

THE LOCAL LANGLANDS CORRESPONDENCE FOR GL_n IN FAMILIES

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1. INTRODUCTION

The goal of this paper is to extend, for any non-archimedean local field E of residue characteristic ℓ , the local Langlands correspondence between n -dimensional Weil-Deligne representations of the Weil group W_E and admissible smooth representations of $GL_n(E)$ to a correspondence defined on p -adic families of representations of the absolute Galois group G_E of E (for primes p distinct from ℓ).

Fix an algebraically closed field K containing \mathbb{Q}_p . Then on one side of the local Langlands correspondence are the n -dimensional, Frobenius-semisimple Weil-Deligne representations of W_E ; that is, pairs (ρ, N) , where ρ is an n -dimensional, semisimple representation of W_E over K , and N is a nilpotent endomorphism of the representation space of ρ such that $\rho(w)N\rho(w)^{-1} = q^{|w|}N$ for all $w \in W_E$.

On the other side are the admissible smooth K -representations of $GL_n(E)$; that is, representations π of $GL_n(E)$ such that every element of π is fixed by some compact open subset U of $GL_n(E)$, and such that, for all such U , the space π^U of U -invariant vectors in π is finite-dimensional.

The most naive thing one could hope for would be the following: given a family of representations of G_E over a complete, Noetherian, local, p -torsion free ring A of characteristic zero and residue characteristic p (that is, a representation $\rho : G_E \rightarrow GL_n(A)$ for such an A), one could hope to find an admissible smooth $A[GL_n(E)]$ -module π such that, for every characteristic zero prime ideal \mathfrak{p} of A , with residue field $\kappa(\mathfrak{p})$, the representation $\pi \otimes_A \kappa(\mathfrak{p})$ of $GL_n(E)$ corresponds, via local Langlands, with the Frobenius-semisimple Weil-Deligne representation attached to $\rho \otimes_A \kappa(\mathfrak{p})$.

A moment's consideration will show that this is far too much to hope for. Indeed, if ρ is a direct sum of two unramified characters that specializes at a single

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characteristic zero point \mathfrak{p}_0 to the trivial character plus the cyclotomic character, then the local Langlands correspondence would tell us that $\pi \otimes_A \kappa(\mathfrak{p}_0)$ would be one-dimensional. On the other hand at general points \mathfrak{p} one would need $\pi \otimes_A \kappa(\mathfrak{p})$ to be infinite-dimensional.

Indeed, as this example suggests, even formulating a precise statement of a “local Langlands correspondence in families” is nontrivial. Our approach is motivated by global considerations, suggested by work of the first author [6].

1.1. Global motivation. In the setting of [6], the ring A is typically a p -adically completed Hecke algebra, and ρ is the two dimensional representation of $G_{\mathbb{Q}}$ over A arising from the theory of p -adic modular forms. For a certain finite set of primes Σ containing p , the Hecke algebra A acts on a suitable localization of p -adically completed cohomology $H^1(X_{\Sigma})$ of the tower of modular curves of levels divisible by primes in Σ . This cohomology is also equipped with commuting actions of $G_{\mathbb{Q}}$, of $\mathrm{GL}_2(\mathbb{Q}_p)$, and of $\mathrm{GL}_2(\mathbb{Q}_{\ell})$ for ℓ a prime of Σ not equal to p .

What is shown in [6] is that the space $H^1(X_{\Sigma})$ has a natural tensor factorization, and that the tensor factor corresponding to ℓ interpolates the local Langlands correspondence in a natural way. More precisely, one has a factorization:

$$H^1(X_{\Sigma}) \cong \rho \otimes \pi_p \otimes \bigotimes_{\ell \in \Sigma \setminus \{p\}} \pi_{\ell}$$

where π_p is a certain representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to $\rho|_{G_{\mathbb{Q}_p}}$ via considerations arising from the p -adic local Langlands correspondence, and the representations π_{ℓ} depend only on the restriction of ρ to $G_{\mathbb{Q}_{\ell}}$, and interpolate the local Langlands correspondence in a natural way. (We refer the reader to Conjecture 6.1.6 of [6] for a precise statement; Proposition 6.2.13 of [6] establishes this conjecture under mild hypotheses.)

The representations π_{ℓ} have a number of nice properties, which suggest the “shape” that the desired “local Langlands correspondence in families” should take:

- π_{ℓ} is the smooth $W(k)$ -dual of a (unique) admissible smooth $A[\mathrm{GL}_2(\mathbb{Q}_{\ell})]$ -module π'_{ℓ} , where k is the residue field of A . (Modules with this property are called “coadmissible” and studied in detail in Appendix C of [6].)
- π_{ℓ} is “ A -cotorsion free” in the sense of Definition C.37 of [6].
- For a Zariski dense set of characteristic zero primes \mathfrak{p} of A , the space $\pi_{\ell}[\mathfrak{p}]$ of \mathfrak{p} -torsion vectors in π_{ℓ} is the representation attached to $\rho \otimes_A \kappa(\mathfrak{p})$ by a “generic” version of the local Langlands correspondence due to Breuil and Schneider. (This correspondence differs from the usual local Langlands correspondence in several ways. Rather than being a bijection it is a map from isomorphism classes of n -dimensional representations of G_E to indecomposable (but not necessarily irreducible) generic representations of $\mathrm{GL}_2(E)$. We refer the reader to Section 4 for the definition and basic properties of the Breuil-Schneider correspondence.)
- The space $(\pi_{\ell}/p\pi_{\ell})[\mathfrak{m}]$ (where \mathfrak{m} is the maximal ideal of A) has an absolutely irreducible generic socle, and no other generic subquotients. (Here recall that the socle of a $k[\mathrm{GL}_n(E)]$ -module is its maximal semisimple submodule; that is, the sum of all of its irreducible submodules. A generic representation of $\mathrm{GL}_n(E)$ is one that whose restriction to the unipotent radical of a Borel contains a generic character; for $\mathrm{GL}_2(\mathbb{Q}_{\ell})$ an irreducible admissible representation is generic if, and only if, it is infinite dimensional.) In the

setting above, the genericity of the socle of $(\pi_\ell/p\pi_\ell)[\mathfrak{m}]$ is a consequence of Ihara's lemma.

Given an arbitrary A , and an arbitrary family $\rho : G_E \rightarrow \mathrm{GL}_n(A)$, it is natural in light of the above properties to ask if one can attach an $A[\mathrm{GL}_n(E)]$ -module π to ρ , with properties similar to those above, and to ask if such properties characterize a such a module uniquely. In fact, in order to avoid the complication of dealing with the theory of coadmissible modules, we will work in a setting dual to the above picture.

1.2. The local Langlands correspondence for GL_n in p -adic families. We are now in a position to state our main result. Before we do so, we introduce further notation: Let K be a field of characteristic zero. Then given $\rho : G_E \rightarrow \mathrm{GL}_n(K)$ as above, we write $\pi(\rho)$ for the representation attached to ρ by the generic local Langlands correspondence of Breuil-Schneider, and we write $\tilde{\pi}(\rho)$ to denote the smooth contragredient of $\pi(\rho)$.

1.2.1. Theorem. *Let A be a reduced complete p -torsion free Noetherian local ring with maximal ideal \mathfrak{m} and finite residue field k of characteristic p . If $\rho : G_E \rightarrow \mathrm{GL}_n(A)$ is continuous (when the target is given its \mathfrak{m} -adic topology), then there exists at most one admissible smooth $\mathrm{GL}_n(E)$ -representation V over A , up to isomorphism, satisfying the following conditions:*

- (1) V is A -torsion free.
- (2) If \mathfrak{a} is a minimal prime of A , with residue field $\kappa(\mathfrak{a})$, then there is a $\kappa(\mathfrak{a})$ -linear $\mathrm{GL}_n(E)$ -equivariant isomorphism

$$\tilde{\pi}(\kappa(\mathfrak{a}) \otimes_A \rho) \xrightarrow{\sim} \kappa(\mathfrak{a}) \otimes_A V.$$

- (3) If we write $\bar{V} := k \otimes_A V$, then the $\mathrm{GL}_n(E)$ cosocle $\mathrm{cosoc}(\bar{V})$ of \bar{V} is absolutely irreducible and generic, while the kernel of the surjection $\bar{V} \rightarrow \mathrm{cosoc}(\bar{V})$ contains no generic Jordan-Hölder factors.

Furthermore, if such a V exists, then:

- (4) There exists an open dense subset U of $\mathrm{Spec} A[\frac{1}{p}]$, such that for each prime \mathfrak{p} in U , there is a $\mathrm{GL}_n(E)$ -equivariant, nonzero surjection

$$\tilde{\pi}(\kappa(\mathfrak{p}) \otimes_A \rho) \rightarrow \kappa(\mathfrak{p}) \otimes_A V,$$

where $\kappa(\mathfrak{p})$ is the residue field of \mathfrak{p} . Moreover, there is an open dense subset U' of U such that if \mathfrak{p} lies in U' then this surjection is an isomorphism.

The admissibility condition on V , together with properties (1) through (3) above, are (roughly) dual to the properties satisfied by the tensor factors π_ℓ . Indeed, we prove a ‘‘recognition theorem’’ (Theorem 6.2.15 below) that states that if the smooth $W(k)$ -dual of V satisfies the properties listed for π_ℓ above (for a given ρ), then V is admissible and satisfies properties (1) through (3).

If a representation V satisfying the conditions of this theorem with respect to a given Galois representation $\rho : G_E \rightarrow \mathrm{GL}_n(A)$ exists, then we write $V := \tilde{\pi}(\rho)$. (Note that the theorem ensures that V is unique up to isomorphism, so that $\tilde{\pi}(\rho)$ is then uniquely determined by ρ , up to isomorphism, if it exists.) We justify this notation by thinking of representations of G_E over fields of characteristic zero as ‘‘families over a single point’’, so that the map $\rho \mapsto \tilde{\pi}(\rho)$, where it is defined, extends

the dual of the Breuil-Schneider correspondence from one-point families to families over local rings A as in the theorem.

Part (4) of the theorem describes the precise sense in which V interpolates the local Langlands correspondences attached to the Galois representations $\kappa(\mathfrak{p}) \otimes_A \rho$ as \mathfrak{p} ranges over the points of $\mathrm{Spec} A[\frac{1}{p}]$. Conjecturally, we can take U equal to all of $\mathrm{Spec} A[\frac{1}{p}]$ in statement (4), although our results fall short of establishing this. (We refer the reader to Theorems 6.2.5 and 6.2.6 for the precise results.) On the other hand, we will give examples in Section 6 showing that the subset U' is, in general, not equal to all of $\mathrm{Spec} A[\frac{1}{p}]$.

1.2.2. Remark. Our convention for the generic local Langlands correspondence is that the $GL_n(\mathcal{K})$ -representation $\pi(\rho)$ attached to a continuous Galois representation $\rho : G_E \rightarrow GL_n(\mathcal{K})$ should have generic socle. It is this convention that seems to fit best with global applications of the type considered in [6] and [7], for example. On the other hand, when working with families, it turns out to be easier to interpolate representations whose cosocle is generic. This is because one may work with admissible $A[GL_n(E)]$ -modules, rather than the less-familiar coadmissible modules of [6]. This explains the appearance of the various contragredient representations in Theorem 1.2.1, and in our notation for the representations that it describes.

1.3. Global applications. Although we have used the results of [6] to motivate our formulation of the local Langlands correspondence in families, in fact Theorem 1.2.1, together with the “recognition theorem” (Theorem 6.2.15) described above, are essential ingredients in the proofs of the main results of [6].

In the language of Theorem 1.2.1, the results of [6] state that (again, under mild hypotheses) when A is a suitable p -adically completed Hecke algebra, and $\rho : G_{\mathbb{Q}} \rightarrow GL_n(A)$ is the representation arising from Eichler-Shimura theory, then the representations $\tilde{\pi}(\rho|_{G_{\mathbb{Q}_\ell}})$ exist for all ℓ not equal to p , and one has a tensor factorization:

$$H^1(X_\Sigma) \cong \rho \otimes \pi_p \otimes \bigotimes_{\ell \in \Sigma \setminus \{p\}} \tilde{\pi}(\rho|_{G_{\mathbb{Q}_\ell}})^\vee.$$

The resulting structure theory for completed cohomology of the modular tower has striking arithmetic applications; in particular [6] establishes many cases of the Fontaine-Mazur conjecture as a corollary of the tensor factorization described above (see in particular [6], corollary 1.2.2.) It thus is natural to attempt to seek a framework in which one can describe completed cohomology of Shimura towers in as broad a context as possible; a primary goal of this paper is to develop a language which should apply to the factors of completed cohomology at primes $\ell \neq p$.

It is important to note that the arguments of [6] do not require one to know, a priori, that representations of the form $\tilde{\pi}(\rho)$ exist. Instead, they rely on Theorem 6.2.15 below to deduce from considerations at a dense set of points (together with a genericity condition that, in the setting of [6], essentially reduces to Ihara’s lemma) that a module with an action of $GL_n(\mathbb{Q}_{\ell_i})$ for each ℓ_i in a finite collection of primes admits a tensor factorization as a product of families of the form $\tilde{\pi}(\rho)$. Whereas the proof of Theorem 1.2.1 is relatively elementary (essentially relying on an integral version of the Bernstein-Zelevinski theory of the derivative), the proof

of Theorem 6.2.15 requires detailed information about the behavior of the generic local Langlands correspondence under specialization, which we develop in section 4.

1.4. Existence of the correspondence: results and conjectures. Just as in the traditional setting, it seems to be easier to characterize the local Langlands correspondence in families than to prove its existence. However, we make the following conjecture.

1.4.1. Conjecture. *If A is a reduced p -torsion free complete Noetherian local ring with maximal ideal \mathfrak{m} and finite residue field k of characteristic p , and if $\rho : G_E \rightarrow \mathrm{GL}_n(A)$ is continuous, then $\tilde{\pi}(\rho)$ exists.*

One instance in which we can verify the conjecture is the case when A is the ring of integers in a finite extension of \mathbb{Q}_p . (In this case, it is a consequence of Theorem 3.3.2.) Forthcoming work of the second author [11], [10] will establish many additional cases of this conjecture, using the second author's theory of the integral Bernstein center [9]. In particular the conjecture holds for $n = 2$ and p odd, and also for p a banal prime; that is, a prime for which the integers $1, q, \dots, q^n$ are distinct modulo p , where q is the order of the residue field of E . (This means that for any fixed n , there are only finitely many p at which the conjecture can fail.)

1.5. A mod p local Langlands correspondence for GL_n . In [20], Vigneras defines a mod p local Langlands correspondence for GL_n . However, this correspondence seems ill-suited for arithmetic applications (and in particular for describing the mod p cohomology of the modular tower.) In particular, it lacks the good behavior of the Breuil-Schneider correspondence under specialization.

In Section 5 we define a “modified” mod p local Langlands correspondence, that is more suited to our needs. It differs from the correspondence of Vigneras in the following ways:

- (a) The input is a Galois representation (not a Weil–Deligne representation).
- (b) The output is an admissible smooth $\mathrm{GL}_n(E)$ -representation that is possibly reducible, but always generic.
- (c) The correspondence is compatible with reduction modulo p in the direct sense given by parts (2) and (3) of Theorem 1.5.1 below. (The Zelevinski involution does not intervene.)

This being said, we rely on the results of [20] for the construction of our correspondence. The key point, whose proof relies on [20], is that for any deformation ρ of $\bar{\rho}$, the representation $\tilde{\pi}(\rho)$ reduces modulo \mathfrak{m} to a representation whose cosocle is absolutely irreducible and generic, and is independent, up to isomorphism, of the choice of ρ . (See the discussion following Corollary 5.1.2 below.)

The key properties of the correspondence of Section 5 are summarized in the following theorem:

1.5.1. Theorem. *There is a map $\bar{\rho} \mapsto \bar{\pi}(\bar{\rho})$ from the set of isomorphism classes of continuous representations $G_E \rightarrow \mathrm{GL}_n(k)$ (where k is a finite field of characteristic p) to the set of isomorphism classes of finite length admissible smooth $\mathrm{GL}_n(E)$ -representations on k -vector spaces, uniquely determined by the following conditions:*

- (1) For any $\bar{\rho}$, the G -socle $\text{soc}(\bar{\pi}(\bar{\rho}))$ of the associated $\text{GL}_n(E)$ -representation $\bar{\pi}(\bar{\rho})$ is absolutely irreducible and generic, and the quotient $\bar{\pi}(\bar{\rho})/\text{soc}(\bar{\pi}(\bar{\rho}))$ contains no generic Jordan–Hölder factors.
- (2) Given $\bar{\rho} : G_E \rightarrow \text{GL}_n(k)$, together with a deformation $\rho : G_E \rightarrow \text{GL}_n(\mathcal{O})$ of $\bar{\rho}$, where \mathcal{O} is a characteristic zero discrete valuation ring with uniformizer ϖ and residue field k' containing k , there is $\text{GL}_n(E)$ -equivariant surjection $\tilde{\pi}(\bar{\rho}) \otimes_k k' \rightarrow \tilde{\pi}(\rho)/\varpi\tilde{\pi}(\rho)$. (Note that $\tilde{\pi}(\rho)$ must exist as \mathcal{O} is a finite extension of \mathbb{Q}_p .)
- (3) The representation $\bar{\pi}(\bar{\rho})$ is minimal with respect to satisfying conditions (1) and (2), i.e. given any continuous representation $\bar{\rho} : G_E \rightarrow \text{GL}_n(k)$ and any representation $\bar{\pi}$ of $\text{GL}_n(E)$ satisfying these two conditions with respect to $\bar{\rho}$, there is a $\text{GL}_n(E)$ -equivariant embedding $\bar{\pi}(\bar{\rho}) \hookrightarrow \bar{\pi}$.

As with the local Langlands correspondence in families, our motivation for introducing this modified mod p local Langlands correspondence is that it arises in global contexts. Indeed, consider the limit $H^1(X_\Sigma, k)$ of the cohomology of the tower X_Σ of modular curves, where the levels of curves in X_Σ are divisible precisely by the primes in Σ . This has an action of a completed Hecke algebra A , and if one considers the maximal ideal of A corresponding to a suitable irreducible representation $\bar{\rho} : G_\mathbb{Q} \rightarrow \text{GL}_2(k)$, then one has an isomorphism:

$$H^1(X_\Sigma, k)[\mathfrak{m}] \cong \bar{\rho} \otimes \bar{\pi}_p \otimes \bigotimes_{\ell \in \Sigma \setminus \{p\}} \bar{\pi}(\bar{\rho}|_{G_{\mathbb{Q}_\ell}}).$$

Thus the modified mod p local Langlands correspondence gives a framework in which one may describe the mod p cohomology of towers of modular curves, and one expects that this should apply to more general towers of Shimura varieties as well.

The modified mod p local Langlands correspondence admits a completely concrete description for $n = 2$ and p odd; we briefly describe some cases of this in section 5, and refer the reader to [12] for the complete picture.

1.5.2. Remark. One can consider the following stronger form of condition (2) of Theorem 1.5.1:

- (2') Given $\bar{\rho} : G_E \rightarrow \text{GL}_n(k)$, together with a deformation $\rho : G_E \rightarrow \text{GL}_n(A)$ of $\bar{\rho}$, where A is a reduced complete Noetherian local $W(k)$ -algebra, flat over $W(k)$, with maximal ideal \mathfrak{m} and residue field k , and $\tilde{\pi}(\rho)$ exists, there is a $\text{GL}_n(E)$ -equivariant surjection $\tilde{\pi}(\bar{\rho}) \rightarrow \tilde{\pi}(\rho)/\varpi\tilde{\pi}(\rho)$.

In some circumstances, we are able to verify that $\bar{\pi}(\bar{\rho})$ is in fact minimal with respect to conditions (1) and (2'). (In other words, $\bar{\pi}(\bar{\rho})$ contains as a submodule the dual of any module that arises by specializing a family $\tilde{\pi}(\rho)$ attached to a deformation of $\bar{\rho}$.) This is essentially a characteristic p analog of Theorems 6.2.5 and 6.2.6. We conjecture that this stronger minimality property holds in general.

1.5.3. Remark. In general, $\bar{\pi}(\bar{\rho})$ is not irreducible, and if this is the case, then it is not possible to strengthen “surjection” to “isomorphism” in the statement of part (2) of Theorem 1.5.1.

1.6. The organization of the paper. We begin by establishing some basic facts about admissible smooth representations of certain topological groups over a Noetherian local ring A ; we apply this machinery in section 2.2 to the study of invariant

lattices in representations of topological groups over the field of fractions of a complete discrete valuation ring. The key result we establish is Lemma 2.2.6, which in certain circumstances allows us to construct an invariant lattice in such a representation whose reduction has a prescribed socle.

Section 3 establishes results specific to the representation theory of $\mathrm{GL}_n(E)$ over certain local rings A . In 3.1 we construct a theory of Kirillov models over a large class of base rings. Our approach essentially follows that of [1], but as we are not working over algebraically closed fields issues of descent arise. In spite of this one recovers almost all of the theory of the Kirillov functors developed in [1, §4]. Particularly useful for us is the notion of a “generic” irreducible representation over an arbitrary field.

Section 3.2 introduces an essential concept for our results: that of an essentially absolutely irreducible and generic representation (“essentially AIG” in short.) These are representations whose socles are absolutely irreducible and generic, and that satisfy a certain finiteness property. The importance of these representations stems from the fact that the “modified Langlands correspondences” we consider send Galois representations to essentially AIG representations. In section 3.3 we apply the results of section 2.2 to establish some basic facts about the reduction theory of essentially AIG representations.

In section 4 we study the behavior of the local Langlands correspondence (over fields of characteristic zero) under specialization. As we have previously discussed, the usual local Langlands correspondence is not suitable for our purposes, and we instead consider a modification of this correspondence due to Breuil and Schneider. Our first main result (Corollary 4.3.3) establishes that the admissible representations of $\mathrm{GL}_n(E)$ produced by the Breuil-Schneider correspondence are essentially AIG. Once we have this, we apply the reduction theory of section 3.3, together with ideas from the Zelevinski classification, to establish Theorem 4.5.7, which relates the behaviour of a Galois representation under specialization to a characteristic zero residue field to the behaviour (under the same specialization) of the corresponding admissible representation constructed by Breuil-Schneider.

Section 5 constructs a “modified local Langlands correspondence” in characteristic p , by analogy with the Breuil-Schneider correspondence; in particular we define this correspondence to be the “minimal” correspondence that satisfies a mod p analogue of Theorem 4.5.7. We refer the reader to Theorem 5.1.5 for the precise definition.

