, we have*

$$\psi_{n,\Xi_w}(c) = \widetilde{\psi}_{n,w}(c)^\iota$$

In particular,

$$\theta^{\mathrm{alg}}_{n,r}(f) = \left(\sum_{\sigma\in\mathcal{G}_n} \left\langle c^\sigma_{n,r-1},\mathrm{res}_p(w_n)\right\rangle_n \sigma^{-1}\right)\cdot \mathrm{char}_\Lambda((X_r)_{\Lambda\text{-tor}})$$

Proof. Let $\Xi_{w,n} : X_{\Gamma_n} \to \Lambda_n$ denote the Γ_n-coinvariants of the composition of the canonical map $\mathrm{ev} : X_r \to R_\Lambda(X_r)$ and Ξ_w. Recall the isomorphisms:

$$(X_r^\Sigma)_{\Gamma_n} = (H_r^\vee)_{\Gamma_n} \xrightarrow{\sim} (H_r^{\Gamma_n})^\vee \xrightarrow{\sim} H^1(\mathbb{Q}_\Sigma/K_n,A_r^*)^\vee. \tag{20}$$

Thus, we may define an element in $(X_r^\Sigma)_{\Gamma_n}$ by defining it as the functional on $H^1(\mathbb{Q}_\Sigma/K_n,A_r^*)$.

For $c \in H^1_e(K_{n,p},T(r))$, consider the element $\varphi_c^\Sigma \in (X_r^\Sigma)_{\Gamma_n}$ defined by first restricting to p and then pairing (via Tate local duality) against c. That is, for $h \in H^1(\mathbb{Q}_\Sigma/K_n,A_r^*)$, set

$$\varphi_c^\Sigma(h) = \langle c,\mathrm{res}_p(h)\rangle\,.$$

Set $\varphi_c \in (X_r)_{\Gamma_n}$ equal to the image of φ_c^Σ under the natural surjection $X_r^\Sigma \to X_r$. Then, by definition, $\psi_{n,\Xi_w}(c) = \Xi_{w,n}(\varphi_c)$.

However, we can also compute $\Xi_{w,n}(\varphi_c)$ directly as the image of $\xi_w(\varphi_c^\Sigma)$ in Λ_n. By the definition of ξ_w, this image equals $\sum_{\sigma\in G_n} b_\sigma\sigma$ where

$$b_\sigma = \varprojlim_{m\geq n} p^m\,\langle c,\mathrm{res}_p(\pi_n^m(w_n^\sigma))\rangle_n\,.$$

But, by definition, this limit equals

$$\langle c,\mathrm{res}_p(w_n^\sigma)\rangle_n = \left\langle c^{\sigma^{-1}},\mathrm{res}_p(w_n)\right\rangle_n$$

as desired. □

12. The main results

In light of Proposition 10.2.2 and Theorem 11.2.3, to compare $\theta^{\mathrm{an}}_{n,r}(f)$ and $\theta^{\mathrm{alg}}_{n,r}(f)$ we need to understand the index of Kato's Euler system in $\mathbb{H}^1(T_r^*)$ compared to $\mathrm{char}_\Lambda X_{\Lambda\text{-tor}}$. This relation will follow from Kato's proof of half of the main conjecture (without p-adic L-functions).

12.1. **Euler system result.** Let $\mathbb{H}^i(T) = \varprojlim_n H^i(\mathbb{Q}_\Sigma/K_n, T)$ and $\mathbb{H}^i_{\mathrm{Iw},w}(T) = \varprojlim_n H^i(K_{n,v_n}, T)$ where w is some place of $\mathbb{Q}_\infty$ and v_n is the place of K_n below w. Further, set

$$\mathbb{H}^2_P(T) = \ker\left(\mathbb{H}^2(T) \to \prod_{w \in \Sigma_\infty} \mathbb{H}^2_{\mathrm{Iw},w}(T)\right)$$

where Σ_∞ is the set of places of K_∞ sitting over Σ.