We finally turn to the study of the local Langlands correspondence for families of admissible representations in section 6. Section 6.2 discusses the main results of our theory; to avoid obscuring this discussion with technicalities we postpone the proofs to section 6.3. Surprisingly little beyond the theory of Kirillov models is necessary to prove the basic uniqueness result of Theorem 6.2.1. On the other hand, establishing more precise results about the structure of the family of admissible representations attached to a given family of Galois representations (for instance, the interpolation theorems 6.2.5 and 6.2.6) requires the full strength of the specialization results in section 4.

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2. REPRESENTATION THEORY — GENERAL BACKGROUND

2.1. Admissible smooth representations. Let A be a Noetherian local ring with maximal ideal \mathfrak{m} and residue field k . In this subsection we recall some basic facts about admissible smooth representations over A .

2.1.1. Definition. An A -linear representation of a topological group H on an A -module V is called smooth if any element of V is fixed by an open subgroup of H .

Clearly any H -invariant sub- or quotient A -module of a smooth H -representation over A is again a smooth H -representation over A .

2.1.2. Definition. A smooth representation of a topological group H on an A -module V is called admissible if for any open subgroup $H_0 \subset H$, the A -module of fixed points V^{H_0} is finitely generated.

Clearly any H -invariant A -submodule of an admissible smooth H -representation over A is again an admissible smooth H -representation over A . (For the case of H -invariant quotients, see Lemma 2.1.6 below.)

Consider the following condition on H :

2.1.3. Condition. H contains a profinite open subgroup, admitting a countable basis of neighbourhoods of the identity, whose pro-order is invertible in A .

Suppose that H satisfies Condition 2.1.3, and let $\{H_i\}_{i \geq 0}$ denote a decreasing sequence of open subgroups of H , each of whose pro-order is invertible in A , and which forms a neighbourhood basis of the identity in H . If V is a smooth H -representation over A , then for each $n \geq 1$, we may define the idempotent projector $\pi_i : V \rightarrow V^{H_i}$ via $v \mapsto \int_{H_i} hvd\mu_i$, where μ_i denotes Haar measure on H_i , normalized so that H_i has total measure 1. If we define $V_i := \ker \pi_i \cap V^{H_{i+1}}$, then the inclusions $V_i \subset V$ induces an isomorphism of A -modules

$$(2.1) \quad \bigoplus_i V_i \xrightarrow{\sim} V.$$

The formation of V_i is evidently functorial on the category of smooth representations of H over A , and thus so is the direct sum decomposition (2.1). In fact one can say something more precise:

2.1.4. Lemma. *Suppose that H satisfies Condition 2.1.3. If W is an H -invariant A -submodule of the smooth H -representation A over V , then the natural maps $W_i \hookrightarrow V_i \cap W$ and $V_i/(V_i \cap W) \rightarrow (V/W)_i$ are isomorphisms.*

Proof. This is evident. \square

2.1.5. Lemma. *Suppose that H satisfies Condition 2.1.3. A smooth H -representation V over A is admissible if and only if each of the A -modules V_i is finitely generated.*

Proof. This follows from the isomorphisms $\bigoplus_{j \leq i} V_j \xrightarrow{\sim} V^{H_i}$ for each i , and the fact that the sequence $\{H_i\}$ is cofinal in the collection of all of open subgroups of H . \square

2.1.6. Lemma. *Suppose that H satisfies Condition 2.1.3. If V is an admissible smooth H -representation over A , and if W is a G -invariant A -submodule of V , then V/W is again an admissible smooth H -representation over A .*

Proof. This follows from the preceding lemma, and the fact that $V_i \rightarrow (V/W)_i$ is surjective. \square

If V is an A -module equipped with an admissible smooth H -representation, then typically V itself will not be finitely generated as an A -module. Nevertheless, the existence of the decomposition (2.1) allows us to extend many results about finitely generated A -modules to the situation of admissible smooth G -representations.

2.1.7. Lemma. *If H satisfies Condition 2.1.3, and if V is an admissible smooth H -representation for which $V/\mathfrak{m}V = 0$, then $V = 0$.*

Proof. The decomposition (2.1) yields the isomorphism $\bigoplus_i V_i/\mathfrak{m}V_i \xrightarrow{\sim} V/\mathfrak{m}$. Thus $V/\mathfrak{m}V = 0$ implies that $V_i/\mathfrak{m}V_i = 0$ for each value of i . Since each V_i is finitely generated over A , this in turn implies that $V_i = 0$ for each i , by Nakayama's lemma. Thus $V = 0$, as claimed. \square

2.1.8. Lemma. *If H satisfies Condition 2.1.3, and if V is an admissible smooth representation such that $V/\mathfrak{m}V$ is finitely generated over $k[H]$, then V is finitely generated over $A[H]$.*

Proof. Let $S \subset A$ be a finite subset whose image in $V/\mathfrak{m}V$ generates this quotient over $k[H]$, and let W be the $A[H]$ -submodule of V generated by S . Lemma 2.1.6 implies that (V/W) is admissible, and by construction we see that $(V/W)/\mathfrak{m}(V/W) = 0$. Thus Lemma 2.1.7 shows that $W = V$, and so V is also finitely generated. \square

It will be technically useful to consider a related notion of admissible representation.

2.1.9. Definition. If V is an A -module equipped with an A -linear representation of H , we say that V is an admissible continuous H -representation if:

- (1) V is \mathfrak{m} -adically complete and separated.
- (2) The H -action on V is continuous, when V is equipped with its \mathfrak{m} -adic topology (i.e. the action map $H \times V \rightarrow V$ is jointly continuous).
- (3) The induced H -representation on $V/\mathfrak{m}V$ (which is automatically smooth, by (1)) is admissible smooth.

2.1.10. Lemma. *Suppose that H satisfies Condition 2.1.3. If V is a continuous admissible H -representation over A , then for each $n > 0$, the induced H -representation on $V/\mathfrak{m}^n V$ is admissible smooth.*

Proof. Condition (2) of Definition 2.1.9 implies that the H -action on $V/\mathfrak{m}^n V$ is continuous, when the latter is equipped with its discrete topology. In other words, $V/\mathfrak{m}^n V$ is a smooth representation of H . Since the formation of H_i -invariants is exact, for any $i \geq 0$, we find that $(V/\mathfrak{m}^n V)^{H_i}/\mathfrak{m}(V/\mathfrak{m}^n V)^{H_i} \xrightarrow{\sim} (V/\mathfrak{m}V)^{H_i}$ is finite dimensional over A/\mathfrak{m} , by Condition (3) of Definition 2.1.9. Lemma 2.1.16 below (applied to the module $(V/\mathfrak{m}^n V)^{H_i}$ of the Artinian local ring A/\mathfrak{m}^n) then shows that $(V/\mathfrak{m}^n V)^{H_i}$ is finitely generated over A/\mathfrak{m}^n . Consequently $V/\mathfrak{m}^n V$ is admissible. \square

2.1.11. Definition. If V is an A -module, we let \widehat{V} denote the \mathfrak{m} -adic completion of V .

2.1.12. Definition. If V is an A -module equipped with an H -representation, we let V_{sm} denote the subset of V consisting of vectors which are smooth, i.e. which are fixed by some open subgroup of H . One immediately verifies that V_{sm} is an A -submodule of V , closed under the action of H . Thus V_{sm} is a smooth H -representation.

2.1.13. Proposition. *Suppose that H satisfies Condition 2.1.3, and let V be an admissible smooth H -representation over A .*

- (1) \widehat{V} is a continuous admissible H -representation over A .
- (2) If A is \mathfrak{m} -adically complete, then the natural map $V \rightarrow \widehat{V}_{\text{sm}}$ is an isomorphism.

Proof. The H -action on V is smooth, and thus so is the H -action on $V/\mathfrak{m}^n V$, for each $n \geq 0$. Passing to the projective limit over n , we find that the H -action on \widehat{V} is \mathfrak{m} -adically continuous. Since $\widehat{V}/\mathfrak{m}\widehat{V} = V/\mathfrak{m}V$, it follows from Lemma 2.1.6 that the H -action on $\widehat{V}/\mathfrak{m}\widehat{V}$ is admissible. Thus \widehat{V} satisfies both the conditions of Definition 2.1.9. This proves (1).

We now turn to proving (2), and so in particular, assume that A is \mathfrak{m} -adically complete. For each $i \geq 0$, we find that

$$\widehat{V}^{H_i} \xrightarrow{\sim} \varprojlim_n (V/\mathfrak{m}^n V)^{H_i} \xrightarrow{\sim} \varprojlim_n V^{H_i}/\mathfrak{m}^n V^{H_i} \xrightarrow{\sim} V^{H_i}$$

(the second isomorphism following from the exactness of the formation of H_i -invariants, and the third following from the fact that V^{H_i} is finitely generated over A , by assumption, and hence \mathfrak{m} -adically complete, since A is \mathfrak{m} -adically complete). Consequently, the map $V^{H_i} \rightarrow \widehat{V}^{H_i}$ is an isomorphism for each $i \geq 0$, and thus, passing to the inductive limit over i , we find that $V \xrightarrow{\sim} \widehat{V}_{\text{sm}}$, as claimed. \square

2.1.14. Proposition. *Suppose that H satisfies Condition 2.1.3 and that A is \mathfrak{m} -adically complete, and let V be an admissible continuous H -representation over A .*

- (1) V_{sm} is an admissible smooth H -representation.
- (2) The natural map $\widehat{V}_{\text{sm}} \rightarrow V$ is an isomorphism.

Proof. Since V is \mathfrak{m} -adically complete and separated, we see that V^{H_i} is \mathfrak{m} -adically complete and separated for each $i \geq 0$. Since the formation of H_i -invariants is exact, we see that $V^{H_i}/\mathfrak{m}V^{H_i} \xrightarrow{\sim} (V/\mathfrak{m}V)^{H_i}$, which by assumption is finite dimensional

over A/\mathfrak{m} . Lemma 2.1.16 below then implies that V^{H_i} is finitely generated over A . Since $(V_{\text{sm}})^{H_i} = V^{H_i}$ by the very definition of V_{sm} , we see that V_{sm} is admissible, proving (1).

If $i \geq 0$ and $n > 0$, then

$$(V_{\text{sm}})^{H_i}/\mathfrak{m}^n(V_{\text{sm}})^{H_i} = V^{H_i}/\mathfrak{m}^n V^{H_i} \xrightarrow{\sim} (V/\mathfrak{m}^n V)^{H_i},$$

the equality holding (as was already noted above) by the very definition of V_{sm} , and the isomorphism following from the exactness of the formation of H_i -invariants. Passing to the inductive limit over i , and taking into account the fact that V_{sm} and $V/\mathfrak{m}^n V$ are both smooth H -representations, we find that $V_{\text{sm}}/\mathfrak{m}^n V_{\text{sm}} \xrightarrow{\sim} V/\mathfrak{m}^n V$. Passing to the projective limit over n , we find that $\widehat{V_{\text{sm}}} \xrightarrow{\sim} V$, proving (2). \square

2.1.15. Remark. It follows from the preceding propositions that if A is \mathfrak{m} -adically complete, then the functors $V \mapsto \widehat{V}$ and $V \mapsto V_{\text{sm}}$ are mutually quasi-inverse, and induce an equivalence of categories between the category of admissible smooth H -representations over A , and the category of admissible continuous H -representations over A .

We close this subsection by recalling a version of Nakayama's lemma in the setting of \mathfrak{m} -adically separated modules over complete local rings.

2.1.16. Lemma. *Suppose that A is \mathfrak{m} -adically complete. If M is an \mathfrak{m} -adically separated A -module such that $M/\mathfrak{m}M$ is finite dimensional over A/\mathfrak{m} , then M is finitely generated over A .*

Proof. Choose $S = \{s_1, \dots, s_m\} \subset M$ to be finite, and such that the image of S in $M/\mathfrak{m}M$ spans M over A/\mathfrak{m} , and let N denote the A -submodule of M generated by S . We then have that $M = N + \mathfrak{m}M$, and so arguing inductively, for each $v \in M$ we may find, for each $i = 1, \dots, s$, a sequence of elements $a_{i,n}$ of A with $a_{i,n} \in \mathfrak{m}^n$ for each n , such that for each n we have

$$v \in \sum_{i=1}^m (a_{i,0} + a_{i,1} + \dots + a_{i,n})s_i + \mathfrak{m}^{n+1}M.$$

Writing $a_i = a_{i,0} + a_{i,1} + \dots$ (a well-defined element of A , since A is \mathfrak{m} -adically complete by assumption), we then find (since M is \mathfrak{m} -adically separated) that $v = \sum_{i=1}^s a_i s_i \in N$, and thus that $M = N$, proving the lemma. \square

2.2. Invariant lattices. Let \mathcal{O} be a complete discrete valuation ring, with field of fractions \mathcal{K} and residue field K of characteristic different from ℓ . Let ϖ be a choice of uniformizer of \mathcal{O} . If V is a \mathcal{K} -vector space, then by a lattice in V we mean an \mathcal{O} -submodule V° which spans V over \mathcal{K} .

2.2.1. Definition. We say that a representation V of a group H over \mathcal{K} is a good integral representation if V contains a ϖ -adically separated H -invariant lattice V° with the property that $\overline{V}^\circ := V^\circ/\varpi V^\circ$ has finite length as a $K[H]$ -module.

We now prove some basic results pertaining to this definition.

2.2.2. Lemma. *Any subrepresentation of a good integral representation V of H over \mathcal{K} is again a good integral representation of H .*

Proof. If V° is a ϖ -adically separated H -invariant lattice in V for which \overline{V}° is of finite length over $K[H]$, then $W^\circ := V^\circ \cap W$ is a ϖ -adically separated H -invariant lattice in W , and since the natural map $\overline{W}^\circ \rightarrow \overline{V}^\circ$ is injective, we see that \overline{W}° also has finite length over $K[H]$. \square

2.2.3. Lemma. *Let V be a good integral representation of a group H , and fix a ϖ -adically separated H -invariant lattice $V^\circ \subset V$ such that \overline{V}° has finite length over $K[H]$. If M is an H -invariant \mathcal{O} -submodule of V/V° , then either M is of bounded exponent as an \mathcal{O} -module, or else M contains a non-zero H -invariant divisible \mathcal{O} -submodule.*

Proof. If M is not of bounded exponent, then the map $M[\varpi^n] \rightarrow M[\varpi^m]$ induced by multiplication by ϖ^{n-m} has non-zero image for each $n \geq m \geq 1$, and hence, since each $M[\varpi^n]$ has finite length, we find that $\varinjlim_n M[\varpi^n] \neq 0$ (the transition maps being given by multiplication by ϖ). If $(m_n)_{n \geq 0}$ is a non-zero element of this projective limit, then the \mathcal{O} -submodule of M generated by the elements m_n is evidently non-zero and divisible. Thus the maximal divisible submodule of M is non-zero; it is also clearly H -invariant. \square

2.2.4. Lemma. *If H is a topological group satisfying Condition 2.1.3, and if V is a good integral admissible smooth representation of H over \mathcal{K} , then:*

- (1) *Any two ϖ -adically separated H -invariant lattices in V are commensurable.*
- (2) *If V° is a ϖ -adically separated H -invariant lattice in V , then V° is finitely generated over $\mathcal{O}[H]$, the $K[H]$ -module $\overline{V}^\circ := V^\circ/\varpi V^\circ$ is of finite length, and the isomorphism class of $(\overline{V}^\circ)^{ss}$ (the semisimplification of \overline{V}° as a $k[H]$ -module) is independent of the choice of V° .*

Proof. Since V is good integral by assumption, we may and do choose an ϖ -adically separated H -invariant lattice $V^\circ \subset V$ such that \overline{V}° is of finite length. Then \overline{V}° is certainly finitely generated over $K[H]$, and so Lemma 2.1.8 implies that V° is finitely generated over $\mathcal{O}[H]$.

Let V^\diamond be another ϖ -adically separated H -invariant lattice in V . We will prove that V^\diamond is commensurable with V° . This will prove (1). An easy (and standard) argument then proves that \overline{V}^\diamond is of finite length over $K[H]$, and that $(\overline{V}^\circ)^{ss}$ and $(\overline{V}^\diamond)^{ss}$ are isomorphic. Also Lemma 2.1.8 will imply that V^\diamond is finitely generated over $\mathcal{O}[H]$. Thus (2) will also follow.

Since V° is finitely generated over $\mathcal{O}[H]$, we may find $m \geq 0$ such that $V^\circ \subset \varpi^{-m}V^\diamond$. In proving the commensurability of V° and V^\diamond , it is clearly no loss of generality to replace V^\diamond by $\varpi^{-m}V^\diamond$, and so we may and do assume for the remainder of the proof that $V^\circ \subset V^\diamond$.

Consider now the quotient $V^\diamond/V^\circ \subset V/V^\circ$. If V^\diamond/V° is not of bounded exponent, then Lemma 2.2.3 shows that it contains a non-zero H -invariant divisible submodule D . The Tate module $T_p D := \varinjlim_n D[\varpi^n]$ (the transition maps being given by multiplication by ϖ) is then a non-zero ϖ -adically complete and separated \mathcal{O} -module, equipped with an action of H , and an injection

$$(2.2) \quad T_p D \hookrightarrow T_p(V/V^\circ) \xrightarrow{\sim} \widehat{V}^\circ.$$

Now

$$T_p D / \varpi T_p D \xrightarrow{\sim} D[\varpi] \subset \frac{1}{\varpi} V^\circ / V^\circ,$$

and hence $T_p D / \varpi T_p D$ is an admissible smooth H -representation. Thus $T_p D$ is an admissible continuous H -representation over \mathcal{O} , and so by Remark 2.1.15, the injection (2.2) is obtained from the induced embedding $(T_p D)_{\text{sm}} \hookrightarrow V^\circ$ by passing to ϖ -adic completions. In particular $(T_p D)_{\text{sm}} \neq 0$. Also, the image of the composite

$$\mathcal{K} \otimes_{\mathcal{O}} (T_p D)_{\text{sm}} \rightarrow V \rightarrow V/V^\circ$$

is precisely D , and so (since $D \subset V^\circ/V^\circ$), we conclude that $\mathcal{K} \otimes_{\mathcal{O}} (T_p D)_{\text{sm}} \subset V^\circ$. But $\mathcal{K} \otimes_{\mathcal{O}} (T_p D)_{\text{sm}}$ is a non-zero \mathcal{K} -vector space, and hence is not ϖ -adically separated. This contradicts our assumption on V° , and hence we conclude that V°/V° is indeed of bounded exponent, and thus that V° and V° are commensurable, as required. \square

2.2.5. Definition. Let H be a topological group satisfying Condition 2.1.3, and let V be a good integral admissible smooth representation of H over \mathcal{K} . If V° is a ϖ -adically separated H -invariant lattice in V such that $\overline{V}^\circ := V^\circ/\varpi V^\circ$ is of finite length as a $K[H]$ -module (which exists, by assumption), then we write \overline{V}^{ss} to denote the semisimplification of \overline{V}° as a $K[H]$ -module. (The preceding lemma shows that, up to isomorphism, \overline{V}^{ss} is independent of the choice of V° .)

The following lemma will allow us to choose lattices in good integral admissible smooth representations whose reductions modulo ϖ have certain specified H -socles.

2.2.6. Lemma. *Let H be a topological group satisfying Condition 2.1.3, and let V be a good integral admissible smooth representation of H over \mathcal{K} . Let \mathcal{S} denote the set of isomorphism classes of Jordan–Hölder factors of \overline{V}^{ss} (as a $K[H]$ -module; the discussion of Definition 2.2.5 shows that this set is well-defined), let \mathcal{T} be a subset of \mathcal{S} , and suppose that V contains no non-zero subrepresentation W (necessarily also good integral, by Lemma 2.2.2) such that every Jordan–Hölder factor of \overline{W}^{ss} belongs to \mathcal{T} . Then there exists a ϖ -adically separated H -invariant lattice V° contained in V° with the property that $\overline{V}^\circ := V^\circ/\varpi V^\circ$ contains no subobject isomorphic to an element of \mathcal{T} .*

Proof. Choose (as we may, by assumption) a ϖ -adically separated H -invariant lattice V° with the property that $\overline{V}^\circ := V^\circ/\varpi V^\circ$ is of finite length as a $K[H]$ -module. Let $M \subset V/V^\circ$ be the maximal $\mathcal{O}[H]$ -submodule all of whose Jordan–Hölder factors are isomorphic to an element of \mathcal{T} .

If we form the projective limit $\varprojlim_n M[\varpi^n] \neq 0$ (the transition maps being given by multiplication by ϖ), then $\varprojlim_n M[\varpi^n] \hookrightarrow \varprojlim_n \frac{1}{\varpi^n} V^\circ/V^\circ \xrightarrow{\sim} \widehat{V}^\circ$, with saturated and ϖ -adically complete image. If we write $W = \mathcal{K} \otimes_{\mathcal{O}} (\varprojlim_n M[\varpi^n])_{\text{sm}}$, then Remark 2.1.15 implies that W vanishes if and only if $\varprojlim_n M[\varpi^n]$ does. On the other hand, by construction W is a subrepresentation of V with the property that all the Jordan–Hölder factors of \overline{W}^{ss} belong to \mathcal{T} , and so by assumption W must vanish. Thus $\varprojlim_n M[\varpi^n] = 0$, and so Lemma 2.2.3 implies that M is of bounded exponent, say $M = M[\varpi^n]$.

Let V^\diamond denote the preimage of M in V . Since $V^\circ \subset V^\diamond \subset \varpi^{-n} V^\circ$, we see that V^\diamond is ϖ -adically separated. Since $\overline{V}^\diamond \xrightarrow{\sim} \varpi^{-1} V^\diamond/V^\diamond \hookrightarrow V/M$, our choice of M ensures that \overline{V}^\diamond contains no subobject isomorphic to an element of \mathcal{T} . \square

3. REPRESENTATION THEORY — THE CASE OF GL_n

3.1. Kirillov models. Let k be a perfect field. Let ℓ be a prime distinct from the characteristic of k , and let \tilde{k} be a Galois extension of k containing all ℓ -power roots of unity.

In this section we set $G = \mathrm{GL}_n(E)$, where E is a non-archimedean local field of residue characteristic ℓ . We will define a notion of Kirillov models for smooth representations of G over a $W(k)$ -algebra A .

We begin by recalling the basic properties of Kirillov models associated to smooth $W(\tilde{k})[G]$ -modules.