Theorem 12.1.1 (Kato)**.** *We have*

(1) $\mathbb{H}^2_P(T_r^*)$ *is a torsion* Λ*-module;*
(2) *if* (Euler) *holds,*

$$\mathrm{char}\,\mathbb{H}^2_P(T_r^*) \text{ divides } \mathrm{char}_\Lambda(\mathbb{H}^1(T_r^*)/\langle z_{\mathrm{K}}\rangle).$$

Proof. This is [20, Theorems 12.4 and 12.5]. See also [24, pg. 217] where the étale cohomology groups in [20] are written in terms of Galois cohomology with local conditions (e.g. $\mathbb{H}^2_P$). □

We must now relate $\mathbb{H}^2_P(T_r^*)$ to $(X_r)_{\Lambda\text{-tor}}$. We begin with a lemma. If M is a Λ-module, we write M^ι for the Λ-module whose underlying set is M, but the group-like element σ acts by σ^{-1}.

Lemma 12.1.2. *If X is a finitely generated torsion Λ-module, then $\mathrm{Ext}^1_\Lambda(X,\Lambda)$ is pseudo-isomorphic to $(X_{\Lambda\text{-tor}})^\iota$.*

Proof. The exact sequence

$$0 \to X_{\Lambda\text{-tor}} \to X \to Z \to 0$$

induces

$$\mathrm{Ext}^1_\Lambda(Z,\Lambda) \to \mathrm{Ext}^1_\Lambda(X,\Lambda) \to \mathrm{Ext}^1_\Lambda(X_{\Lambda\text{-tor}},\Lambda) \to \mathrm{Ext}^2_\Lambda(Z,\Lambda).$$

We first claim that $\mathrm{Ext}^i_\Lambda(Z,\Lambda)$ is pseudo-null for $i = 1, 2$.

To see this, let $R_\Lambda(Z)$ denote the reflective-hull of Z and consider the exact sequence

$$0 \to Z \to R_\Lambda(Z) \to H \to 0$$

with H finite. We then have

$$0 = \mathrm{Ext}^i_\Lambda(R_\Lambda(Z),\Lambda) \to \mathrm{Ext}^i_\Lambda(Z,\Lambda) \to \mathrm{Ext}^i_\Lambda(H,\Lambda)$$

since $R_\Lambda(Z)$ is free. So it suffices to show that $\mathrm{Ext}^i_\Lambda(H,\Lambda)$ is finite for $i = 1, 2$. To see this, note that $\mathrm{Ann}_\Lambda(H)$ is finite-index in Λ and thus the same is true for the annihilator of $\mathrm{Ext}^i_\Lambda(H,\Lambda)$. But since this module is finitely-generated, it must then be finite as desired.

Thus, $\mathrm{Ext}^1_\Lambda(X,\Lambda)$ is pseudo-isomorphic to $\mathrm{Ext}^1_\Lambda(X_{\Lambda\text{-tor}},\Lambda)$. Lastly, by [37, pg. 474], $\mathrm{Ext}^1_\Lambda(Y,\Lambda)$ is pseudo-isomorphic to Y^ι for any finitely generated Λ-module which completes the proof. □

Theorem 12.1.3. *Assume* (Irred) *holds. If $X_{\Lambda\text{-tor}}$ denotes the Λ-torsion submodule of X, we have*

$$\operatorname{char}_\Lambda(X_r)_{\Lambda\text{-tor}} = \operatorname{char}_\Lambda(\mathbb{H}^2_P(T_r^*)^\iota).$$

Proof. By the two exact sequences in [28] appearing right after (0.13.2), we have a pseudo-isomorphism

$$(\widetilde{H}^1_{f,\mathrm{Iw}}(A_r^*)^\vee)_{\Lambda\text{-tor}} \cong \operatorname{Ext}^1_\Lambda(\widetilde{H}^2_{f,\mathrm{Iw}}(\mathbb{Q}_\infty/\mathbb{Q}, T_r^*), \Lambda)$$

Here $\widetilde{H}^i_{f,\mathrm{Iw}}$ are Selmer complexes where, for A_r^*, there is no local condition at p (and thus for T_r^* we impose the harshest local condition). Relating these back to classical Selmer groups (as in [28, (0.10)], we have a pseudo-isomorphism

$$X_{\Lambda\text{-tor}} \cong \operatorname{Ext}^1_\Lambda(\mathbb{H}^2_P(T_r^*), \Lambda).$$

By Theorem 12.1.1 and Lemma 12.1.2, we then have a pseudo-isomorphisms

$$X_{\Lambda\text{-tor}} \cong \mathbb{H}^2_P(T_r^*)^\iota$$

as desired. □

12.2. **Divisibility.** We begin with a simple lemma whose proof we leave to the reader.