In the case of smooth $\mathbb{C}[G]$ -modules, these results are found in [1, §4]; over more general algebraically closed fields they can be found in [17, Ch. III.1]. The extension of these results to coefficients in $W(\tilde{k})$ is more or less immediate. We summarize the key facts:

Define subgroups P_n and N_n of $\mathrm{GL}_n(E)$ by setting:

$$\begin{aligned} P_n &= \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathrm{GL}_{n-1}(E), \quad b \in E^{n-1} \right\}, \\ N_n &= \left\{ \begin{pmatrix} \mathrm{Id}_{n-1} & b \\ 0 & 1 \end{pmatrix} \mid b \in E^{n-1} \right\}, \end{aligned}$$

We consider $\mathrm{GL}_{n-1}(E)$ as a subgroup of P_n in the obvious way, and identify N_n with E^{n-1} . Note that $P_n = \mathrm{GL}_{n-1}(E)N_n$. Any character $\psi : E^{n-1} \rightarrow W(\tilde{k})^\times$ induces a character of N_n via

$$\begin{pmatrix} \mathrm{Id}_{n-1} & b \\ 0 & 1 \end{pmatrix} \mapsto \psi(b),$$

which we again denote by ψ .

We fix, for the remainder of this section, a character $\psi : E \rightarrow W(\tilde{k})^\times$ whose kernel is equal to the subgroup \mathcal{O}_E of E . We consider ψ as a character of E^{n-1} by setting $\psi(e_1, \dots, e_{n-1}) = \psi(e_{n-1})$, and also as a character of N_n via the isomorphism of E^{n-1} with N_n . The subgroup $\mathrm{GL}_{n-1}(E)$ of P_n normalizes N_n , and therefore acts on the set of characters of N_n by conjugation. The stabilizer of ψ under this action is the subgroup P_{n-1} of $\mathrm{GL}_{n-1}(E)$.

3.1.1. Definition. For a $W(\tilde{k})$ -algebra A , let $\mathrm{Rep}_A(G)$ denote the category of smooth $A[G]$ -modules. Define functors $\Psi^-, \Psi^+, \Phi^-, \Phi^+, \hat{\Phi}^+$ by:

- $\Psi^- : \mathrm{Rep}_{W(\tilde{k})}(P_n) \rightarrow \mathrm{Rep}_{W(\tilde{k})}(\mathrm{GL}_{n-1}(E))$ is given by $\Psi^-(V) = V_{N_n}$, the module of N_n -coinvariants of V .
- $\Psi^+ : \mathrm{Rep}_{W(\tilde{k})}(\mathrm{GL}_{n-1}(E)) \rightarrow \mathrm{Rep}_{W(\tilde{k})}(P_n)$ is the functor that takes a GL_{n-1} -module V and extends the action of GL_{n-1} to P_n by letting N_n act trivially.
- $\Phi^- : \mathrm{Rep}_{W(\tilde{k})}(P_n) \rightarrow \mathrm{Rep}_{W(\tilde{k})}(P_{n-1})$ is given by $\Phi^-(V) = V_\psi$, where V_ψ is the largest quotient of V on which N_n acts by ψ . (As P_{n-1} normalizes ψ , V_ψ is naturally a P_{n-1} -module.)
- $\Phi^+ : \mathrm{Rep}_{W(\tilde{k})}(P_{n-1}) \rightarrow \mathrm{Rep}_{W(\tilde{k})}(P_n)$ is given by

$$\Phi^+(V) = \mathrm{c-Ind}_{P_{n-1}N_n}^{P_n} V',$$

where V' is the $P_{n-1}N_n$ -module obtained from V by letting N_n act via ψ , and $\mathrm{c-Ind}$ denotes smooth induction with compact support.

- $\hat{\Phi}^+ : \text{Rep}_{W(\tilde{k})}(P_{n-1}) \rightarrow \text{Rep}_{W(\tilde{k})}(P_n)$ is given by

$$\hat{\Phi}^+(V) = \text{Ind}_{P_{n-1}N_n}^{P_n} V'.$$

The natural surjections of V onto Ψ^-V and of V onto Φ^-V are $\text{GL}_{n-1}(E)$ and P_{n-1} -equivariant, respectively.

3.1.2. Remark. Note that these functors differ from the ones defined in [1] in that they are not “normalized.” More precisely, the functors defined in [1] are twists of the above functors by the square roots of certain modulus characters. This makes them unsuitable for most of our purposes, as the descent arguments we make at the end of this section do not apply to the twisted functors defined in [1]. We will thus use the “non-normalized” functors defined above throughout the bulk of the paper.

An unfortunate exception to this is in the proof of Proposition 4.3.2. While it would in principle be possible to give a proof of Proposition 4.3.2 using the non-normalized functors that we use elsewhere, the normalization of [1] simplifies the combinatorics immensely. We have thus chosen to adopt this normalization for the purposes of that proof only.

The arguments of [1, §3.2] carry over to this setting to show:

- 3.1.3. Proposition.** (1) *The functors Ψ^- , Ψ^+ , Φ^- , Φ^+ , $\hat{\Phi}^+$ are exact.*
(2) *$\hat{\Phi}^+$ is left adjoint to Φ^- , Ψ^- is left adjoint to Ψ^+ , and Φ^- is left adjoint to $\hat{\Phi}^+$.*
(3) *$\Psi^- \hat{\Phi}^+ = \Phi^- \Psi^+ = 0$.*
(4) *The composite functors $\Psi^- \Psi^+$, $\Phi^- \hat{\Phi}^+$, and $\Phi^- \Phi^+$ are naturally isomorphic to identity functors.*
(5) *One has an exact sequence of functors:*

$$0 \rightarrow \hat{\Phi}^+ \Phi^- \rightarrow \text{Id} \rightarrow \Psi^+ \Psi^- \rightarrow 0.$$

For our purposes, it will be necessary to have versions of these functors for representations over $W(k)$, rather than $W(\tilde{k})$. The key difficulty is that the character ψ is not defined over $W(k)$. Nonetheless, one has:

3.1.4. Proposition. *The functors Ψ^- , Ψ^+ , Φ^- , Φ^+ , $\hat{\Phi}^+$ descend to functors defined on representations over $W(k)$. That is, one has a functor:*

$$\Psi^- : \text{Rep}_{W(k)}(P_n) \rightarrow \text{Rep}_{W(k)}(\text{GL}_{n-1}(E))$$

such that for any $W(k)[P_n]$ -module V , one has

$$\Psi^-(V \otimes_{W(k)} W(\tilde{k})) = \Psi^-(V) \otimes_{W(k)} W(\tilde{k}),$$

and similarly for the remaining functors. Moreover, the statements of Proposition 3.1.3 apply to these functors.

Proof. For the functors Ψ^- and Ψ^+ this is clear, as the character ψ does not intervene in their definition. We thus begin with the functor Φ^- .

Note that $\text{Gal}(\tilde{k}/k)$ acts naturally on $W(\tilde{k})$, and fixes $W(k)$. Moreover, $W(\tilde{k})$ is faithfully flat over $W(k)$. For $\sigma \in \text{Gal}(\tilde{k}/k)$, let ψ^σ be the character $\sigma \circ \psi$ of E^\times . There exists a unique $e_\sigma \in \mathcal{O}_E^\times$ such that $\psi^\sigma(e) = \psi(e_\sigma e)$. The map $\sigma \mapsto e_\sigma$ is a homomorphism from $\text{Gal}(\tilde{k}/k)$ to \mathcal{O}_E^\times .

Consider each e_σ as an element of P_{n-1} via the inclusions:

$$\mathcal{O}_E^\times \subset E^\times \subset \mathrm{GL}_{n-1}(E) \subset P_{n-1}.$$

If we consider ψ as a character of N_n , we have $\psi^\sigma(u) = \psi(e_\sigma u e_\sigma^{-1})$ for all $u \in N_n$.

Now let V be a smooth P_n -representation over $W(k)$, and let \tilde{V} be the representation $V \otimes_{W(k)} W(\tilde{k})$. Then we have a $W(\tilde{k})$ -semilinear action of $\mathrm{Gal}(\tilde{k}/k)$ on \tilde{V} , that fixes V . By definition, $\Phi^- \tilde{V}$ is the quotient of \tilde{V} by the $W(\tilde{k})[P_{n-1}]$ -submodule generated by all vectors of the form $uv - \psi(u)v$, for $u \in N_n$ and $v \in \tilde{V}$.

The Galois action on \tilde{V} does not descend to $\Phi^- \tilde{V}$, but a twist of it does. For an element $[v] \in \Phi^- \tilde{V}$, represented by an element v of \tilde{V} , define $\sigma[v] = [e_\sigma \sigma v]$. This is well-defined, since if $v = uw - \psi(u)w$ for some $w \in \tilde{V}$ and $u \in N_n$, we can set $w' = e_\sigma \sigma w$, $u' = e_\sigma u e_\sigma^{-1}$, and then $e_\sigma \sigma v = u' w' - \psi(u') w'$. We thus obtain a $W(\tilde{k})$ -semilinear action of $\mathrm{Gal}(\tilde{k}/k)$ on $\Phi^- \tilde{V}$; we define $\Phi^- V$ to be the invariants under this action. This is clearly functorial with the desired properties. (Note, however, that the surjection $\tilde{V} \rightarrow \Phi^- \tilde{V}$ does *not* descend to a natural surjection of V onto $\Phi^- V$.)

Now let V be a smooth P_{n-1} -representation over $W(k)$. Then $\Phi^+ \tilde{V}$ and $\hat{\Phi}^+ \tilde{V}$ can both be realized as spaces of functions: $f : P_n \rightarrow \tilde{V}$, such that for any $h \in P_{n-1}$ and any u in N_n , we have $f(ghu) = \psi(u)hf(g)$. Define an action of $\mathrm{Gal}(\tilde{k}/k)$ on the space of such functions by setting $(\sigma f)(g) = e_\sigma^{-1} \sigma f(g)$. This preserves the identity $f(ghu) = \psi(u)hf(g)$, and so defines a $W(\tilde{k})$ -semilinear action of $\mathrm{Gal}(\tilde{k}/k)$ on $\Phi^+ \tilde{V}$ and $\hat{\Phi}^+ \tilde{V}$. We set $\Phi^+ V$ and $\hat{\Phi}^+ V$ to be the invariants of this action in $\Phi^+ \tilde{V}$ and $\hat{\Phi}^+ \tilde{V}$, respectively. Note that if V is a smooth P_n -representation, then the natural maps: $\Phi^+ \Phi^- \tilde{V} \rightarrow \tilde{V}$ and $\tilde{V} \rightarrow \hat{\Phi}^+ \Phi^- \tilde{V}$ are $\mathrm{Gal}(\tilde{k}/k)$ -equivariant, and hence descend to V .

Using this, one easily verifies the adjointness property (2) of Proposition 3.1.3 for the functors over $W(k)$. Properties (1), (3), (4), and (5) then follow by base change and the fact that $W(\tilde{k})$ is faithfully flat over $W(k)$. \square

Note that if A is a Noetherian $W(k)$ -algebra, and V is a smooth representation of P^{n-1} over A , then the modules $\Psi^- V$, $\Phi^- V$ obtained by treating V as a representation of P^{n-1} over $W(k)$ and applying the appropriate functors inherit an A -module structure. We can thus define the functors Ψ^- , Ψ^+ , Φ^- , etc. on suitable categories of smooth representations over A . It is then clear that if B is a Noetherian A -algebra, one has $\Psi^-(V \otimes_A B) = \Psi^-(V) \otimes_A B$, and similarly for the remaining functors.

Finally, observe that the functors Ψ^- , Ψ^+ , Φ^- , Φ^+ , $\hat{\Phi}^+$ commute with tensor products; that is, if M is an A -module, then $\Psi^-(V \otimes_A M) \cong \Psi^-(V) \otimes_A M$, and similarly for the other functors.

We now define the ‘‘derivatives’’ of a smooth P_n -representation V . For $0 \leq r \leq n$, we set $V^{(r)} = \Psi^-(\Phi^-)^{r-1} V$; $V^{(r)}$ is a representation of $\mathrm{GL}_{n-r}(E)$. If A is a $W(\tilde{k})$ -algebra, then one has a $\mathrm{GL}_{n-r}(E)$ -equivariant surjection $V \rightarrow V^{(r)}$ (but this is not true if A is only a $W(k)$ -algebra.)

Note that $V^{(n)}$ is simply an A -module. The adjointness properties of Proposition 3.1.3 give, for any V , maps:

$$V \rightarrow (\hat{\Phi}^+)^{(n-1)} \Psi^+(V^{(n)}).$$

$$(\Phi^+)^{n-1}\Psi^+(V^{(n)}) \rightarrow V.$$

(Note that in the setting of the second of these maps, the Ψ^+ that appears, and the Ψ^- that arises in the definition of $V^{(n)}$, are both the identity functor, and thus the fact that Ψ^+ is usually a right adjoint to Ψ^- rather than a left adjoint is not an issue.) The image of V in $(\hat{\Phi}^+)^{(n-1)}\Psi^+(V^{(n)})$ is called the Kirillov model of V .

The exact sequence of Proposition 3.1.3 implies that the map

$$(\Phi^+)^{n-1}\Psi^+(V^{(n)}) \rightarrow V$$

is injective; we denote its image by $\mathfrak{J}(V)$. The space $\mathfrak{J}(V)$ is often referred to as the space of Schwartz functions in V .

3.1.5. Lemma. *Let V be a smooth P_n -module over A . Set $\tilde{A} = A \otimes_{W(k)} W(\tilde{k})$, and let $\tilde{V} = V \otimes_{W(k)} W(\tilde{k})$. The modules V and $\mathfrak{J}(V)$ each contain an A -submodule W such that W is isomorphic to $V^{(n)}$, and $W \otimes_A \tilde{A}$ maps isomorphically to $\tilde{V}^{(n)}$ under the surjection $\tilde{V} \rightarrow \tilde{V}^{(n)}$.*

Proof. It suffices to show that $\mathfrak{J}(V)$ contains such a submodule, as $\mathfrak{J}(V)$ embeds in V . We have

$$\mathfrak{J}(V) = (\Phi^+)^{n-1}\Psi^+V^{(n)} = [(\Phi^+)^{n-1}\Psi^+W(k)] \otimes_{W(k)} V^{(n)}.$$

Moreover, the map $\mathfrak{J}(\tilde{V}) \rightarrow \tilde{V}^{(n)}$ arises from the surjection:

$$(\Phi^+)^{n-1}\Psi^+W(\tilde{k}) \rightarrow W(\tilde{k})$$

by tensoring over $W(\tilde{k})$ with $\tilde{V}^{(n)}$. It thus suffices to construct a submodule W of $[(\Phi^+)^{n-1}\Psi^+W(k)]$ that is free of rank one over $W(k)$ and such that $W \otimes_{W(k)} W(\tilde{k})$ maps isomorphically onto $W(\tilde{k})$ under the surjection

$$(\Phi^+)^{n-1}\Psi^+W(\tilde{k}) \rightarrow W(\tilde{k}).$$

This amounts to simply choosing any element of $(\Phi^+)^{(n-1)}\Psi^+W(k)$ that maps to an element of $W(\tilde{k}) \setminus \varpi W(\tilde{k})$, where ϖ is the uniformizer of $W(\tilde{k})$. Such an element clearly exists, as otherwise the image of the composition:

$$[(\Phi^+)^{n-1}\Psi^+W(k)] \otimes_{W(k)} W(\tilde{k}) \xrightarrow{\sim} (\Phi^+)^{n-1}\Psi^+W(\tilde{k}) \rightarrow W(\tilde{k})$$

would be contained in $\varpi W(\tilde{k})$. \square

Over a field, the top derivatives are multiplicative with respect to parabolic induction:

3.1.6. Proposition. *Let V and W be admissible k -representations of $\mathrm{GL}_n(E)$ and $\mathrm{GL}_m(E)$, respectively. Let $P \subset \mathrm{GL}_{n+m}(E)$ be the parabolic subgroup of G given by:*

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a \in \mathrm{GL}_n(E), \quad d \in \mathrm{GL}_m(E) \right\},$$

and consider $V \otimes W$ as a representation of P by letting the unipotent radical of P act trivially. Then

$$(\mathrm{Ind}_P^{\mathrm{GL}_{n+m}(E)} V \otimes W)^{(n+m)} = V^{(n)} \otimes W^{(m)}.$$

Proof. This is a special case of [17, Lem. 1.10]. Note that the derivative functors used in [17] are normalized differently than ours; this normalization does not affect the top derivative $V^{(n)}$ of a representation V of $\mathrm{GL}_n(E)$. \square

We also have:

3.1.7. Theorem. *Let V be an absolutely irreducible admissible representation of $\mathrm{GL}_n(E)$ over a field K that is a $W(k)$ -algebra. Then $V^{(n)}$ is either zero or a one-dimensional K -vector space, and is one-dimensional if V is cuspidal.*

Proof. This is [17, III.5.10], over $\overline{\mathbb{F}}_p$ or $\overline{\mathbb{Q}}_p$; the same argument works over an arbitrary field containing the ℓ -power roots of unity. The general case follows by extending scalars. \square

3.1.8. Definition. *We say that an absolutely irreducible admissible representation V of $\mathrm{GL}_n(E)$ over a field K that is a $W(k)$ -algebra is generic if $V^{(n)}$ is one-dimensional over K .*

We now turn to finiteness properties of the derivative.

3.1.9. Lemma. *Let A be a Noetherian local ring, with maximal ideal \mathfrak{m} , and suppose that M is a submodule of a direct sum of finitely generated A -modules. If $M/\mathfrak{m}M$ is finite dimensional, then M is finitely generated.*

Proof. Our assumption on $M/\mathfrak{m}M$ allows us to choose a finitely generated submodule N of M such that $N + \mathfrak{m}M = M$, or equivalently such that $\mathfrak{m}(M/N) = M/N$. Nakayama's Lemma then shows that any finitely generated quotient of M/N must vanish. Since by assumption M embeds into a direct sum of finitely generated A -modules, we may find a finitely generated A -module X , and an A -module Y which is a direct sum of finitely generated A -modules, and an embedding $M \subset X \oplus Y$, such that $N \subset X$. On the one hand $M/(M \cap X)$ is a quotient of M/N , and hence has no non-vanishing finitely generated quotients. On the other, the projection of $X \oplus Y$ onto Y induces an embedding of $M/(M \cap X)$ into Y . Thus $M/(M \cap X) = 0$, and so $M \subset X$ is finitely generated. \square

We note for future reference the following corollary of the preceding lemma.

3.1.10. Corollary. *Let A be a Noetherian local ring, and suppose that M is a submodule of a direct sum of finitely generated A -modules. If $M/\mathfrak{m}M$ vanishes, then M itself vanishes.*

Proof. The preceding result implies that M is finitely generated. The corollary thus follows from Nakayama's Lemma. \square

3.1.11. Theorem. *Suppose that A is a Noetherian local $W(k)$ -algebra with maximal ideal \mathfrak{m} . Let V be an admissible representation of GL_n over A , and suppose that $(V/\mathfrak{m}V)^{(n)}$ is finite dimensional over A/\mathfrak{m} . Then $V^{(n)}$ is a finitely generated A -module.*

Proof. As derivatives commute with tensor products, we have

$$V^{(n)}/\mathfrak{m}V^{(n)} \cong (V/\mathfrak{m}V)^{(n)}.$$

On the other hand, we have already observed that $\mathfrak{J}(V)$ (and hence V) contains an A -submodule isomorphic to $V^{(n)}$. As V is a direct sum of finitely generated A -modules, the result follows from Lemma 3.1.9. \square

3.1.12. Remark. In the setting of Theorem 3.1.11, if $V/\mathfrak{m}V$ has finite length, then $(V/\mathfrak{m}V)^{(n)}$ is finite dimensional over A/\mathfrak{m} by Theorem 3.1.7. Thus Theorem 3.1.11 applies to all admissible representations of GL_n over A such that $V/\mathfrak{m}V$ has finite length.

Theorem 3.1.11 allows us to establish the following extension of Proposition 3.1.6:

3.1.13. Corollary. *Let A be a reduced Noetherian local $W(k)$ -algebra with maximal ideal \mathfrak{m} . Let V and W be admissible smooth A -representations of $\mathrm{GL}_n(E)$ and $\mathrm{GL}_m(E)$, and let P be the parabolic subgroup of $\mathrm{GL}_{n+m}(E)$ defined in the statement of Proposition 3.1.6. Then, if $V^{(n)}$ and $W^{(m)}$ are free of rank one over A , so is the A -module:*

$$(\mathrm{Ind}_P^{\mathrm{GL}_{n+m}(E)} V \otimes W)^{(n+m)}.$$

Proof. For each minimal prime \mathfrak{a} of A , let $\kappa(\mathfrak{a})$ be its residue field. Set $V_{\mathfrak{a}} = V \otimes_A \kappa(\mathfrak{a})$ and $W_{\mathfrak{a}} = W \otimes_A \kappa(\mathfrak{a})$. We have isomorphisms:

$$(\mathrm{Ind}_P^{\mathrm{GL}_{n+m}(E)} V \otimes W)^{(n+m)} \otimes_A \kappa(\mathfrak{a}) \xrightarrow{\sim} (\mathrm{Ind}_P^{\mathrm{GL}_{n+m}(E)} V_{\mathfrak{a}} \otimes W_{\mathfrak{a}})^{(n+m)},$$

and the latter is one-dimensional by Proposition 3.1.6. Thus in particular the annihilator of $(\mathrm{Ind}_P^{\mathrm{GL}_{n+m}(E)} V \otimes W)^{(n+m)}$ as an A -module is the zero ideal of A .

On the other hand, let $\bar{V} = V/\mathfrak{m}V$ and $\bar{W} = W/\mathfrak{m}W$. Then we have isomorphisms:

$$(\mathrm{Ind}_P^{\mathrm{GL}_{n+m}(E)} V \otimes W)^{(n+m)} \otimes_A A/\mathfrak{m} \xrightarrow{\sim} (\mathrm{Ind}_P^{\mathrm{GL}_{n+m}(E)} \bar{V} \otimes \bar{W})^{(n+m)},$$

and the latter is again one-dimensional by Proposition 3.1.6. Theorem 3.1.11 shows that $(\mathrm{Ind}_P^{\mathrm{GL}_{n+m}(E)} V \otimes W)^{(n+m)}$ is furthermore finitely generated over A , and thus it follows by Nakayama's lemma that $(\mathrm{Ind}_P^{\mathrm{GL}_{n+m}(E)} V \otimes W)^{(n+m)}$ is a cyclic A -module, and hence (taking into account that it is faithful, as we proved above) is free of rank one. \square

We will also need a generalization of this machinery to a product of $\mathrm{GL}_n(E_i)$ for various local fields E_i of residue characteristics ℓ_i , all prime to the residue characteristic of k . Fix a finite collection of such E_i , indexed by a set S , and let G be the product of the groups $\mathrm{GL}_n(E_i)$ for all i . Let P_n be the product of the subgroups $P_n(E_i)$ of $\mathrm{GL}_n(E_i)$.

Now if we fix for each i a character $\psi_i : N_n(E_i) \rightarrow W(\tilde{k})^\times$, we can define functors $\Psi^{-,i}, \Phi^{-,i}, \Psi^{+,i}, \Phi^{+,i}, \hat{\Phi}^{+,i}$ as follows: if H is any topological group, and V is a $P_n(E_i) \times H$ -module, then $\Psi^{-,i}(V)$ is the $\mathrm{GL}_{n-1}(E_i) \times H$ -module defined by applying Ψ^- to V (considered as a $P_n(E_i)$ -module,) and then taking the natural action of H on $\Psi^-(V)$. The other functors are defined similarly. Note that if V is a G -module over A (or even a module over the product of the $P_n(E_i)$), then $\Psi^{-,i}\Psi^{-,j}V = \Psi^{-,j}\Psi^{-,i}V$ (here the equality denotes a natural isomorphism), and the other functors have similar commutativity properties. If S' is a subset of S , the composition of functors $\Psi^{-,i}$ for all $i \in S'$ is thus a functor that takes an A -module V over the product of the groups $P_n(E_i)$ to an A -module with an action of $P_{n-1}(E_i)$ for each $i \in S'$ and of $P_n(E_i)$ for each i not in S' . This composition is independent (up to natural isomorphism) of the order in which we compose the functors; we denote it by $\Psi^{-,S'}$. Similarly define $\Phi^{-,S'}, \Psi^{+,S'}$, etc. Finally, if V is an $A[G]$ -module, define $V^{(n),S'}$ to be the representation $\Psi^{-,S'}(\Phi^{-,S'})^{n-1}V$. Note that the functors $\Phi^{-,S'}, \Psi^{+,S'}$, etc. satisfy analogues of properties (1)-(4) of Proposition 3.1.3.

When it is clear from the context what S and E_i we are working with, we will denote $\Phi^{+,S}, \Psi^{+,S}$, etc. by Φ^+, Ψ^+ , and so forth.

Similarly, if V is an $A[G]$ -module, we define $\mathfrak{J}_i(V)$ to be space of Schwartz functions in V for the action of $\mathrm{GL}_n(E_i)$ on V ; this has an action of $P_n(E_i)$ and of $\mathrm{GL}_n(E_j)$ for j not equal to i . Note that $\mathfrak{J}_i\mathfrak{J}_j(V) = \mathfrak{J}_j\mathfrak{J}_i(V)$, so that we can define, for any $S' \subset S$, the functor $\mathfrak{J}_{S'}$ to be the composition (in any order) of the functors \mathfrak{J}_i for i in S' . Then $\mathfrak{J}_{S'}(V)$ is the smallest A -submodule of V , stable under $P_n(E_i)$ for i in S' and $\mathrm{GL}_n(E_i)$ for i not in S' , such that the map $\mathfrak{J}_S(V)^{(n),S'} \rightarrow V^{(n),S'}$ is an isomorphism. By construction, the functor $\mathfrak{J}_{S'}$ is left adjoint to the functor $V \mapsto V^{(n),S'}$ (when the latter is considered as a functor from $A[P_n]$ -modules to A -modules).

Now, precisely as in the proof of Theorem 3.1.11, we have:

3.1.14. Theorem. *Let V be an admissible representation of G over a Noetherian local $W(k)$ -algebra A , and suppose that $(V/\mathfrak{m}V)^{(n)}$ is finite dimensional over $A/\mathfrak{m}A$. Then $V^{(n)}$ is a finitely generated A -module.*

We also have an analogue of Theorem 3.1.7

3.1.15. Theorem. *Let V be an absolutely irreducible admissible representation of G over k . Then $V^{(n)}$ is either zero or a one-dimensional k -vector space.*

Proof. The representation V splits as a tensor product of absolutely irreducible representations V_i of $\mathrm{GL}_i(E_i)$ for all $i \in I$. It follows that $V^{(n)}$ is the tensor product of the $V_i^{(n)}$. Hence this result is an immediate consequence of Theorem 3.1.7. \square

3.1.16. Proposition. *Let V be an $A[G]$ -module, and suppose that $V^{(n)}$ is free of rank one over A . Then the map $A \rightarrow \mathrm{End}_{P_n}(\mathfrak{J}(V))$ is an isomorphism.*

Proof. By the adjointness properties of the functors Ψ^+ and Φ^+ we have natural isomorphisms:

$$\mathrm{End}_{P_n}(\mathfrak{J}(V)) \xrightarrow{\sim} \mathrm{Hom}_A(V^{(n)}, \Psi^-(\Phi^-)^{n-1}\mathfrak{J}(V)) \xrightarrow{\sim} \mathrm{End}_A(V^{(n)}).$$

The result follows immediately. \square

3.2. Essentially AIG representations. Let K be a field of characteristic different from ℓ .

3.2.1. Definition. We say that a smooth representation V of $G := \mathrm{GL}_n(E)$ is essentially absolutely irreducible and generic (“essentially AIG” for short) if:

- (1) The G -socle $\mathrm{soc}(V)$ is absolutely irreducible and generic.
- (2) The quotient $V/\mathrm{soc}(V)$ contains no generic Jordan–Hölder factors; equivalently, $(V/\mathrm{soc}(V))^{(n)} = 0$.
- (3) The representation V is the sum (or equivalently, the union) of its finite length submodules.

3.2.2. Lemma. (1) *If V is an essentially AIG smooth representation of G , and if $\chi : E^\times \rightarrow k^\times$ is a continuous character, then $(\chi \circ \det) \otimes V$ is again essentially AIG.*

(2) *If V is an essentially AIG smooth G -representation, and if $U \subset V$ is a non-zero smooth G -subrepresentation, then U is also essentially AIG, and furthermore $\mathrm{soc}(U) = \mathrm{soc}(V)$.*

(3) *If U and V are essentially AIG admissible smooth G -representations, then restricting to socles induces an embedding*

$$\mathrm{Hom}_G(U, V) \hookrightarrow \mathrm{Hom}_G(\mathrm{soc}(U), \mathrm{soc}(V)).$$

- (4) *Any non-zero G -equivariant homomorphism between essentially AIG admissible smooth G -representations is an embedding.*

Proof. Claim (1) is clear.

If $U \subset V$ is as in (2), then $0 \neq \text{soc}(U) \subset \text{soc}(V)$. Since the latter is absolutely irreducible, we find that $\text{soc}(U) = \text{soc}(V)$, and so in particular $\text{soc}(U)$ is absolutely irreducible and generic. Furthermore, we see that $U/\text{soc}(U) \hookrightarrow V/\text{soc}(V)$. Since the latter representation contains no generic Jordan–Hölder factors, neither does the former. Finally, every element of U is contained in a finite length submodule of V ; the intersection of this with U is also finite length. Thus U is the union of its finite length submodules, and is therefore essentially AIG, proving (2).

Now suppose that $\phi : U \rightarrow V$ is a map of essentially AIG representations, as in (3). If $\phi(\text{soc}(U)) = 0$, then ϕ factors to induce a map $U/\text{soc}(U) \rightarrow V$. But the source of this map has no generic Jordan–Hölder factors, while its target has generic socle. Thus this map vanishes, and hence ϕ vanishes. This proves (3).

To prove (4), suppose given $\phi : U \rightarrow V$ as above. If $\ker \phi \neq 0$, then it has a non-zero socle. As $\text{soc}(U)$ is irreducible, we conclude that $\text{soc}(U) \subset \ker \phi$. Part (3) then implies that $\phi = 0$. \square

3.2.3. Lemma. *If V is an essentially AIG smooth representation of G over K , and if U is a non-zero submodule of V , then $\text{Hom}_G(U, V)$ is one-dimensional over K . In particular, $\text{End}_G(V) = K$.*

Proof. Part (2) of the preceding lemma shows that U is again essentially AIG and that $\text{soc}(U) = \text{soc}(V)$. Part (3) of the same lemma shows that restriction to socles induces an embedding

$$\text{Hom}_G(U, V) \hookrightarrow \text{Hom}_G(\text{soc}(U), \text{soc}(V)) = \text{End}_G(\text{soc}(V)) = K$$

(where the first equality follows from the already noted equality of socles, and the second equality following from the absolute irreducibility of $\text{soc}(V)$). Since $\text{Hom}_G(U, V)$ is non-zero (as U embeds into V by assumption), it must therefore be one-dimensional, as claimed. \square

3.2.4. Lemma. *If V is an essentially AIG smooth representation of G over K , then V admits a central character.*

Proof. The preceding lemma shows that $\text{Aut}_G(V) = K^\times$. Since the centre Z of G acts as automorphisms of V , the lemma follows. \square

3.2.5. Lemma. *Let V and W be essentially AIG smooth representations of G over K , and let K' be a finite separable extension of K . For any map $f : V \otimes_K K' \rightarrow W \otimes_K K'$, there exists a scalar $c \in (K')^\times$ such that cf descends uniquely to a map $V \rightarrow W$.*

Proof. We may assume K' is Galois over K , and that f is nonzero (and thus injective). For $\sigma \in \text{Gal}(K'/K)$, define f^σ by $f^\sigma(x) = \sigma f(\sigma^{-1}x)$. Then $f^\sigma = c_\sigma f$ for a scalar $c_\sigma \in (k')^\times$. The c_σ give a cocycle in $H^1(\text{Gal}(K'/K), (K')^\times)$ and are therefore a coboundary; that is, there exists a $c \in (K')^\times$ such that $c_\sigma = \frac{c}{\sigma c}$ for all σ . Then cf is Galois-equivariant, and thus descends to K . \square

3.2.6. Definition. If V is an essentially AIG admissible smooth G -representation, then we say that a smooth representation W is an essentially AIG envelope of V if:

- (1) W is itself essentially AIG.

- (2) There is an G -equivariant embedding $V \hookrightarrow W$ (which Lemma 3.2.3 shows is then unique up to multiplication by a non-zero scalar).
- (3) W is maximal with respect to properties (1) and (2), i.e. if $V \hookrightarrow Y$ is any G -equivariant embedding with Y essentially AIG admissible smooth, then there is a G -equivariant embedding $Y \hookrightarrow W$.

3.2.7. Proposition. *If V is an essentially AIG admissible smooth G -representation, then V admits an essentially AIG envelope, which is unique up to isomorphism.*

Proof. Let $V \hookrightarrow I$ be an injective envelope of V in the category of smooth representations. Let U denote the subrepresentation of I/V obtained by taking the sum of all the non-generic subrepresentations of I/V (so U is the maximal subrepresentation of I/V for which $U^{(n)} = 0$), and define X to be the preimage of U in I . Let W be the sum of all of the finite length submodules of X . By construction, the socle of W is generic, $(W/\text{soc}(W))^{(n)} = 0$, and W is the sum of its finite length submodules, so W is essentially AIG.

If $V \hookrightarrow Y$ is an embedding as in (3), then (since I is injective) we may extend the embedding of V into I to an embedding of Y into I . Since every Jordan–Hölder constituent of Y/V is nongeneric, the image of Y lies in X . Moreover, Y is the sum of its finite length submodules, so the image of Y lies in W . \square

If V is an essentially AIG smooth G -representation, then we write $\text{env}(V)$ to denote the essentially AIG envelope of V (which by the preceding proposition exists, and is unique up to isomorphism).

3.2.8. Lemma. *Let V be an essentially AIG admissible smooth G -representation. If $\chi : E^\times \rightarrow K^\times$ is a continuous character, then there is an isomorphism*

$$\text{env}((\chi \circ \det) \otimes V) \xrightarrow{\sim} (\chi \circ \det) \otimes \text{env}(V);$$

i.e. the formation of essentially AIG envelopes is compatible with twisting.

Proof. This is immediate from Lemma 3.2.2. \square

In fact, essentially AIG representations actually have finite length. This will be proven in forthcoming work of the second author [10]. In this paper, we will content ourselves with somewhat weaker results whose proofs are more elementary. In the case when K is of characteristic zero, this finiteness follows from Theorem 4.3.9 below, while in the case when $n = 2$, it is easy to establish for arbitrary K . (See Proposition 3.2.18 below.) In the remainder of this section, we establish a weaker finiteness result for essentially AIG representations that will suffice for our purposes. The key tool will be the notions of cuspidal and supercuspidal support; we recall the definitions below.

Let $\{\pi_1, \dots, \pi_r\}$ be a multiset of irreducible cuspidal representations of the groups $\text{GL}_{n_1}(E), \dots, \text{GL}_{n_r}(E)$, for n_1, \dots, n_r such that $\sum n_i = n$.

3.2.9. Definition. *Let π be an irreducible representation of G over \overline{K} . The multiset $\{\pi_1, \dots, \pi_r\}$ is a cuspidal support of π if there exists a parabolic subgroup $P = MU$ of G , with M isomorphic to the product of the GL_{n_i} , such that π is isomorphic to a quotient of the normalized parabolic induction*

$$\text{Ind}_P^G \pi_1 \otimes \cdots \otimes \pi_r,$$

for some choice of ordering of π_1, \dots, π_r .

It is an easy consequence of the “geometric lemma” of Bernstein-Zelevinski, and adjointness of parabolic induction and restriction, that every irreducible π has a cuspidal support, and this cuspidal support is uniquely determined (as a multiset of isomorphism classes of irreducible representations.) We refer the reader to [17], II.2.4, for the details. By contrast, the notion of supercuspidal support lies considerably deeper:

3.2.10. Definition. *Let π be an irreducible representation of G over \overline{K} . The multiset $\{\pi_1, \dots, \pi_r\}$ is a supercuspidal support of π if each π_i is supercuspidal, and there exists a parabolic subgroup $P = MU$ of G , with M isomorphic to the product of the GL_{n_i} , such that π is isomorphic to a Jordan–Hölder constituent of the normalized parabolic induction*

$$\mathrm{Ind}_P^G \pi_1 \otimes \cdots \otimes \pi_r.$$

In characteristic zero the notions of cuspidal and supercuspidal support coincide. This is false, however, in characteristic p , where there are cuspidal representations that are not supercuspidal. In this setting it is a deep result of Vigneras ([18], V.4) that every irreducible π has a supercuspidal support that is uniquely determined by π .

Let $\mathrm{scs}(\pi)$ (resp. $\mathrm{cs}(\pi)$) denote the supercuspidal support (resp. cuspidal support) of π .

3.2.11. Proposition. *Let $P = MU$ be a parabolic subgroup of G , with M isomorphic to $\prod_i \mathrm{GL}_{n_i}$. Then:*

- (1) *Let π_i be an irreducible admissible representation of GL_{n_i} for all i . If π is an irreducible submodule or quotient of*

$$\mathrm{Ind}_P^G \pi_1 \otimes \cdots \otimes \pi_r,$$

then $\mathrm{cs}(\pi)$ is the multiset sum of $\mathrm{cs}(\pi_i)$ for all i .

- (2) *Let π_i be an irreducible admissible representation of GL_{n_i} for all i . If π is a Jordan–Hölder constituent of*

$$\mathrm{Ind}_P^G \pi_1 \otimes \cdots \otimes \pi_r,$$

then $\mathrm{scs}(\pi)$ is the multiset sum of $\mathrm{scs}(\pi_i)$ for all i .

- (3) *Let π be an irreducible admissible representation of G over \overline{k} , and let $\pi' = \pi_1 \otimes \cdots \otimes \pi_r$ be a Jordan–Hölder constituent of $\mathrm{Res}_G^P \pi$. Then $\mathrm{scs}(\pi)$ is the multiset sum of $\mathrm{scs}(\pi_i)$ for all i .*

Proof. Statement (1) and (2) are easy consequences of the definitions, together with the transitivity of parabolic induction. The details of the proof of (1) can be found in [17], II.2.20, for instance. Statement (3) follows from the “geometric lemma” of Bernstein-Zelevinski (which holds in characteristic p as well as characteristic zero by [17], II.2.19.) \square

3.2.12. Proposition. *Let $\{\pi_1, \dots, \pi_r\}$ be a multiset of supercuspidal representations of GL_{n_i} over \overline{K} . There exists, up to isomorphism, exactly one irreducible generic representation π of G over \overline{K} with supercuspidal support equal to $\{\pi_1, \dots, \pi_r\}$.*

Proof. A representation π with supercuspidal support $\{\pi_1, \dots, \pi_r\}$ is isomorphic to a generic Jordan–Hölder constituent of $\mathrm{Ind}_P^G \pi_1 \otimes \cdots \otimes \pi_r$. By Theorem 3.1.7 and Proposition 3.1.6, the top derivative of $(\mathrm{Ind}_P^G \pi_1 \otimes \cdots \otimes \pi_r)$ is one-dimensional, so it has exactly one generic Jordan–Hölder constituent. \square

We are now in a position to show:

3.2.13. Theorem. *Let π, π' be irreducible admissible representations of G over \overline{K} , or more generally of a Levi subgroup M of G . If $\text{Ext}^i(\pi, \pi')$ is nonzero for some i , then π and π' have the same supercuspidal support.*

Proof. By [18, IV.6.2], the category of smooth representations of M factors as a product of blocks; two irreducible representations of M are in the same block if, and only if, their supercuspidal supports are *inertially equivalent*, that is, if and only if they coincide up to twisting by unramified characters. We thus reduce to the case where the supercuspidal supports of π and π' differ by at most an unramified twist.

First consider the case in which π and π' are both cuspidal representations of G . In this case, Vigneras' classification of the supercuspidal supports of cuspidal representations (see for instance the proof of [18], V.4) shows that π' is an unramified twist of π . In this case, there exists a maximal distinguished cuspidal type $(K, \kappa \otimes \sigma)$ (in the sense of [18], section IV) contained in both π and π' .

We recall some basic facts from [18] about the pair $(K, \kappa \otimes \sigma)$. In particular, K is a compact open subgroup of G , with a normal pro- ℓ subgroup K_1 . Moreover, there exists a finite extension E' of E , an embedding of $\text{GL}_r(\mathcal{O}_{E'})$ into K , and an isomorphism of K/K_1 with $\overline{G} := \text{GL}_r(\mathcal{O}_{E'}/\varpi_{E'})$ such that the composition

$$\text{GL}_r(\mathcal{O}_{E'}) \rightarrow K \rightarrow K/K_1 \cong \overline{G}$$

is equal to the reduction map $\text{GL}_r(\mathcal{O}_{E'}) \rightarrow \overline{G}$. Moreover, κ is a representation of K whose restriction to K_1 is irreducible, and σ is a cuspidal representation of \overline{G} viewed as a representation of K trivial on K_1 .

The composition

$$\mathcal{O}_{E'}^\times \rightarrow \text{GL}_r(\mathcal{O}_{E'}) \rightarrow K$$

extends to a map $(E')^\times \rightarrow G$ such that the G -intertwining of $(K, \kappa \otimes \sigma)$ is precisely $(E')^\times K$. It follows that there are extensions τ and τ' of $\kappa \otimes \sigma$ to $(E')^\times K$ such that $\pi = c - \text{Ind}_{(E')^\times K}^G \tau$ and $\pi' = c - \text{Ind}_{(E')^\times K}^G \tau'$.

The choice of extension τ defines an action of $(E')^\times K/K_1$ on $\text{Hom}_{K_1}(\kappa|_{K_1}, V)$ for any representation V of $(E')^\times K$; this makes the functor $\text{Hom}_{K_1}(\kappa|_{K_1}, -)$ into an equivalence between the category of representations of $(E')^\times K$ whose restriction to K_1 is a direct sum of copies of κ and the category of representations of $(E')^\times K/K_1$.

Frobenius reciprocity and the Mackey formula yield an isomorphism of $\text{Ext}_G^i(\pi, \pi')$ with $\text{Ext}_{(E')^\times K/K_1}^i(\tau, \tau')$. Applying the equivalence of the previous paragraph, and using the fact that τ' is an unramified twist of τ , we find that the latter is isomorphic to $\text{Ext}_{(E')^\times K/K_1}^i(\sigma, \sigma \otimes \chi)$, for some unramified character χ of $(E')^\times K/K_1$. On the other hand, we have an isomorphism of $(E')^\times K/K_1$ with $(\varpi_{E'})^\mathbb{Z} \times \overline{G}$. It is thus clear that $\text{Ext}_{(E')^\times K/K_1}^i(\sigma, \sigma \otimes \chi)$ can only be nonzero if $\varpi_{E'}$ acts on σ and $\sigma \otimes \chi$ via the same scalar; if this happens then σ and $\sigma \otimes \chi$ are isomorphic and so τ is isomorphic to τ' . Then π and π' are isomorphic as claimed.

The case in which π and π' are cuspidal representations of M now follows, as any such representation is a tensor product of cuspidal representations of the factors GL_{n_i} of M .

We handle the remaining cases by induction on i ; the case $i = 0$ is clear. Now assume exactly one of π and π' is cuspidal; without loss of generality (taking duals if necessary) we may assume π is cuspidal. We then have a proper parabolic subgroup

$P' = M'U'$ of M , and a cuspidal representation σ of M' , such that π' arises as a submodule of $\text{Ind}_{P'}^M \sigma$. We thus have an exact sequence:

$$0 \rightarrow \pi' \rightarrow \text{Ind}_{P'}^M \sigma \rightarrow C \rightarrow 0$$

where C is the cokernel of the inclusion of π' in $\text{Ind}_{P'}^M \sigma$. By Frobenius reciprocity $\text{Ext}^j(\pi, \text{Ind}_{P'}^M \sigma) = \text{Ext}^j(\text{Res}_M^{P'} \pi, \sigma)$ for all j , and the latter vanishes because π is cuspidal. Thus $\text{Ext}^i(\pi, \pi')$ is isomorphic to $\text{Ext}^{i-1}(\pi, C)$. By the inductive hypothesis, the latter vanishes unless π has the same supercuspidal support as some subquotient of C ; since every subquotient of C has the same supercuspidal support as π' the claim follows in this case.