Lemma 12.2.1. *For $\lambda \in \Lambda$, we have*

$$\lambda \cdot \sum_{\sigma\in\mathcal{G}_n} \left\langle c^\sigma_{n,r-1}, \operatorname{res}_p(w_n)\right\rangle_n \sigma = \sum_{\sigma\in\mathcal{G}_n} \left\langle c^\sigma_{n,r-1}, \operatorname{res}_p(\lambda w_n)\right\rangle_n \sigma.$$

We conclude with our main theorem which immediately implies Theorem 10.1.4.

Theorem 12.2.2. *If* (Irred)*,* (No pole at $s = r$)*, and* (Euler) *hold, then there exists a non-zero constant $C \in \mathcal{O}_L$ (independent of n and r) such that*

$$\theta^{\mathrm{alg}}_{n,r}(f) \mid C \cdot \theta^{\mathrm{an}}_{n,r}(f)$$

in Λ_n for all $n \geq 0$.

Proof. Recall, $w = (w_n)$ is our fixed generator of $\mathbb{H}^1(T_r^*)$. By Theorem 11.2.3, we have

$$\theta^{\mathrm{alg}}_{n,r}(f) = \operatorname{char}_\Lambda\left((X_r)_{\Lambda\text{-tor}}\right) \cdot \sum_{\sigma\in\mathcal{G}_n} \left\langle c^\sigma_{n,r-1}, \operatorname{res}_p(w_n)\right\rangle_n \sigma^{-1}$$

Applying the involution $\iota : \Lambda_n \to \Lambda_n$ which sends group-like elements to their inverse, yields

$$\iota(\theta^{\mathrm{alg}}_{n,r}(f)) = \operatorname{char}_\Lambda \mathbb{H}^2_P(T_r^*) \cdot \sum_{\sigma\in\mathcal{G}_n} \left\langle c^\sigma_{n,r-1}, \operatorname{res}_p(w_n)\right\rangle_n \sigma \tag{21}$$

by Theorem 12.1.3. Now write $z_{\mathrm{K}} = \alpha\cdot w$ for some $\alpha \in \Lambda$. Thus, $\alpha = \operatorname{char}_\Lambda(\mathbb{H}^1(T_r^*)/\langle z_K\rangle)$, and $\operatorname{char}_\Lambda \mathbb{H}^2_P(T_r^*)$ divides α by Theorem 12.1.1.

Scaling (21) by α, applying Lemma 12.2.1 and then Proposition 10.2.2 gives

$$\begin{aligned}\alpha \cdot \iota(\theta^{\mathrm{alg}}_{n,r}(f)) &= \operatorname{char}_\Lambda \mathbb{H}^2_P(T_r^*) \cdot \sum_{\sigma\in\mathcal{G}_n} \left\langle c^\sigma_{n,r-1}, \operatorname{res}_p(z_{\mathrm{K},n})\right\rangle_n \sigma \\ &= \operatorname{char}_\Lambda \mathbb{H}^2_P(T_r^*) \cdot C \cdot \iota(\theta^{\mathrm{an}}_{n,r}(f))\end{aligned}$$

Thus

$$\iota(\theta^{\mathrm{alg}}_{n,r}(f)) \cdot \frac{\alpha}{\operatorname{char}_\Lambda \mathbb{H}^2_P(T_r^*)} = C \cdot \iota(\theta^{\mathrm{an}}_{n,r}(f))$$

for some non-zero C independent of n and r. In particular, $\theta^{\rm alg}_{n,r}(f)$ divides $C\cdot\theta^{\rm an}_{n,r}(f)$ in Λ_n. $\square$

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