Finally, assume that neither π nor π' is cuspidal. We again fix a proper parabolic subgroup $P' = M'U'$ of M , and a cuspidal representation σ of M' , such that π' arises as a submodule of $\text{Ind}_{P'}^M \sigma$. Now the exact sequence:

$$0 \rightarrow \pi' \rightarrow \text{Ind}_{P'}^M \sigma \rightarrow C \rightarrow 0$$

yields an exact sequence:

$$\text{Ext}^{i-1}(\pi, C) \rightarrow \text{Ext}^i(\pi, \pi') \rightarrow \text{Ext}^i(\pi, \text{Ind}_{P'}^M \sigma).$$

By the inductive hypothesis $\text{Ext}^{i-1}(\pi, C)$ is nonzero only if π has the same supercuspidal support as π' . On the other hand, we again have an isomorphism of $\text{Ext}^i(\pi, \text{Ind}_{P'}^M \sigma)$ with $\text{Ext}^i(\text{Res}_M^{P'} \pi, \sigma)$, and (as σ is cuspidal), the previous case shows that this too vanishes unless π and π' have the same supercuspidal support. \square

3.2.14. Corollary. *If V is an essentially AIG representation of G over \overline{K} , then all the Jordan–Hölder constituents of V have the same supercuspidal support.*

Proof. Suppose otherwise. As V is the sum of its finite length submodules, there is then a finite length submodule W of V that is minimal among submodules of V that have a Jordan–Hölder constituent with supercuspidal support different from that of $\text{soc}(V)$. Let W' be the kernel of the map $W \rightarrow \text{cosoc}(W)$. The minimality of W implies that $\text{cosoc}(W)$ is irreducible and that every Jordan–Hölder constituent of W' has the same supercuspidal support as $\text{soc}(V)$. Thus $\text{Ext}^i(W', \text{cosoc}(W))$ vanishes for all i , by the preceding theorem; in particular $\text{cosoc}(W)$ is a direct summand of W . This is impossible, since Lemma 3.2.3 implies that any essentially AIG representation is indecomposable. \square

3.2.15. Corollary. *Let V be an essentially AIG representation of G over K . If V is admissible, then V has finite length.*

Proof. By Corollary 3.2.14 there are only finitely many isomorphism classes of Jordan–Hölder constituents of V , and one can bound the number of times any given Jordan–Hölder constituent appears in terms of the dimension of the U -invariants of V for a sufficiently small compact open subgroup U of G . \square

Corollary 3.2.14 has additional finiteness implications for essentially AIG representations. More precisely, for a smooth representation V of G , define $\text{soc}_c(V)$ inductively by setting $\text{soc}_1(V) = \text{soc}(V)$, and defining $\text{soc}_c(V)$ to be the preimage of $\text{soc}(V/\text{soc}_{c-1}(V))$ under the surjection

$$V \rightarrow V/\text{soc}_{c-1}(V).$$

We then have:

3.2.16. Theorem. *Let V be an essentially AIG representation of G over \overline{K} . Then $\text{soc}_c(V)$ has finite length for all c .*

Proof. By induction it suffices to show that $\text{soc}_c(V)/\text{soc}_{c-1}(V)$ has finite length for all $c \geq 2$. The space $\text{soc}_c(V)/\text{soc}_{c-1}(V)$ is semisimple, and every irreducible summand of $\text{soc}_c(V)/\text{soc}_{c-1}(V)$ is an irreducible non-generic representation of G with the same supercuspidal support as $\text{soc}(V)$. There are finitely many isomorphism classes of such representations. It thus suffices to show, for every irreducible non-generic representation π of G with the same supercuspidal support as $\text{soc}(V)$, that $\text{Hom}(\pi, \text{soc}_c(V)/\text{soc}_{c-1}(V))$ is finite dimensional. We have an exact sequence:

$$0 \rightarrow \text{soc}_{c-1}(V) \rightarrow \text{soc}_c(V) \rightarrow \text{soc}_c(V)/\text{soc}_{c-1}(V) \rightarrow 0.$$

As the socle of $\text{soc}_c(V)$ is generic, we have $\text{Hom}(\pi, \text{soc}_c(V)) = 0$. We thus obtain an injection:

$$\text{Hom}(\pi, \text{soc}_c(V)/\text{soc}_{c-1}(V)) \rightarrow \text{Ext}^1(\pi, \text{soc}_{c-1}(V)).$$

But $\text{soc}_{c-1}(V)$ has finite length by the induction hypothesis, and so by [19], Corollary 2.12, $\text{Ext}^1(\pi, \text{soc}_{c-1}(V))$ is finite dimensional. \square

3.2.17. Corollary. *Let V be an essentially AIG representation of G , let c be a positive integer, and let V_i be an arbitrary collection of submodules of V of length less than or equal to c . Then the sum of the V_i has finite length.*

Proof. Each V_i is contained in $\text{soc}_c(V)$, so their sum is as well. The result thus follows immediately from the theorem above. \square

We close this subsection with the following result treating essentially AIG representations in the case $n = 2$.

3.2.18. Proposition. *Any essentially AIG representation over $\text{GL}_2(E)$ is of finite length.*

Proof. Let V be an essentially AIG representation over $\text{GL}_2(E)$; we must show that V has finite length. Clearly we may check this after making an extension of scalars, and so without loss of generality we may and do assume that $K = \overline{K}$.

If $V/\text{soc}(V)$ is trivial then $V = \text{soc}(V)$ is irreducible, and we are done. Thus we assume from now on that $V/\text{soc}(V)$ is non-trivial. The quotient $V/\text{soc}(V)$ contains no generic constituent, hence its Jordan–Hölder factors are all one-dimensional, and so each is of the form $\chi \circ \det$ for some character χ . Moreover, if there exist Jordan–Hölder factors of $V/\text{soc}(V)$ isomorphic to $\chi \circ \det$ and $\chi' \circ \det$, then Corollary 3.2.14 implies $\chi \circ \det$ and $\chi' \circ \det$ have the same supercuspidal support. From this it is easy to see that $\chi^2 = (\chi')^2$. Replacing V by an appropriate twist, we may thus assume that the center E^\times of G acts trivially on each Jordan–Hölder factor of $V/\text{soc}(V)$.

Since $V/\text{soc}(V)$ is the sum of its finite length subrepresentations (as V is; this is one of the conditions of being essentially AIG), and since each of its Jordan–Hölder factors is one-dimensional, V is a sum of *finite-dimensional* representations of $\text{GL}_2(E)$. The action of $\text{GL}_2(E)$ on any finite dimensional representation factors through \det , so the action of $\text{GL}_2(E)$ on $V/\text{soc}(V)$ must also factor through \det . In this way we regard $V/\text{soc}(V)$ as a representation of E^\times . Since V admits a central character, by Lemma 3.2.4, and since the centre acts trivially on each Jordan–Hölder factor of $V/\text{soc}(V)$, we see that the centre must act trivially on V . (This is where we use the non-triviality of $V/\text{soc}(V)$.) Thus $V/\text{soc}(V)$ is in fact a representation

of the group $E^\times/(E^\times)^2$. Theorem 3.2.16 shows that the socle of $V/\text{soc}(V)$ is finite length, and it follows from the fact that $E^\times/(E^\times)^2$ is finite that $V/\text{soc}(V)$ itself is of finite length, as required. \square

3.3. Invariant lattices. We now prove some results about the reduction of finite length essentially AIG representations of $\text{GL}_n(E)$. Let \mathcal{O} be a complete discrete valuation ring, with field of fractions \mathcal{K} and residue field K of characteristic different from ℓ . Fix a uniformizer ϖ of \mathcal{O} .

If V is an integral admissible smooth representation of $\text{GL}_n(E)$ of finite length, then by the ‘‘Brauer-Nesbitt Theorem’’ of ([17, Ch. II.5.11]), V is a good integral representation in the sense of Definition 2.2.1, and hence Lemma 2.2.4 shows that if V° is a $\text{GL}_n(E)$ -invariant \mathcal{O} -lattice in V , then $(V^\circ/\varpi V^\circ)^{\text{ss}}$ is independent of V° ; we denote it \overline{V}^{ss} .

3.3.1. Proposition. *If V is an essentially AIG admissible smooth representation of $\text{GL}_n(E)$ over \mathcal{K} which is integral and of finite length, then \overline{V}^{ss} contains a unique irreducible generic summand.*

Proof. Fix an invariant \mathcal{O} -lattice V° in V . Then $(V^\circ)^{(n)}$ is a finitely generated \mathcal{O} -submodule of $V^{(n)}$, and the latter is a one-dimensional \mathcal{K} -vector space. Thus $(V^\circ)^{(n)}$ is free of rank one over \mathcal{O} . As the derivative commutes with tensor products, it follows that $(\overline{V}^{\text{ss}})^{(n)}$ is a one-dimensional K -vector space; the result follows. \square

3.3.2. Theorem. *If V is an essentially AIG admissible smooth representation of $\text{GL}_n(E)$ over \mathcal{K} which is integral and of finite length, then V admits a ϖ -adically separated $\text{GL}_n(E)$ -invariant lattice V° which is admissible as a $\text{GL}_n(E)$ -representation, and such that $\overline{V}^\circ := V^\circ/\varpi V^\circ$ is essentially AIG. Moreover, V° is unique up to homothety.*

Proof. We apply Lemma 2.2.6 to V , taking \mathcal{T} to consist of all the nongeneric Jordan–Hölder factors. This yields an \mathcal{O} -lattice V° , such that \overline{V}° contains no nongeneric subrepresentations. As \overline{V}^{ss} has only one irreducible generic submodule, this submodule is the socle of \overline{V}^{ss} , and $(\overline{V}^{\text{ss}}/\text{soc}(\overline{V}^{\text{ss}}))^{(n)} = 0$. If H is any open subgroup of G , then $(V^\circ)^H$ is ϖ -adically separated, and its E -span coincides with V^H , which is finite dimensional, since V is admissible. It follows that $(V^\circ)^H$ is finitely generated over \mathcal{O} , and so V° is an admissible smooth representation of $\text{GL}_n(E)$. Thus \overline{V}° is admissible smooth, and therefore essentially AIG.

Suppose now that V^\diamond is a second lattice in V satisfying the conditions of the corollary. Scaling it appropriately, we may assume that $V^\diamond \subset V^\circ$, but that the induced map $\overline{V}^\diamond \rightarrow \overline{V}^\circ$ is non-zero. Since both source and target are essentially AIG, this map is necessarily injective by Lemma 3.2.2, and hence (since source and target are of the same length) an isomorphism. \square

4. THE LOCAL LANGLANDS CORRESPONDENCE IN CHARACTERISTIC ZERO

Let F be an algebraically closed field of characteristic zero. The local Langlands correspondence for $\text{GL}_n(E)$ [8] establishes a certain bijection between the set of isomorphism classes of irreducible admissible smooth representations of $\text{GL}_n(E)$ on F -vector spaces, and the set of isomorphism classes of n -dimensional Frobenius semisimple Weil–Deligne representations over F (as defined in [5, §8] or [16, §4]).

In fact there are various choices of correspondence, depending on the desired normalization. The so-called unitary correspondence is uniquely determined by the requirement that the local L - and ε -factors attached to a pair of corresponding isomorphism classes should coincide. On the other hand, this correspondence depends on the choice of a square root of ℓ in F , and (because of this) is not compatible in general with change of coefficients (although a suitably chosen twist will be; we refer the reader to 4.2 for details.)

However, even if we normalize the local Langlands correspondence to be compatible with change of coefficients, the correspondence as usually defined is not suitable for the applications we have in mind. In particular, the usual local Langlands correspondence fails to be compatible with specialization. More precisely, let \mathcal{O} be a complete discrete valuation ring containing \mathbb{Q}_p , with field of fractions \mathcal{K} and residue field K , and let $\rho : G_E \rightarrow \mathrm{GL}_n(\mathcal{O})$ be a continuous Galois representation. Then the local Langlands correspondence associates admissible representations π and $\bar{\pi}$ to the Weil–Deligne representations induced by $\rho \otimes_{\mathcal{O}} \mathcal{K}$ and $\rho \otimes_{\mathcal{O}} K$, but there need not be a close relationship between $\bar{\pi}$ and the reduction of π . (For example, $\bar{\pi}$ could be a character even if π is infinite-dimensional.)

We therefore work with a modification of the usual local Langlands correspondence, which we describe fully in 4.2. We denote this correspondence by $\rho \mapsto \pi(\rho)$, where ρ is a continuous n -dimensional representation of G_E over an extension K of \mathbb{Q}_p . The correspondence $\rho \mapsto \pi(\rho)$ is essentially the generic local Langlands correspondence introduced by Breuil and Schneider in [2]. Unlike more standard formulations of local Langlands, the representation $\pi(\rho)$ of $\mathrm{GL}_n(E)$ will in general be reducible (in fact, it will be an essentially AIG representation of $\mathrm{GL}_n(E)$). The map $\rho \mapsto \pi(\rho)$ will not be a bijection in any meaningful sense but simply a map from isomorphism classes of n -dimensional representations of G_E over K to indecomposable admissible representations of $\mathrm{GL}_n(E)$ over K . The advantage of this choice is that the map $\rho \mapsto \pi(\rho)$ will be compatible with change of coefficients (in the sense that $\pi(\rho \otimes_K K')$ will be isomorphic to $\pi(\rho) \otimes_K K'$ for an extension K' of K), and also compatible with specialization (in the sense of Theorem 4.5.7 below.)

4.1. Galois representations and Weil–Deligne representations. In order to give a precise description of the map $\rho \mapsto \pi(\rho)$, we first recall some basic facts about Frobenius-semisimple Weil–Deligne representations. Recall that a Weil–Deligne representation over a field K containing \mathbb{Q}_p is a pair (ρ', N) , where $\rho' : W_E \rightarrow \mathrm{GL}_n(K)$ is a smooth representation of W_E with coefficients in K and N is a nilpotent endomorphism of K^n satisfying $\rho'(w)N\rho'(w)^{-1} = |w|N$. The representation (ρ', N) is called Frobenius-semisimple if ρ' is absolutely semisimple.

We first consider absolutely irreducible representations $\rho' : W_E \rightarrow \mathrm{GL}_n(K)$. Let I_E be the inertia subgroup of E . Then $\rho'(I_E)$ is a finite group, and so all of its irreducible representations are defined over a finite extension K_0 of \mathbb{Q}_p . After replacing K with an algebraic extension we may assume K contains a subfield isomorphic to K_0 . Then the restriction of ρ' to I_E splits as a direct sum of absolutely irreducible representations τ_i of I_E over K_0 .

Let Φ be a Frobenius element of W_E , and let V be an I_E -stable subspace of K^n isomorphic to $\tau_1 \otimes_{K_0} K$ as an I_E -representation. Then I_E acts on ΦV by the conjugate τ_1^Φ of τ_1 . In fact, we have:

4.1.1. **Lemma.** *Let r be the order of the orbit of τ_1 under the action of Φ on the set of isomorphism classes of absolutely irreducible representations of I_E over K_0 . Then we have a direct sum decomposition:*

$$\rho'|_{I_E} = \bigoplus_{i=0}^{r-1} \tau_1^{\Phi^i} \otimes_{K_0} K$$

and the action of Φ on this decomposition permutes the summands.

Proof. As I_E is normal in W_E , this is a standard result in Clifford theory. \square

In particular, the vector space $\mathrm{Hom}_{K[I_E]}(\tau_1 \otimes_{K_0} K, \rho'|_{I_E})$ is one-dimensional. If we fix an isomorphism $\tau_1 \xrightarrow{\sim} \tau_1^{\Phi^r}$, then we get an endomorphism Ψ of this vector space via:

$$\mathrm{Hom}_{K[I_E]}(\tau_1 \otimes_{K_0} K, \rho') \xrightarrow{\Phi^r} \mathrm{Hom}_{K[I_E]}(\tau_1^{\Phi^r} \otimes_{K_0} K, \rho') \xrightarrow{\sim} \mathrm{Hom}_{K[I_E]}(\tau_1 \otimes_{K_0} K, \rho').$$

The action of Ψ is given by a scalar λ in K^\times .

4.1.2. **Lemma.** *For any $\lambda \in K^\times$ there is a unique absolutely irreducible representation ρ' over K (up to isomorphism) such that $\rho'|_{I_E}$ contains $\tau_1 \otimes_{K_0} K$ and Ψ acts on $\mathrm{Hom}_{K[I_E]}(\tau_1 \otimes_{K_0} K, \rho')$ via λ .*

Proof. If $r = 1$, then the restriction of ρ' to I_E is given by $\tau_1 \otimes_{K_0} K$, and so to determine ρ' it suffices to give an action of Φ on the representation space of τ_1 , compatible with the action of I_E . This amounts to giving an isomorphism $\tau_1 \otimes_{K_0} K \xrightarrow{\sim} (\tau_1 \otimes_{K_0} K)^\Phi$. As we have already fixed an isomorphism $\tau_1 \xrightarrow{\sim} \tau_1^\Phi$, such an isomorphism is determined by λ .

If $r > 1$, let E' be the unramified extension of E of degree r . The restriction of ρ' to $W_{E'}$ breaks up as a sum of irreducible representations $\rho'_0, \dots, \rho'_{r-1}$ such that the restriction of ρ'_i to I_E is isomorphic to $\tau_1^{\Phi^i}$. Thus ρ'_i is determined by λ and τ_1 , and ρ' is isomorphic to $\mathrm{Ind}_{W_{E'}}^{W_E} \rho'_0$ by Frobenius reciprocity. \square

4.1.3. **Lemma.** *Let K be a field containing \mathbb{Q}_p , and let ρ' be an absolutely irreducible representation of W_E over K . Then there exists an unramified character $\chi : W_E \rightarrow \overline{K}^\times$ such that the twist $\rho' \otimes \chi$ is defined over $\overline{\mathbb{Q}_p}$.*

Proof. The representation τ_1 is defined over a finite extension of \mathbb{Q}_p , so it suffices to show that after a twist we can take the scalar λ to be in $\overline{\mathbb{Q}_p}$. Twisting by an unramified χ changes λ to $\chi(\Phi)^r \lambda$, so this is clear. \square

4.1.4. **Definition.** *Let ρ' be an absolutely irreducible smooth representation $W_E \rightarrow \mathrm{GL}_n(K)$, and let d be a positive integer. The special representation $\mathrm{Sp}_{\rho', d}$ is the pair*

$$\mathrm{Sp}_{\rho', d} = (V_0 \oplus \dots \oplus V_{d-1}, N),$$

where W_E acts on V_i by $| \cdot |^i \rho'$ and N maps V_i isomorphically onto V_{i+1} for $0 \leq i \leq d-2$.

The representation $\mathrm{Sp}_{\rho', d}$ is well-defined up to isomorphism, and is an absolutely indecomposable Weil–Deligne representation. If K is algebraically closed, then every indecomposable Frobenius-semisimple Weil–Deligne representation has the form $\mathrm{Sp}_{\rho', d}$ for a unique absolutely irreducible representation ρ' of W_E over K . Combining this with the previous lemma, we find:

4.1.5. Lemma. *Let K be a field containing \mathbb{Q}_p , and let (ρ', N) be an indecomposable Frobenius-semisimple Weil–Deligne representation over K . Then there exists a character $\chi : W_E \rightarrow \overline{K}^\times$ such that the twist $(\rho' \otimes \chi, N)$ is defined over $\overline{\mathbb{Q}_p}$.*

In those situations in which we will apply the local Langlands correspondence, we will be beginning not with Weil–Deligne representations, but with Galois representations. Thus we recall the recipe of Deligne for associating a Weil–Deligne representation to a continuous Galois representation, in a slightly broader context than that in which it is usually considered.

Let A be a complete Noetherian local domain of residue characteristic p different from ℓ , maximal ideal \mathfrak{m} , and field of fractions \mathcal{K} of characteristic zero. Let R be any subring of \mathcal{K} containing A and $\frac{1}{p}$. (In most applications, R will equal either \mathcal{K} , or else a complete discrete valuation ring \mathcal{O} containing A and contained in \mathcal{K} .)

For any $n \geq 0$, we say that a representation $\rho : G_E \rightarrow \mathrm{GL}_n(R)$ is continuous if we can find a finitely generated A -submodule M of R^n that is invariant under $\rho(G_E)$, spans R^n over R , and such that the induced G_E -action on M is \mathfrak{m} -adically continuous. (Note that if $R = \mathcal{K}$, and \mathcal{K} is a finite extension of \mathbb{Q}_p , then this coincides with the usual notion of continuity of a G_E -representation.)

As in [16, (4.2)], we fix a non-zero homomorphism $t_p : I_E \rightarrow \mathbb{Q}_p$. (When comparing the present discussion with that of [16], note that the roles of ℓ and p are reversed.) This homomorphism is uniquely determined up to scaling by an element of \mathbb{Q}_p^\times . The following result then extends a theorem of Deligne [5, §8], [16, Thm. (4.2.1)] (which treats the case when the coefficient field is a finite extension of \mathbb{Q}_p).

4.1.6. Proposition. *A continuous representation $\rho : G_E \rightarrow \mathrm{GL}_n(R)$ uniquely determines the following data:*

- (1) *a representation $\rho' : W_E \rightarrow \mathrm{GL}_n(R)$ that is continuous when the target is equipped with its discrete topology;*
- (2) *a nilpotent matrix $N \in \mathrm{M}_n(R)$;*
subject to the following condition:
- (3) *$\rho(\Phi^r \sigma) = \rho'(\Phi^r \sigma) \exp(t_p(\sigma)N)$ for all $\sigma \in I_E$ and $r \in \mathbb{Z}$.*

Furthermore, as a Weil–Deligne representation, the pair (ρ', N) is independent, up to isomorphism, of the choice of t_p and Φ .

Proof. Let (ρ'_1, N_1) and (ρ'_2, N_2) be two Weil–Deligne representations satisfying the condition of the proposition. Then there is an open subgroup of I_E on which both ρ'_1 and ρ'_2 are trivial; we can thus find an element σ of I_E for which $\rho'_1(\sigma)$ and $\rho'_2(\sigma)$ are the identity but $t_p(\sigma)$ is nonzero. Then $N_1 = \frac{1}{t_p(\sigma)} \log \rho(\sigma) = N_2$. The identity

$$\rho(\Phi^r \sigma) = \rho'_i(\Phi^r \sigma) \exp(t_p(\sigma)N_i)$$

then forces $\rho'_1 = \rho'_2$.

It thus suffices to construct a (ρ', N) as above. Choose a finitely-generated A -submodule M of R^n that is preserved by ρ and spans R^n over R . Then G_E acts via ρ on $M/\mathfrak{m}^{i+1}M$ for all i , and these A -modules are discrete with respect to the \mathfrak{m} -adic topology. In particular for each i the subgroup H_i of I_E that acts trivially on M/\mathfrak{m}^iM is a compact open subgroup of I_E .

The group of automorphisms of $M/\mathfrak{m}^{i+1}M$ that reduce to the identity in M/\mathfrak{m}^iM is an abelian p -group for all $i \geq 2$. Thus the action of H_2 on M factors through

the map $t_p : I_E \rightarrow \mathbb{Q}_p$. Let σ be an element of H_2 ; the action of σ on M yields an element of α of $\text{End}(M)$ that is congruent to the identity modulo \mathfrak{m}^2 . The power series $\log(\alpha)$ thus converges in the \mathfrak{m} -adic topology on $\text{End}(M)$; set $N = \frac{1}{t_p(\sigma)} \log(\alpha)$. Then any $\tau \in H_2$ acts on M via $\exp(t_p(\tau)N)$. It follows that for all $\tau \in G_E$, $\rho(\tau)N\rho(\tau)^{-1} = |\tau|N$. In particular N must be nilpotent.

We can then set $\rho'(\Phi^r \sigma) = \rho(\Phi^r \sigma) \exp(t_p(\sigma)N)^{-1}$ for all $\sigma \in I_E$ and $r \in \mathbb{Z}$; this gives a well-defined ρ' that is trivial on the compact open subgroup H_2 of W_E . \square

Let \mathcal{O} be a discrete valuation ring containing A and contained in \mathcal{K} , with residue field K of characteristic zero and uniformizer ϖ . We will be interested in the reduction mod ϖ of both Galois representations and Weil–Deligne representations over \mathcal{O} . One has:

4.1.7. Lemma. *Let $\rho' : W_E \rightarrow \text{GL}_n(\mathcal{O})$ be a representation of W_E over \mathcal{O} such that $\rho' \otimes_{\mathcal{O}} K$ is absolutely irreducible. Then $\bar{\rho}' := \rho' \otimes_{\mathcal{O}} K$ is also absolutely irreducible.*

Proof. By Lemma 4.1.1, over a finite extension of \mathcal{K} , the restriction of ρ' to I_E splits as a direct sum of absolutely irreducible representations ρ'_i of I_E , each of which factors through a finite quotient of I_E and is defined over $\overline{\mathbb{Q}_p}$. The representations ρ'_i are distinct and cyclically permuted by conjugation by Φ . As K has characteristic zero, the ρ'_i remain irreducible and distinct after “reduction mod ϖ ”.

Thus, over a finite extension of K , $\bar{\rho}'$ splits as a direct sum of absolutely irreducible representations $\bar{\rho}'_i$ which are distinct and cyclically permuted under conjugation by Φ . It is thus clear that $\bar{\rho}'$ is absolutely irreducible. \square

It follows that if $\rho' : W_E \rightarrow \text{GL}_n(\mathcal{K})$ is absolutely irreducible and contains an \mathcal{O} -lattice L , then the mod ϖ reduction of L is independent, up to isomorphism, of the lattice L . We denote this reduction by $\bar{\rho}'$.

The passage from Galois representations to Weil–Deligne representations commutes with reduction modulo ϖ :

4.1.8. Lemma. *Let $\rho : G_E \rightarrow \text{GL}_n(\mathcal{O})$ be a continuous Galois representation, with mod ϖ reduction $\bar{\rho}$, and let (ρ', N) and $(\bar{\rho}', \bar{N})$ be the Weil–Deligne representations attached to ρ and $\bar{\rho}$, respectively. Then $(\bar{\rho}', \bar{N})$ is isomorphic to $(\rho' \otimes_{\mathcal{O}} K, N \otimes_{\mathcal{O}} K)$.*

Proof. This follows immediately from the identities

$$\begin{aligned} \rho(\Phi^r \sigma) &= \rho'(\Phi^r \sigma) \exp(t_p(\sigma)N) \\ \bar{\rho}(\Phi^r \sigma) &= \bar{\rho}'(\Phi^r \sigma) \exp(t_p(\sigma)\bar{N}) \end{aligned}$$

and the fact that the latter identity characterizes $(\bar{\rho}', N)$ up to isomorphism. \square

Given a Weil–Deligne representation (ρ', N) over \mathcal{K} , one can associate a natural Frobenius-semisimple representation $(\rho', N)^{\text{F-ss}}$, (the Frobenius-semisimplification of (ρ', N) .) We recall the definition; see [5, 8.5] for details.

The matrix $\rho'(\Phi)$ factors uniquely as a product su , with s and u elements of $\text{GL}_n(\mathcal{K})$ that are semisimple and unipotent, respectively, and commute with each other. Moreover, if $\rho'(\Phi)$ lies in $\text{GL}_n(\mathcal{O})$, then so do s and u . The element u then commutes with N , and one defines $(\rho')^{\text{F-ss}}$ to be the representation of W_F that satisfies $(\rho')^{\text{F-ss}}(\Phi^r \sigma) = u^{-r} \rho'(\Phi^r \sigma)$. Then $(\rho')^{\text{F-ss}}$ is a semisimple representation of W_F over \mathcal{K} , and the pair $((\rho')^{\text{F-ss}}, N)$ is a Frobenius-semisimple Weil–Deligne representation which we write $(\rho', N)^{\text{F-ss}}$.

It will be necessary for us to understand how Frobenius-semisimplification commutes with reduction modulo ϖ . Note that even if (ρ', N) is a Frobenius-semisimple Weil–Deligne representation over \mathcal{O} , its reduction modulo ϖ need not be, as the mod ϖ reduction of a semisimple element of $\mathrm{GL}_n(\mathcal{O})$ need not be semisimple.

4.1.9. Lemma. *Let (ρ', N) be a Weil–Deligne representation over \mathcal{O} , and let $(\bar{\rho}', \bar{N})$ be its reduction mod ϖ . Then $(\rho', N)^{\mathrm{F-ss}}$ is defined over \mathcal{O} . Moreover, the reduction mod ϖ of $(\rho', N)^{\mathrm{F-ss}}$ has Frobenius-semisimplification $(\bar{\rho}', \bar{N})^{\mathrm{F-ss}}$.*

Proof. If $\rho'(\Phi)$ decomposes as su , with s and u as above, then s and u lie in $\mathrm{GL}_n(\mathcal{O})$, so $(\rho', N)^{\mathrm{F-ss}}$ is defined over \mathcal{O} . Thus $\bar{\rho}'(\Phi) = \bar{s}\bar{u}$, where \bar{s} and \bar{u} are the mod ϖ reductions of s and u .

The element \bar{s} decomposes uniquely as $\bar{s}'\bar{u}'$, where \bar{s}' is semisimple and \bar{u}' is unipotent and commutes with \bar{s}' . As \bar{u} commutes with \bar{s} , \bar{s} also decomposes as a product of $\bar{u}\bar{s}'\bar{u}^{-1}$ with $\bar{u}\bar{u}'\bar{u}^{-1}$; the uniqueness of this decomposition shows that these two decompositions coincide. That is, \bar{u} commutes with \bar{s}' and \bar{u}' .

We have $\bar{\rho}'(\Phi) = \bar{s}'\bar{u}'\bar{u}$, and the unipotent element $\bar{u}'\bar{u}$ commutes with \bar{s}' . Thus the Frobenius-semisimplification of $(\bar{\rho}', \bar{N})$ sends Φ to \bar{s}' . On the other hand, the reduction of $(\rho', N)^{\mathrm{F-ss}}$ takes Φ to \bar{s} , which equals $\bar{s}'\bar{u}'$. Hence the Frobenius-semisimplification of the reduction of $(\rho', N)^{\mathrm{F-ss}}$ takes Φ to \bar{s} , and therefore coincides with $(\bar{\rho}', \bar{N})^{\mathrm{F-ss}}$. \square

4.2. The generic local Langlands correspondence of Breuil and Schneider.

We are now in a position to describe the “generic local Langlands correspondence” of Breuil and Schneider [2, pp. 162–164]. This is a map $(\rho', N) \mapsto \pi(\rho', N)$ from Frobenius-semisimple Weil–Deligne representations over a finite extension K of \mathbb{Q}_p to indecomposable admissible representations of $\mathrm{GL}_n(E)$ over K . Fix a choice of $\ell^{\frac{1}{2}}$ in $\overline{\mathbb{Q}_p}$ (and thus a choice of square root of the character $|\cdot| \circ \det$ of $\mathrm{GL}_n(E)$, as well as a unitary local Langlands correspondence for representations over $\overline{\mathbb{Q}_p}$). With this choice, the properties of this correspondence can be summarized as follows (c.f. [2, 4.2]):

- (1) For any character $\chi : W_F \rightarrow \mathbb{Q}_p^\times$, one has $\pi(\rho' \otimes \chi, N) = \pi(\rho', N) \otimes \chi$.
- (2) If K' is a finite extension of K , then $\pi(\rho' \otimes_K K', N) = \pi(\rho', N) \otimes_K K'$.
- (3) If (ρ', N) is a direct sum of representations of the form $\mathrm{Sp}_{\rho'_i, n_i}$ over $\overline{\mathbb{Q}_p}$, then $\pi(\rho', N)$ is defined by the parabolic induction:

$$\pi(\rho', N) = (|\cdot| \circ \det)^{-\frac{n-1}{2}} \mathrm{Ind}_Q^{\mathrm{GL}_n(E)} \mathrm{St}_{\pi_1, n_1} \otimes \cdots \otimes \mathrm{St}_{\pi_r, n_r},$$

where π_i corresponds to ρ_i under the unitary local Langlands correspondence, St_{π_i, n_i} is the generalized Steinberg representation (and thus corresponds to $\mathrm{Sp}_{\rho_i, n_i}$ under unitary local Langlands,) and Q is the upper triangular parabolic subgroup of $\mathrm{GL}_n(E)$ whose Levi subgroup is block diagonal with block sizes $(n_1 \dim \rho'_1, \dots, n_r \dim \rho'_r)$. The symbol $\mathrm{Ind}_Q^{\mathrm{GL}_n(E)}$ denotes *normalized* parabolic induction. The representations $\mathrm{St}_{\rho_i, n_i}$ are ordered so that the condition of [14, Def. 1.2.4] holds. (As long as this condition is satisfied, the resulting parabolic induction is independent, up to isomorphism, of the precise choice of ordering, as well as of the choice the square root of ℓ needed to define $(|\cdot| \circ \det)^{\frac{1}{2}}$.)

These properties uniquely characterize the generic local Langlands correspondence. We will need a slight extension of this correspondence to the case of coefficients in an arbitrary field extension K of \mathbb{Q}_p . Let (ρ', N) be a Frobenius-semisimple Weil–Deligne representation over K , and suppose that it decomposes over \overline{K} as a direct sum of representations of the form $\mathrm{Sp}_{\rho'_i, n_i}$. Then by Lemma 4.1.5, there exist characters $\chi_i : W_E \rightarrow \overline{K}^\times$ such that $\rho'_i \otimes \chi_i$ is defined over $\overline{\mathbb{Q}}_p$. For such representations the unitary local Langlands correspondence is defined, and we can take π_i to be the representation over K such that $\pi_i \otimes \chi_i$ corresponds to $\rho'_i \otimes \chi_i$ via the unitary local Langlands correspondence over $\overline{\mathbb{Q}}_p$.

$$\mathrm{St}_{\pi_i, n_i} = \mathrm{St}_{\pi'_i \otimes \chi_i, n_i} \otimes (\chi_i^{-1} \circ \det).$$

$$\pi(\rho', N) = (|\cdot| \circ \det)^{-\frac{n-1}{2}} \mathrm{Ind}_Q^{\mathrm{GL}_n(E)} \mathrm{St}_{\pi_1, n_1} \otimes \cdots \otimes \mathrm{St}_{\pi_r, n_r},$$

where the St_{π_i, n_i} are ordered as before. *A priori*, this is a representation of $\mathrm{GL}_n(E)$ over \overline{K} , but the argument of [2, Lem. 4.2] shows that $\pi(\rho', N)$ is defined over K itself. Moreover, $\pi(\rho', N)$ is independent of the choices of χ_i .

As was the case over finite extension of \mathbb{Q}_p , the map $(\rho', N) \mapsto \pi(\rho', N)$ is compatible with twists, and also with arbitrary field extensions.

We extend this definition to a map from representations of G_E to admissible representations of $\mathrm{GL}_n(E)$ as follows:

4.2.1. Definition. *Let ρ be a continuous n -dimensional representation of G_E over K , and let (ρ', N) be the corresponding Weil–Deligne representation. We define $\pi(\rho)$ to be $\pi((\rho', N)^{\mathrm{F-ss}})$.*

4.3. Segments and the Zelevinski classification. Our next goal is to establish key properties of the generic local Langlands correspondence (in particular, we will show that $\pi(\rho', N)$ is essentially AIG).

Following [21], we define a segment to be a set of supercuspidal representations of the form: $[\pi, (|\cdot| \circ \det)\pi, \dots, (|\cdot| \circ \det)^{r-1}\pi]$, where π is an irreducible supercuspidal representation of $\mathrm{GL}_n(E)$ over \overline{K} . We think of the segment Δ given by $[\pi, (|\cdot| \circ \det)\pi, \dots, (|\cdot| \circ \det)^{r-1}\pi]$ as corresponding to the generalized Steinberg representation $\mathrm{St}_{\pi, r}$; this gives a bijection between segments and generalized Steinberg representations. If $\mathrm{St}_{\pi, r}$ corresponds to a segment Δ , we will often write St_Δ for $\mathrm{St}_{\pi, r}$. Similarly, we will write Sp_Δ for the indecomposable Weil–Deligne representation $\mathrm{Sp}_{\rho, r}$, where ρ is the irreducible Weil–Deligne representation corresponding to π under the unitary local Langlands correspondence.

Two segments Δ, Δ' are said to be *linked* if neither contains the other, and if $\Delta \cup \Delta'$ is a segment. The segment Δ *precedes* Δ' if Δ and Δ' are linked and Δ' has the form $[(|\cdot| \circ \det)\pi, \dots, (|\cdot| \circ \det)^{r-1}\pi]$ for some π in Δ .

We consider the following condition on a sequence $\vec{\mathcal{S}}$ of segments Δ_i :

4.3.1. Condition. *For all $i < j$, the segment Δ_i does not precede the segment Δ_j .*

It is clear that any multiset of segments can be given an ordering that satisfies Condition 4.3.1. If \mathcal{S} is a multiset of segments (henceforth referred to as a multisegment), then we let $\pi(\mathcal{S})$ denote the normalized parabolic induction

$$\mathrm{Ind}_Q^{\mathrm{GL}_n(E)} \mathrm{St}_{\Delta_1} \otimes \cdots \otimes \mathrm{St}_{\Delta_n},$$

where $\Delta_1, \dots, \Delta_n$ are the segments in \mathcal{S} , taken with multiplicities and ordered so that Condition 4.3.1 holds, and Q is a suitable, block upper triangular parabolic

subgroup of $\mathrm{GL}_n(E)$. By [21, Prop. 6.4], the representation $\pi(\mathcal{S})$ does not depend, up to isomorphism, on the order of the collection of segments in \mathcal{S} (as long as Condition 4.3.1 holds). Note that if (ρ', N) is an n -dimensional Frobenius-semisimple Weil–Deligne representation that decomposes as the direct sum of Sp_{Δ_i} for $\Delta_i \in \mathcal{S}$, then $\pi(\rho', N)$ is isomorphic to $(|\cdot| \circ \det)^{-\frac{n-1}{2}} \pi(\mathcal{S})$. By [14, 1.2.5], $\pi(\mathcal{S})$ admits a unique irreducible quotient $Q(\mathcal{S})$, and $Q(\mathcal{S})$ is the irreducible representation corresponding to (ρ', N) under the unitary local Langlands correspondence.

4.3.2. Proposition. *If \mathcal{S} is an multisegment, then every irreducible submodule of $\pi(\mathcal{S})$ is generic.*

Proof. We will prove a stronger statement — namely, that every irreducible P_n -submodule of the restriction $\pi(\mathcal{S})|_{P_n}$ is generic. (In other words, $\pi(\mathcal{S})$ embeds in its Kirillov model.) Over the complex numbers this is a result of Jacquet-Shalika [13]. Their argument does not seem to adapt easily to other fields of characteristic zero. One could reduce this proposition to their result by choosing an isomorphism of \overline{K} with \mathbb{C} ; we instead give an algebraic argument over \overline{K} that is an adaptation of the argument of [1, 4.15]. Their argument necessarily uses Ψ^\pm and Φ^\pm functors that are normalized differently from ours, to avoid unpleasant combinatorial issues. Therefore, *for the purposes of this proof only* we take the functors Ψ^\pm and Φ^\pm to be normalized as in [1], rather than as in section 3.1.

Let \mathcal{S} be the multisegment $(\Delta_1, \dots, \Delta_n)$, where Δ_i does not precede Δ_j for any $j > i$. We can assume without loss of generality that the Δ_i are ordered so that if $\Delta_i = [\pi_i, (|\cdot| \circ \det)\pi_i, \dots, (|\cdot| \circ \det)^{r_i-1}\pi_i]$, where π_i is a supercuspidal representation of $\mathrm{GL}_{n_i}(E)$, then $(|\cdot| \circ \det)^{r_i}\pi_i$ is not contained in any segment Δ_j with $j > i$; clearly for such an ordering Δ_i never precedes a Δ_j with $j > i$. We proceed by induction on the sum of the lengths of the segments Δ_i . Note that the result is clear for a single segment, as St_{Δ_i} is absolutely irreducible and generic. Let \mathcal{S}' be the multisegment $(\Delta_2, \dots, \Delta_n)$; by the induction hypothesis every irreducible submodule of $\pi(\mathcal{S}')$ is generic.

Suppose we have an irreducible, non-generic submodule ω of $\pi(\mathcal{S})|_{P_n}$. We have $\pi(\mathcal{S}) = \mathrm{St}_{\Delta_1} \times \pi(\mathcal{S}')$, where “ \times ” is the product defined in [1, 4.12]. By [1, 4.13a], we have an exact sequence:

$$0 \rightarrow (\mathrm{St}_{\Delta_1})|_{P_{r_1 n_1}} \times \pi(\mathcal{S}') \rightarrow \pi(\mathcal{S})|_{P_n} \rightarrow \mathrm{St}_{\Delta_1} \times \pi(\mathcal{S}')|_{P_{n-r_1 n_1}} \rightarrow 0.$$

In particular, ω is a submodule of one of $(\mathrm{St}_{\Delta_1})|_{P_{r_1 n_1}} \times \pi(\mathcal{S}')$ or $\mathrm{St}_{\Delta_1} \times \pi(\mathcal{S}')|_{P_{n-r_1 n_1}}$.

Suppose first that ω is contained in $(\mathrm{St}_{\Delta_1})|_{P_{r_1 n_1}} \times \pi(\mathcal{S}')$. By [21, 9.6], $\mathrm{St}_{\Delta_i}^{(k)}$ is zero if k is not divisible by n_i , whereas $\mathrm{St}_{\Delta_i}^{(kn_i)}$ is $\mathrm{St}_{\Delta_i^{(k)}}$, where $\Delta_i^{(k)}$ is the segment $[(|\cdot| \circ \det)^k \pi_i, \dots, (|\cdot| \circ \det)^{r-1} \pi_i]$. It follows by [1, 4.13c], that, for $i < n_1$,

$$(\Phi^-)^i((\Phi^-)^{kn_1}(\mathrm{St}_{\Delta_1}|_{P_{r_1 n_1}}) \times \pi(\mathcal{S}')) = (\Phi^-)^{kn_1+i}(\mathrm{St}_{\Delta_1}|_{P_{r_1 n_1}}) \times \pi(\mathcal{S}'),$$

so that for such i , $((\Phi^-)^{kn_1}(\mathrm{St}_{\Delta_1}|_{P_{r_1 n_1}}) \times \pi(\mathcal{S}'))^{(i)} = 0$. For $i = n_1$, [1, 4.13c] shows that the representation $(\Phi^-)^{(k+1)n_1}(\mathrm{St}_{\Delta_1}|_{P_{r_1 n_1}}) \times \pi(\mathcal{S}')$ is instead a proper submodule of $(\Phi^-)^{n_1}((\Phi^-)^{kn_1}(\mathrm{St}_{\Delta_1}|_{P_{r_1 n_1}}) \times \pi(\mathcal{S}'))$; the quotient of the latter by the former is isomorphic to $\mathrm{St}_{\Delta_1^{(k+1)}} \times \pi(\mathcal{S}')|_{P_{n-r_1 n_1}}$.

Since ω is contained in $(\mathrm{St}_{\Delta_1})|_{P_{r_1 n_1}} \times \pi(\mathcal{S}')$, we have $\omega^{(i)} = 0$ for $i < n_1$. As ω has at least one nonzero derivative it follows that $(\Phi^-)^{n_1-1}\omega$ is nonzero. On the

other hand, we have $(\Phi^-)^{n_1-1}\omega \subset (\Phi^-)^{n_1-1}((\text{St}_{\Delta_1})|_{P_{r_1 n_1}} \times \pi(\mathcal{S}'))$; by [1, 4.13d] it follows that $(\Phi^-)^{n_1}\omega$ is nonzero.

Then $(\Phi^-)^{n_1}\omega$ is a non-generic submodule of $(\Phi^-)^{n_1}((\text{St}_{\Delta_1})|_{P_{r_1 n_1}} \times \pi(\mathcal{S}'))$, and is therefore a non-generic submodule of either $(\Phi^-)^{n_1}((\text{St}_{\Delta_1})|_{P_{r_1 n_1}}) \times \pi(\mathcal{S}')$, or $\text{St}_{\Delta_1^{(1)}} \times \pi(\mathcal{S}')|_{P_{n-r_1 n_1}}$. It is easy to rule out the latter case: by the inductive hypothesis $\pi(\mathcal{S}')|_{P_{n-r_1 n_1}}$ has no non-generic submodules; by [21, 5.3] neither does $\text{St}_{\Delta_1^{(1)}} \times \pi(\mathcal{S}')|_{P_{n-r_1 n_1}}$.

Thus $(\Phi^-)^{n_1}\omega$ is a non-generic submodule of $(\Phi^-)^{n_1}((\text{St}_{\Delta_1})|_{P_{r_1 n_1}}) \times \pi(\mathcal{S}')$. In particular $((\Phi^-)^{n_1}\omega)^{(i)} = 0$ for $i < n_1$; it follows as above that $(\Phi^-)^{2n_1-1}\omega$ is nonzero, and by [1, 4.13d], that $(\Phi^-)^{2n_1}\omega$ is nonzero. Then $(\Phi^-)^{2n_1}\omega$ is a nonzero non-generic submodule of $(\Phi^-)^{n_1}((\Phi^-)^{n_1}((\text{St}_{\Delta_1})|_{P_{r_1 n_1}}) \times \pi(\mathcal{S}'))$, and hence (with another use of the inductive hypothesis and [21, 5.3]), is a nonzero non-generic submodule of $(\Phi^-)^{2n_1}((\text{St}_{\Delta_1})|_{P_{r_1 n_1}}) \times \pi(\mathcal{S}')$.

Proceeding in this fashion we find that $(\Phi^-)^{kn_1}\omega$ is a nonzero non-generic submodule of $(\Phi^-)^{kn_1}((\text{St}_{\Delta_1})|_{P_{r_1 n_1}}) \times \pi(\mathcal{S}')$ for all k , which is impossible since the latter vanishes for large k .

We have thus ruled out the possibility that ω is contained in $(\text{St}_{\Delta_1})|_{P_{r_1 n_1}} \times \pi(\mathcal{S}')$. The other alternative is that ω is contained in $\text{St}_{\Delta_1} \times \pi(\mathcal{S}')|_{P_{n-r_1 n_1}}$. Suppose this were the case, and let k be the largest integer such that $\omega^{(k)}$ is nonzero. Then $\omega^{(k)}$ is nonzero and embeds in the k -th derivative of $\text{St}_{\Delta_1} \times \pi(\mathcal{S}')|_{P_{n-r_1 n_1}}$, which is $\text{St}_{\Delta_1} \times \pi(\mathcal{S}')^{(k)}$. It follows that the supercuspidal support of $\omega^{(k)}$ contains that of St_{Δ_1} ; in particular it contains $(| \cdot | \circ \det)^{r-1}\pi_1$. By [1, 4.7b], it follows that $(| \cdot | \circ \det)^r\pi_1$ is contained in the supercuspidal support of $\pi(\mathcal{S})$; this is impossible by our choice of ordering on the Δ_i . \square

4.3.3. Corollary. *If π is an admissible representation of GL_n over a field K of characteristic zero, such that $\pi \otimes_K \overline{K}$ is isomorphic to $\pi(\mathcal{S})$ for some \mathcal{S} satisfying Condition 4.3.1, then π is essentially AIG. In particular, every representation $\pi(\rho', N)$ over a field K of characteristic zero is essentially AIG.*

Proof. It suffices to show that $\pi \otimes_K \overline{K}$ is essentially AIG, as then the socle of π must be absolutely irreducible and generic, and π must contain no other irreducible generic subquotients. But $\pi \otimes_K \overline{K}$ has the form $\pi(\mathcal{S})$ for some \mathcal{S} , so the previous proposition shows that the socle of $\pi(\mathcal{S})$ is a direct sum of irreducible generic representations. It thus suffices to show that $\pi(\mathcal{S})^{(n)}$ is one-dimensional; this follows from the fact that St_{Δ_i} is irreducible and generic, together with Theorem 3.1.7 and Proposition 3.1.6. \square

If \mathcal{S} and \mathcal{S}' are two multisegments, we say \mathcal{S}' arises from \mathcal{S} by an elementary operation if \mathcal{S}' is obtained from \mathcal{S} by replacing a pair of linked segments Δ, Δ' in \mathcal{S} with the pair $\Delta \cup \Delta', \Delta \cap \Delta'$. We say that $\mathcal{S}' \preceq \mathcal{S}$ if \mathcal{S}' can be obtained from \mathcal{S} by a sequence of elementary operations. This partial order contains information about the Jordan–Hölder constituents of a given $\pi(\mathcal{S})$. In particular, one has the following:

4.3.4. Theorem ([15], Theorem 5.3). *If \mathcal{S} is a multisegment, then every Jordan–Hölder constituent of $\pi(\mathcal{S})$ is isomorphic to $Q(\mathcal{S}')$ for some $\mathcal{S}' \preceq \mathcal{S}$, and each such $Q(\mathcal{S}')$ appears with positive multiplicity in $\pi(\mathcal{S})$.*

In fact, the relationship between $\pi(\mathcal{S})$ and $\pi(\mathcal{S}')$ is considerably stronger than the theorem above suggests. We will construct maps of $\pi(\mathcal{S}')$ into $\pi(\mathcal{S})$ for all $\mathcal{S}' \preceq \mathcal{S}$, and show that any nonzero such map is an embedding, and unique up to scaling. Before we do so, however, we need a preliminary result about the partial order \preceq . Let \mathcal{S} be a multisegment, and suppose that \mathcal{S}' is obtained from \mathcal{S} by a single elementary operation. We say this elementary operation is *primitive* if there is no multisegment \mathcal{S}'' with $\mathcal{S}' \preceq \mathcal{S}'' \preceq \mathcal{S}$ other than $\mathcal{S}'' = \mathcal{S}$ and $\mathcal{S}'' = \mathcal{S}'$.

4.3.5. Lemma. *Let \mathcal{S} be a multisegment, let Δ and Δ' be two linked segments in \mathcal{S} , such that Δ precedes Δ' . Suppose that the elementary operation that replaces Δ and Δ' with $\Delta \cap \Delta'$, $\Delta \cup \Delta'$ is primitive. Then there exists an ordering on \mathcal{S} that satisfies Condition 4.3.1, and in which Δ' and Δ appear consecutively.*

Proof. Choose an ordering on \mathcal{S} that satisfies Condition 4.3.1, and that minimizes the number of segments that appear between Δ' and Δ . Suppose there is a segment Δ'' between Δ and Δ' .

By our assumption on the chosen ordering, the ordering on \mathcal{S} obtained by moving Δ'' after Δ fails to satisfy Condition 4.3.1. There must thus be a segment Δ''' that appears between Δ'' and Δ in the chosen ordering, for which Δ''' precedes Δ'' . Similarly, Δ'' must precede a segment that appears between Δ' and Δ'' in the chosen ordering.

Applying these considerations repeatedly we obtain a chain:

$$\Delta' = \Delta_0, \Delta_1, \dots, \Delta_r = \Delta$$

such that each Δ_i precedes Δ_{i-1} , and appears after Δ_{i-1} in the chosen order on \mathcal{S} . Moreover, since Δ precedes Δ' , it follows that Δ precedes Δ_1 . The elementary operation on \mathcal{S} that replaces Δ and Δ' with $\Delta \cap \Delta'$ and $\Delta \cup \Delta'$ then factors as:

- (1) Replace Δ and Δ_1 with $\Delta \cup \Delta_1$ and $\Delta \cap \Delta_1$.
- (2) Replace Δ' and $\Delta \cup \Delta_1$ with $\Delta \cup \Delta_1 \cup \Delta'$ and $\Delta' \cap (\Delta \cup \Delta_1)$. (Note that $\Delta \cup \Delta_1 \cup \Delta'$ is equal to $\Delta \cup \Delta'$.)
- (3) Replace $\Delta' \cap (\Delta \cup \Delta_1)$ and $\Delta \cap \Delta_1$ with $[\Delta' \cap (\Delta \cup \Delta_1)] \cup [\Delta \cap \Delta_1]$ (which is equal to Δ_1), and $[\Delta' \cap (\Delta \cup \Delta_1)] \cap [\Delta \cap \Delta_1]$ (which is equal to $\Delta \cap \Delta'$.)

In particular the elementary operation that replaces Δ and Δ' with $\Delta \cap \Delta'$ and $\Delta \cup \Delta'$ is not primitive, as required. \square

4.3.6. Proposition. *Suppose that \mathcal{S} and \mathcal{S}' are multisegments, and that $\mathcal{S}' \preceq \mathcal{S}$. Then $\text{Hom}_{\overline{K}[\text{GL}_n(E)]}(\pi(\mathcal{S}'), \pi(\mathcal{S}))$ is one-dimensional over \overline{K} , and every nonzero map $\pi(\mathcal{S}') \rightarrow \pi(\mathcal{S})$ is an embedding.*

Proof. As $\pi(\mathcal{S}')$ and $\pi(\mathcal{S})$ are essentially AIG, it suffices to show that there exists a nonzero map $\pi(\mathcal{S}') \rightarrow \pi(\mathcal{S})$. Moreover, we may reduce to the case where \mathcal{S}' and \mathcal{S} differ by a single, primitive, elementary operation. Let \mathcal{S}' differ from \mathcal{S} by replacing Δ, Δ' with $\Delta \cup \Delta', \Delta \cap \Delta'$, where Δ' precedes Δ . By the above lemma we may choose an ordering on \mathcal{S} that satisfies Condition 4.3.1 in which Δ and Δ' are adjacent. We obtain from this ordering on \mathcal{S} an ordering on \mathcal{S}' in which $\Delta \cap \Delta'$ replaces Δ and $\Delta \cup \Delta'$ replaces Δ' ; this ordering also satisfies Condition 4.3.1. Let \mathcal{S}_0 be the multisegment consisting of the segments that appear in \mathcal{S} before Δ and Δ' in this chosen ordering, and let \mathcal{S}_1 be the multisegment consisting of the segments that appear in \mathcal{S} after Δ and Δ' .

By [15, Prop. 4.3], we find that $\pi(\Delta \cup \Delta', \Delta \cap \Delta')$ embeds in $\pi(\Delta, \Delta')$. But $\pi(\mathcal{S}')$ is isomorphic to

$$\mathrm{Ind}_P^{\mathrm{GL}_n} \pi(\mathcal{S}_0) \otimes \pi(\Delta \cup \Delta', \Delta \cap \Delta') \otimes \pi(\mathcal{S}_1),$$

for P a suitable block upper triangular parabolic subgroup of GL_n , and $\pi(\mathcal{S})$ is isomorphic to

$$\mathrm{Ind}_P^{\mathrm{GL}_n} \pi(\mathcal{S}_0) \otimes \pi(\Delta, \Delta') \otimes \pi(\mathcal{S}_1).$$

The embedding of $\pi(\Delta \cup \Delta', \Delta \cap \Delta')$ in $\pi(\Delta, \Delta')$ thus gives rise to a nonzero map of $\pi(\mathcal{S}')$ into $\pi(\mathcal{S})$, as required. \square

Moreover, the embeddings of $\pi(\mathcal{S}')$ into $\pi(\mathcal{S})$ constructed above descend to fields of definition:

4.3.7. Proposition. *Let π and π' be admissible representations over K , and suppose there are multisegments \mathcal{S} and \mathcal{S}' , with $\mathcal{S}' \preceq \mathcal{S}$, such that $\pi \otimes_K \overline{K}$ is isomorphic to $\pi(\mathcal{S})$ and $\pi' \otimes_K \overline{K}$ is isomorphic to $\pi(\mathcal{S}')$. Then $\mathrm{Hom}_{K[\mathrm{GL}_n(E)]}(\pi', \pi)$ is one-dimensional over K , and every nonzero map $\pi' \rightarrow \pi$ is an embedding.*

Proof. As π and π' are essentially AIG, $\mathrm{Hom}_{K[\mathrm{GL}_n(E)]}(\pi', \pi)$ is either zero or one-dimensional over K , and every nonzero map $\pi' \rightarrow \pi$ is an embedding. It thus suffices to construct a nonzero map from π' to π . Let $\phi : \pi' \otimes_K \overline{K} \rightarrow \pi \otimes_K \overline{K}$ be an embedding. By Lemma 3.2.5, a scalar multiple of ϕ descends to the desired embedding of π' in π . \square

We immediately deduce:

4.3.8. Corollary. *Let ρ be a continuous n -dimensional representation of G_E over K , and let π be an admissible representation of $\mathrm{GL}_n(E)$ over K , such that $\pi \otimes_K \overline{K}$ is isomorphic to $\pi(\rho \otimes_K \overline{K})$. Then π is isomorphic to $\pi(\rho)$.*

The above results allow us to establish some useful facts about essentially AIG envelopes in characteristic zero, that will be useful in the proof of Proposition 6.2.8.

4.3.9. Theorem. *Let π be an irreducible generic representation of $\mathrm{GL}_n(E)$ over an algebraically closed field K of characteristic zero, and let π_1, \dots, π_r be the supercuspidal support of π , ordered so that Condition 4.3.1 holds (when the π_i are treated as one-element segments.) Then the parabolic induction*

$$\mathrm{Ind}_P^{\mathrm{GL}_n(E)} \pi_1 \otimes \cdots \otimes \pi_r$$

is an essentially AIG envelope of π . (Here $P = MU$ is a suitable block upper triangular parabolic subgroup of $\mathrm{GL}_n(E)$, with Levi subgroup M and unipotent radical U .)

Proof. By Corollary 4.3.3, the representation

$$\mathrm{Ind}_P^{\mathrm{GL}_n(E)} \pi_1 \otimes \cdots \otimes \pi_r$$

is essentially AIG. Its socle is thus an irreducible generic representation with the same supercuspidal support as π , and is therefore isomorphic to π , by Proposition 3.2.12. It thus remains to show that any essentially AIG representation W whose socle is isomorphic to π embeds in

$$\mathrm{Ind}_P^{\mathrm{GL}_n(E)} \pi_1 \otimes \cdots \otimes \pi_r.$$

Note that as any map of essentially AIG representations is injective, it suffices to construct a map:

$$W \rightarrow \mathrm{Ind}_P^{\mathrm{GL}_n(E)} \pi_1 \otimes \cdots \otimes \pi_r.$$

By Frobenius Reciprocity, this is equivalent to constructing a map:

$$\mathrm{Res}_{\mathrm{GL}_n(E)}^P W \rightarrow \pi_1 \otimes \cdots \otimes \pi_r.$$

As every Jordan–Hölder constituent of W has supercuspidal support $\{\pi_1, \dots, \pi_r\}$ by Corollary 3.2.14, it follows that every Jordan–Hölder constituent of $\mathrm{Res}_{\mathrm{GL}_n(E)}^P W$ is a supercuspidal representation of M , and at least one of these Jordan–Hölder constituents is isomorphic to $\pi_1 \otimes \cdots \otimes \pi_r$. By Theorem 3.2.13, $\pi_1 \otimes \cdots \otimes \pi_r$ only admits nontrivial extensions (as an M -representation) with irreducible representations isomorphic to $\pi_1 \otimes \cdots \otimes \pi_r$. Thus $\mathrm{Res}_{\mathrm{GL}_n(E)}^P W$ admits a quotient isomorphic to $\pi_1 \otimes \cdots \otimes \pi_r$ and the result follows. \square

4.3.10. Corollary. *Let W be an essentially AIG representation of $\mathrm{GL}_n(E)$ over a field K of characteristic zero. Then W has finite length.*

Proof. Let W' be the essentially AIG envelope of $\mathrm{soc}(W)$. By the preceding lemma, $W' \otimes_K \overline{K}$ has finite length, so W' , and hence W , has finite length. \square

It will be useful later, in the proof of Proposition 6.2.8, to have control over the multiplicities of Jordan–Hölder constituents of essentially AIG representations. We observe:

4.3.11. Corollary. *Let W be an essentially AIG representation of $\mathrm{GL}_2(E)$ or $\mathrm{GL}_3(E)$ over a field K of characteristic zero. Then no Jordan–Hölder constituent of W appears with multiplicity greater than one.*

Proof. Theorem 4.3.9 above shows that W embeds in some parabolic induction

$$\mathrm{Ind}_P^{\mathrm{GL}_n(E)} \pi_1 \otimes \cdots \otimes \pi_r$$

with each π_i cuspidal. Zelevinski’s computations of multiplicities of the Jordan–Hölder constituents of such inductions ([21, §11]) shows that when $r \leq 3$, each Jordan–Hölder constituent of such an induction occurs with multiplicity one. The result follows immediately. \square

4.3.12. Remark. In contrast to the preceding proposition, if $n = 4$, and if we choose a Levi subgroup of the form $(E^\times)^4$ of $\mathrm{GL}_4(E)$,

$$\mathrm{Ind}_P^{\mathrm{GL}_n(E)} | \cdot |^2 \otimes | \cdot | \otimes | \cdot | \otimes 1$$

has a Jordan–Hölder constituent that appears with multiplicity two.

4.4. Reduction of $\pi(\mathcal{S})$. We now turn to integrality considerations. We continue to suppose that \mathcal{O} is a discrete valuation ring, with residue field K of characteristic zero, uniformizer ϖ , and field of fractions \mathcal{K} . We say an admissible representation π over \mathcal{K} is \mathcal{O} -integral if it contains a ϖ -adically separated \mathcal{O} -lattice.

4.4.1. Lemma. *Let π be an absolutely irreducible supercuspidal representation of $\mathrm{GL}_n(E)$ over \mathcal{K} . Then π is \mathcal{O} -integral if and only if its central character takes values in \mathcal{O}^\times . In this case there is a ϖ -adically separated $\mathrm{GL}_n(E)$ -stable \mathcal{O} -lattice π° in π , unique up to homothety, such that the reduction $\pi^\circ / \varpi \pi^\circ$ is absolutely irreducible and supercuspidal.*

Proof. Clearly if π is \mathcal{O} -integral, then its central character takes values in \mathcal{O}^\times . Let \mathcal{K}' be a finite Galois extension of \mathcal{K} , such that there exists a character $\chi : E^\times \rightarrow \mathcal{K}^\times$ whose n th power is the central character of π . If the central character of π takes values in \mathcal{O}^\times , then χ takes values in $(\mathcal{O}')^\times$, where \mathcal{O}' is the integral closure of \mathcal{O} in \mathcal{K}' .

The central character of $\pi \otimes \chi^{-1} \circ \det$ is trivial. By [17, II.4.9], $\pi \otimes \chi^{-1} \circ \det$ is defined over a finite extension F of \mathbb{Q}_p , contained in \mathcal{K}' . That is, there exists an admissible representation π_0 over F such that $\pi_0 \otimes_F \mathcal{K}'$ is isomorphic to $\pi \otimes \chi^{-1} \circ \det$. As \mathcal{O}' has residue characteristic zero, F is contained in \mathcal{O}' . Thus $\pi^\circ := (\pi_0 \otimes_F \mathcal{O}') \otimes (\chi \circ \det)$ is a ϖ' -adically separated \mathcal{O}' -lattice in $\pi \otimes_{\mathcal{K}} \mathcal{K}'$, where ϖ' is a uniformizer of \mathcal{O}' .

The reduction modulo ϖ' of π° is $(\pi_0 \otimes_F K') \otimes (\bar{\chi} \circ \det)$, where K' is the residue field of \mathcal{O}' and $\bar{\chi}$ is the reduction of χ modulo ϖ' . In particular $\pi^\circ / \varpi \pi^\circ$ is absolutely irreducible and supercuspidal (and therefore π° is unique up to homothety.) It follows that π° is stable under the action of $\text{Gal}(\mathcal{K}'/\mathcal{K})$, and hence descends to a $\text{GL}_n(E)$ -stable lattice in π (which must also be unique up to homothety.) \square

Given an \mathcal{O} -integral absolutely irreducible supercuspidal representation π of $\text{GL}_n(E)$ over \mathcal{K} , we can thus define $\bar{\pi}$ to be the reduction mod ϖ of any ϖ -adically separated $\text{GL}_n(E)$ -stable \mathcal{O} -lattice in π . For a segment $\Delta = [\pi, (| \circ \det)\pi, \dots, (| \circ \det)^{r-1}\pi]$, let $\bar{\Delta}$ be the segment $[\bar{\pi}, (| \circ \det)\bar{\pi}, \dots, (| \circ \det)^{r-1}\bar{\pi}]$. If \mathcal{S} is a multiset of integral segments, define $\bar{\mathcal{S}}$ to be the multiset consisting of the segments $\bar{\Delta}_i$ for $\Delta_i \in \mathcal{S}$.

4.4.2. Lemma. *Let π be an \mathcal{O} -integral, absolutely irreducible supercuspidal representation of $\text{GL}_n(E)$ over \mathcal{K} , and let Δ be the segment $[\pi, (| \circ \det)\pi, \dots, (| \circ \det)^{r-1}\pi]$. There is a ϖ -adically separated, $\text{GL}_n(E)$ -stable \mathcal{O} -lattice St_Δ° in St_Δ , unique up to homothety, and $\text{St}_\Delta^\circ / \varpi \text{St}_\Delta^\circ$ is isomorphic to $\text{St}_{\bar{\Delta}}$.*

Proof. This follows by precisely the same argument as in Lemma 4.4.1. \square

If we want to consider the reduction mod ϖ of representations of the form $\pi(\mathcal{S})$, then the situation is more complicated, as $\pi(\mathcal{S})$ typically contains more than one homothety class of lattices. However, Theorem 3.3.2 allows us to single out a preferred such homothety class.

4.4.3. Proposition. *If \mathcal{S} is a multiset over \mathcal{K} that are \mathcal{O} -integral, then there is an \mathcal{O} -lattice $\pi(\mathcal{S})^\circ$ in $\pi(\mathcal{S})$, unique up to homothety, such that $\pi(\mathcal{S})^\circ / \varpi \pi(\mathcal{S})^\circ$ is essentially AIG. Moreover, $\pi(\mathcal{S})^\circ / \varpi \pi(\mathcal{S})^\circ$ is isomorphic to $\pi(\bar{\mathcal{S}})$.*

Proof. Theorem 3.3.2 shows that $\pi(\mathcal{S})^\circ$ exists and is unique up to homothety. Let $\Delta_1, \dots, \Delta_r$ be the segments in \mathcal{S} , and fix for each i a ϖ -adically separated \mathcal{O} -lattice L_i in St_{Δ_i} . Then $L_i / \varpi L_i$ is isomorphic to $\text{St}_{\bar{\Delta}_i}$. Recall that

$$\pi(\mathcal{S}) = \text{Ind}_P^{\text{GL}_n(E)} \text{St}_{\Delta_1} \otimes \dots \otimes \text{St}_{\Delta_r},$$

and hence contains the integral induction $\text{Ind}_P^{\text{GL}_n(E)} L_1 \otimes \dots \otimes L_r$ as a lattice. The mod ϖ reduction of this lattice is clearly isomorphic to $\pi(\bar{\mathcal{S}})$, which is essentially AIG. Thus $\text{Ind}_P^{\text{GL}_n(E)} L_1 \otimes \dots \otimes L_r$ is homothetic to $\pi(\mathcal{S})^\circ$, and hence $\pi(\mathcal{S})^\circ / \varpi \pi(\mathcal{S})^\circ$ is indeed isomorphic to $\pi(\bar{\mathcal{S}})$, as claimed. \square

4.4.4. Corollary. *Let \mathcal{K}' be a finite Galois extension of \mathcal{K} , and let \mathcal{O}' be the integral closure of \mathcal{O} in \mathcal{K}' . Let π be an admissible representation of $\text{GL}_n(E)$ over \mathcal{K} , and*

let \mathcal{S} be a multisegment consisting of \mathcal{O}' -integral segments over \mathcal{K}' . Suppose that $\pi \otimes_{\mathcal{K}} \mathcal{K}'$ is isomorphic to $\pi(\mathcal{S})$. Then π is \mathcal{O} -integral, and there is a ϖ -adically separated \mathcal{O} -lattice π° in π , unique up to homothety, such that $\pi^\circ / \varpi \pi^\circ$ is essentially AIG. Moreover, $\pi^\circ / \varpi \pi^\circ \otimes_{\mathcal{K}} \mathcal{K}'$ is isomorphic to $\pi(\overline{\mathcal{S}})$.

Proof. It suffices to show that the lattice $\pi(\mathcal{S})^\circ$ constructed in the previous proposition is stable under the action of $\text{Gal}(\mathcal{K}'/\mathcal{K})$. This is clear since $\pi(\mathcal{S})^\circ$ is unique up to homothety. \square

4.5. Compatibility with specialization. We now use the results of the previous sections to understand the relationship between $\pi(\rho \otimes_{\mathcal{O}} \mathcal{K})$ and $\pi(\rho \otimes_{\mathcal{O}} K)$, where $\rho : G_E \rightarrow \text{GL}_n(\mathcal{O})$ is a continuous Galois representation. The key idea is a geometric interpretation of the partial order \preceq on multisegments, due to Zelevinski [22].

Let $V = \bigoplus_{\pi} V_{\pi}$ be a finite-dimensional vector space over a field F , “graded” by the set of isomorphism classes of irreducible supercuspidal representations of $\text{GL}_m(E)$ over $\overline{\mathcal{K}}$, for all m . We denote the automorphisms of V as a graded F -vector space by $\text{Aut}(V)$, and let $\text{End}^+(V)$ denote the space of F -linear endomorphisms of V that take V_{π} to $V_{(| \cdot | \circ \det)_{\pi}}$ for all π . Let N_V be an element of $\text{End}^+(V)$; it is a nilpotent endomorphism of V .

We construct a bijection between the set of isomorphism classes of pairs (V, N_V) and the set of multisegments \mathcal{S} over $\overline{\mathcal{K}}$, as follows: For any segment $\Delta = [\pi, (|\cdot| \circ \det)_{\pi}, \dots, (|\cdot| \circ \det)^{r-1}_{\pi}]$, let $V_{\Delta, F}$ be the vector space defined by $(V_{\Delta, F})_{\pi'} = F$ if π' is in Δ , and zero otherwise. We define an endomorphism $N_{\Delta, F}$ of $V_{\Delta, F}$ that is an isomorphism $(V_{\Delta, F})_{(| \cdot | \circ \det)^i_{\pi}} \rightarrow (V_{\Delta, F})_{(| \cdot | \circ \det)^{i+1}_{\pi}}$ for $0 \leq i < r-1$, and zero otherwise.

For a multisegment \mathcal{S} , we define:

$$(V_{\mathcal{S}, F}, N_{\mathcal{S}, F}) = \bigoplus_{\Delta \in \mathcal{S}} (V_{\Delta, F}, N_{\Delta, F}).$$

It is easy to see (for instance, by the structure theory of graded $F[N]/N^r$ -modules) that the association $\mathcal{S} \mapsto (V_{\mathcal{S}, F}, N_{\mathcal{S}, F})$ yields a bijection between multisegments and isomorphism classes of pairs (V, N_V) .

4.5.1. Theorem ([22, §2]). *Let \mathcal{S}' and \mathcal{S} be multisegments over $\overline{\mathcal{K}}$. Then $\mathcal{S}' \preceq \mathcal{S}$ if and only if $V_{\mathcal{S}, F}$ is isomorphic to $V_{\mathcal{S}', F}$ as a graded F -vector space, and $N_{\mathcal{S}, F}$ is in the closure of the orbit of $N_{\mathcal{S}', F}$ under the action of $\text{Aut}(V_{\mathcal{S}', F})$ on $\text{End}^+(V_{\mathcal{S}', F})$.*

As a result, if $\mathcal{S}' \preceq \mathcal{S}$, then, for all i , the rank of $N_{\mathcal{S}, F}^i$ is less than or equal to that of $N_{\mathcal{S}', F}^i$. These ranks are equal for all i if, and only if, $\mathcal{S}' = \mathcal{S}$.

If (ρ', N) is a Frobenius-semisimple Weil–Deligne representation over $\overline{\mathcal{K}}$, and if \mathcal{S} is the multisegment such that $\pi(\mathcal{S}) = (|\cdot| \circ \det)^{\frac{n-1}{2}} \pi(\rho', N)$, then the pair $(V_{\mathcal{S}, \overline{\mathcal{K}}}, N_{\mathcal{S}, \overline{\mathcal{K}}})$ can be described easily in terms of (ρ', N) . Indeed, one has:

4.5.2. Lemma. *For any supercuspidal representation π of $\text{GL}_m(E)$ over $\overline{\mathcal{K}}$, there is a natural isomorphism:*

$$(V_{\mathcal{S}, \overline{\mathcal{K}}})_{\pi} \xrightarrow{\sim} \text{Hom}_{\overline{\mathcal{K}}[W_E]}(\rho, \rho'),$$

where ρ is the absolutely irreducible representation of W_E that corresponds to π under the unitary local Langlands correspondence. Moreover, under these isomorphisms, the map

$$N_{\mathcal{S}, \overline{\mathcal{K}}} : (V_{\mathcal{S}, \overline{\mathcal{K}}})_{\pi} \rightarrow (V_{\mathcal{S}, \overline{\mathcal{K}}})_{(| \cdot | \circ \det)_{\pi}}$$

is identified with the map

$$\mathrm{Hom}_{\overline{\mathcal{K}}[W_E]}(\rho, \rho') \rightarrow \mathrm{Hom}_{\overline{\mathcal{K}}[W_E]}(| \cdot | \otimes \rho, \rho')$$

induced by N .

Proof. This is true by construction if (ρ', N) is indecomposable, and extends to the general case by taking direct sums. \square

Zelevinski's result strongly suggests a connection between the Zelevinski partial order and reduction of Weil–Deligne representations. In order to make this connection precise we need a compatibility between the reduction mod ϖ and local Langlands:

4.5.3. Lemma. *If ρ' is absolutely irreducible, and $\pi \otimes_{\mathcal{K}} \overline{\mathcal{K}}$ corresponds to $\rho' \otimes_{\mathcal{K}} \overline{\mathcal{K}}$ under the unitary local Langlands correspondence, then $\overline{\pi} \otimes_K \overline{K}$ and $\overline{\rho'} \otimes_K \overline{K}$ correspond under the unitary local Langlands correspondence.*

Proof. We first translate this into a statement in terms of the generic local Langlands correspondence. From this point of view the representation $\pi(\rho')$ is isomorphic to $(| \cdot | \circ \det)^{-\frac{n-1}{2}} \pi$, and we must show that $\pi(\overline{\rho'})$ is isomorphic to $(| \cdot | \circ \det)^{-\frac{n-1}{2}} \overline{\pi}$.

There is a finite extension \mathcal{K}' of \mathcal{K} , and a character $\chi : W_E \rightarrow (\mathcal{K}')^\times$, such that $\rho' \otimes \chi$ is defined over \mathbb{Q}_p ; as ρ' is integral χ takes values in $(\mathcal{O}')^\times$, where \mathcal{O}' is the integral closure of \mathcal{O} in \mathcal{K}' . In particular, there is a finite extension K_0 of \mathbb{Q}_p , contained in \mathcal{K}' , and a representation $\rho_0 : W_E \rightarrow \mathrm{GL}_n(K_0)$, such that $\rho_0 \otimes_{K_0} \mathcal{K}'$ is isomorphic to $\rho' \otimes \chi$. If we let K' be the residue field of \mathcal{O}' , and let $\overline{\chi}$ be the reduction mod ϖ' of the character χ , then K_0 is contained in K' and $\rho_0 \otimes_{K_0} K'$ is isomorphic to $\overline{\rho'} \otimes \chi$. It follows that $\pi(\overline{\rho'}) \otimes \chi$ is isomorphic to $(| \cdot | \circ \det)^{-\frac{n-1}{2}} \pi(\rho_0) \otimes \chi$.

Let $\pi_0 = (| \cdot | \circ \det)^{-\frac{n-1}{2}} \pi(\rho_0)$. As the generic local Langlands correspondence is compatible with twists and base change, the representation $\pi \otimes \chi$ is isomorphic to $\pi_0 \otimes_{K_0} \mathcal{K}'$. Thus $\overline{\pi} \otimes \chi$ is isomorphic to $\pi_0 \otimes_{K_0} K'$, and hence to $(| \cdot | \circ \det)^{-\frac{n-1}{2}} \pi(\overline{\rho'}) \otimes \chi$. The result follows. \square

If $\overline{\mathcal{S}}$ is obtained from a multisegment \mathcal{S} by reduction mod ϖ , the pairs $(V_{\mathcal{S}, F}, N_{\mathcal{S}, F})$ and $(V_{\overline{\mathcal{S}}, F}, N_{\overline{\mathcal{S}}, F})$ are related by $(V_{\overline{\mathcal{S}}, F})_{\overline{\pi}} = \bigoplus_{\pi'} (V_{\mathcal{S}, F})_{\pi'}$, where the sum is over π' with $\overline{\pi'} = \overline{\pi}$.

Now let (ρ', N) be a Weil–Deligne representation over \mathcal{O} such that the restriction of $\rho' \otimes_{\mathcal{O}} \mathcal{K}$ to I_E is a direct sum of absolutely irreducible representations of I_E over \mathcal{K} , and such that $\rho' \otimes_{\mathcal{O}} \mathcal{K}$ is a direct sum of absolutely irreducible representations ρ'_i of W_E . (We can always arrange this by replacing \mathcal{K} with a finite extension.) Let \mathcal{S} be the segment associated to $(\rho', N) \otimes_{\mathcal{O}} \mathcal{K}$; we have

$$(V_{\mathcal{S}, \mathcal{K}})_{\pi_i} = \mathrm{Hom}_{\mathcal{K}[W_E]}(\rho_i, \rho' \otimes_{\mathcal{O}} \mathcal{K}),$$

where π_i corresponds to ρ_i under unitary local Langlands. We also consider the K -vector-space $V_{\overline{\mathcal{S}}, K}$.

Let $(\overline{\rho'}, \overline{N})$ be the Weil–Deligne representation $(\rho', N) \otimes_{\mathcal{O}} K$, and let \mathcal{S}' be the multisegment associated to $(\overline{\rho'}, \overline{N})^{\mathrm{F-ss}}$. Our goal is to compare \mathcal{S}' to $\overline{\mathcal{S}}$; we will do this by comparing $V_{\overline{\mathcal{S}}, \mathcal{K}}$ to $V_{\mathcal{S}', K}$. The key difficulty is to construct an \mathcal{O} -lattice in $V_{\overline{\mathcal{S}}, \mathcal{K}}$, stable under $N_{\overline{\mathcal{S}}, \mathcal{K}}$, whose reduction mod ϖ is isomorphic to $V_{\mathcal{S}', K}$.

By Lemma 4.1.1 there is a finite extension K_0 of \mathbb{Q}_p and representations τ_1, \dots, τ_n of I_E over K_0 , each in its own orbit under conjugation by Φ , such that the restriction

$\rho' \otimes_{\mathcal{O}} \mathcal{K}$ is a direct sum of Φ -conjugates of the τ_i , each with multiplicity one. For each i , let L_i be the \mathcal{O} -module $\mathrm{Hom}_{\mathcal{O}[I_E]}(\tau_i \otimes_{K_0} \mathcal{O}, \rho')$.

For each j such that the restriction of ρ'_j to I_E contains a copy of $\tau_i \otimes_{K_0} \mathcal{K}$, the restriction map

$$\mathrm{Hom}_{\mathcal{K}[W_E]}(\rho'_j, \rho') \rightarrow \mathrm{Hom}_{\mathcal{K}[I_E]}(\tau_i \otimes_{K_0} \mathcal{K}, \rho')$$

is an injection. We thus obtain an isomorphism:

$$L_i \otimes_{\mathcal{O}} \mathcal{K} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}[I_E]}(\tau_i \otimes_{K_0} \mathcal{K}, \rho') \xrightarrow{\sim} \bigoplus_j \mathrm{Hom}_{\mathcal{K}[W_E]}(\rho'_j, \rho'),$$

where the latter sum is over those j such that the restriction of ρ'_j to I_E contains a copy of $\tau_i \otimes_{K_0} \mathcal{K}$. This condition is satisfied if, and only if, the restriction of $\bar{\rho}'_j$ to I_E contains a copy of $\tau_i \otimes_{K_0} K$. We can thus view L_i as a sublattice of $\bigoplus_j (V_{\bar{\mathcal{S}}, \mathcal{K}})_{\bar{\pi}_j}$, where the sum is over those j such that the restriction of $\bar{\rho}'_j$ to I_E contains a copy of $\tau_i \otimes_{K_0} K$. For any such j , let $L_{\bar{\pi}_j}$ be the intersection $L_i \cap (V_{\bar{\mathcal{S}}, \mathcal{K}})_{\bar{\pi}_j}$.

In fact, this gives a direct sum decomposition of L_i . To see this, first observe:

4.5.4. Lemma. *Let M be a free \mathcal{O} -module of finite rank, and Ψ is an \mathcal{O} -linear endomorphism of M that acts semisimply on $M \otimes_{\mathcal{O}} \mathcal{K}$. Suppose that all of the eigenvalues of $\Psi \otimes_{\mathcal{O}} \mathcal{K}$ lie in \mathcal{K} , and for each eigenvalue $\tilde{\lambda}$ of $\Psi \otimes_{\mathcal{O}} \mathcal{K}$, let $M_{\tilde{\lambda}}$ be the intersection of M with the $\tilde{\lambda}$ -eigenspace of $\Psi \otimes_{\mathcal{O}} \mathcal{K}$. Then M decomposes as*

$$M = \bigoplus_{\lambda} M_{\lambda},$$

where λ runs over the eigenvalues of $\bar{\Psi} : M/\varpi M \rightarrow M/\varpi M$ and M_{λ} is the sum of $M_{\tilde{\lambda}}$ for those $\tilde{\lambda}$ congruent to λ modulo ϖ . The endomorphism Ψ acts on $M_{\lambda}/\varpi M_{\lambda}$ as the product of λ with a unipotent endomorphism of $M_{\lambda}/\varpi M_{\lambda}$.

Proof. Let $P(t)$ be the minimal polynomial of Ψ , and consider M as an $\mathcal{O}[t]/P(t)$ -module on which t acts by Ψ . The connected components of $\mathrm{Spec} \mathcal{O}[t]/P(t)$ are in bijection with the roots λ of the mod ϖ reduction $\bar{\Psi}$ of Ψ ; these are the eigenvalues of $\bar{\Psi}$. Thus, considered as a sheaf on $\mathrm{Spec} \mathcal{O}[t]/P(t)$, M decomposes as a direct sum of sheaves M_{λ} supported on each connected component. On each M_{λ} , the minimal polynomial of $\bar{\Psi}$ is a power of $t - \lambda$, so $\lambda^{-1} \bar{\Psi}$ is unipotent on M_{λ} . \square

4.5.5. Lemma. *We have a direct sum decomposition: $L_i = \bigoplus_j L_{\bar{\pi}_j}$, where the sum is over those j such that the restriction of $\bar{\rho}'_j$ to I_E contains a copy of $\tau_i \otimes_{K_0} K$.*

Proof. Let r be the size of the orbit of τ_i under the conjugation action of Φ , and fix an isomorphism $\tau_i^{\Phi^r} \xrightarrow{\sim} \tau_i$. This isomorphism induces an endomorphism Ψ of $\mathrm{Hom}_{\mathcal{O}[I_E]}(\tau_i \otimes_{K_0} \mathcal{O}, \rho')$ via

$$\mathrm{Hom}_{\mathcal{O}[I_E]}(\tau_i \otimes_{K_0} \mathcal{O}, \rho') \xrightarrow{\Phi^r} \mathrm{Hom}_{\mathcal{O}[I_E]}(\tau_i^{\Phi^r} \otimes_{K_0} \mathcal{O}, \rho') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}[I_E]}(\tau_i \otimes_{K_0} \mathcal{O}, \rho').$$

Let ρ'_j be an absolutely irreducible summand of $\rho' \otimes_{\mathcal{O}} \mathcal{K}$ whose restriction to I_E contains a copy of $\tau_i \otimes_{K_0} \mathcal{K}$. This copy is unique, and yields a restriction map:

$$\mathrm{Hom}_{\mathcal{K}[W_E]}(\rho'_j, \rho' \otimes_{\mathcal{O}} \mathcal{K}) \rightarrow \mathrm{Hom}_{\mathcal{K}[I_E]}(\tau_i, \rho' \otimes_{\mathcal{O}} \mathcal{K}).$$

This restriction map is injective, and its image can be characterized in terms of Ψ . In particular, the endomorphism:

$$\mathrm{Hom}_{\mathcal{K}[I_E]}(\tau_i \otimes_{K_0} \mathcal{K}, \rho'_j) \xrightarrow{\Phi^r} \mathrm{Hom}_{\mathcal{K}[I_E]}(\tau_i^{\Phi^r} \otimes_{K_0} \mathcal{K}, \rho'_j) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}[I_E]}(\tau_i \otimes_{K_0} \mathcal{K}, \rho'_j).$$

is an endomorphism of one-dimensional \mathcal{K} vector spaces and is thus given by a scalar $\tilde{\lambda}$; it follows by Lemma 4.1.2 that ρ'_j is determined by $\tilde{\lambda}$ and τ_i , and that the image of the map:

$$\mathrm{Hom}_{\mathcal{K}[W_E]}(\rho'_j, \rho' \otimes_{\mathcal{O}} \mathcal{K}) \rightarrow \mathrm{Hom}_{\mathcal{K}[I_E]}(\tau_i, \rho' \otimes_{\mathcal{O}} \mathcal{K})$$

is the $\tilde{\lambda}$ -eigenspace of Ψ .

Now let $\bar{\rho}_j$ be an absolutely irreducible summand of $\rho' \otimes_{\mathcal{O}} K$ whose restriction to I_E contains a copy of $\tau_i \otimes_{K_0} K$, and let $\bar{\pi}_j$ be the corresponding admissible representation. Then the endomorphism Ψ of $\mathrm{Hom}_{\mathcal{K}[I_E]}(\tau_i \otimes_{K_0} K, \bar{\rho}_j)$ is a scalar λ , and, by the same reasoning as above, $(V_{\bar{\mathcal{S}}, \mathcal{K}})_{\bar{\pi}_j}$ is the sum of the $\tilde{\lambda}$ -eigenspaces of Ψ for those $\tilde{\lambda}$ congruent to λ modulo ϖ . Thus, by the preceding lemma, $L_{\bar{\pi}_j}$ is a direct summand of L_i . \square

Let L be the lattice in $V_{\bar{\mathcal{S}}, \mathcal{K}}$ defined by:

$$L = \bigoplus_{\pi} L_{\pi}.$$

Note that as $N_{\bar{\mathcal{S}}, \mathcal{K}}$ preserves each L_i , it also preserves L .

4.5.6. Lemma. *There is a natural isomorphism $L/\varpi L \xrightarrow{\sim} V_{\mathcal{S}', K}$. Moreover, the endomorphism $N_{\bar{\mathcal{S}}, \mathcal{K}}$ of L reduces to $N_{\mathcal{S}', K}$ under this isomorphism.*

Proof. Recall that \mathcal{S}' is the multisegment associated to $(\bar{\rho}', \bar{N})^{\mathrm{F-ss}}$. Let $\bar{\rho}_i$ be any absolutely irreducible Jordan–Hölder constituent $\bar{\rho}_i$ of $(\bar{\rho}')^{\mathrm{ss}}$, corresponding to an admissible representation $\bar{\pi}_i$ under unitary local Langlands. Then $(V_{\mathcal{S}', K})_{\bar{\pi}_i}$ is equal to $\mathrm{Hom}_{\mathcal{K}[W_E]}(\bar{\rho}_i, (\bar{\rho}')^{\mathrm{ss}})$. It thus suffices to construct, for each i , a natural isomorphism of $(L/\varpi L)_{\bar{\pi}_i}$ with $(V_{\mathcal{S}', K})_{\bar{\pi}_i}$.

Let τ be an absolutely irreducible representation of I_E over K_0 such that $\bar{\rho}_i$ contains $\tau \otimes_{K_0} K$; let r be the order of the orbit of τ under conjugation by Φ , and fix an isomorphism of τ with τ^{Φ^r} . Let $\lambda \in K^{\times}$ be the scalar giving the action of Φ^r on $\mathrm{Hom}_{\mathcal{K}[I_E]}(\tau \otimes_{K_0} K, \bar{\rho}_i|_{I_E})$ under this identification.

We also have an action of Φ^r on $\mathrm{Hom}_{\mathcal{K}[I_E]}(\tau \otimes_{K_0} K, (\bar{\rho}')^{\mathrm{ss}}|_{I_E})$; this yields a linear endomorphism $\bar{\Psi}^{\mathrm{ss}}$ of $\mathrm{Hom}_{\mathcal{K}[I_E]}(\tau \otimes_{K_0} K, (\bar{\rho}')^{\mathrm{ss}}|_{I_E})$. The natural map

$$(V_{\mathcal{S}', K})_{\bar{\pi}_i} \rightarrow \mathrm{Hom}_{\mathcal{K}[I_E]}(\tau \otimes_{K_0} K, (\bar{\rho}')^{\mathrm{ss}}|_{I_E})$$

identifies $(V_{\mathcal{S}', K})_{\bar{\pi}_i}$ with the λ -eigenspace of $\bar{\Psi}^{\mathrm{ss}}$.

On the other hand, the previous lemma shows that $L_{\bar{\pi}_i}$ is the sum of the $\tilde{\lambda}$ -eigenspaces of Ψ on $\mathrm{Hom}_{\mathcal{O}[I_E]}(\tau \otimes_{K_0} \mathcal{O}, \rho'|_{I_E})$; it follows that $L_{\bar{\pi}_i}/\varpi L_{\bar{\pi}_i}$ is the λ -generalized eigenspace of $\Psi/\varpi\Psi$ on $\mathrm{Hom}_{\mathcal{K}[I_E]}(\tau \otimes_{K_0} K, \bar{\rho}'|_{I_E})$.

Finally, observe that $\bar{\Psi}^{\mathrm{ss}}$ is the semisimplification of $\Psi/\varpi\Psi$, so that the λ -generalized eigenspace of $\Psi/\varpi\Psi$ is equal to the λ -eigenspace of $\bar{\Psi}^{\mathrm{ss}}$, and hence to $(V_{\mathcal{S}', K})_{\bar{\pi}_i}$. One verifies easily that these identifications are all compatible with the monodromy operators. \square

By Theorem 4.5.1 it follows that for \mathcal{S} and \mathcal{S}' as in Lemma 4.5.6, we must have $\bar{\mathcal{S}} \preceq \mathcal{S}'$. Moreover, we have equality if, and only if, the ranks of the operators $N_{\bar{\mathcal{S}}, \mathcal{K}}^i$ and $N_{\mathcal{S}', K}^i$ agree for all i . We are thus finally in a position to prove:

4.5.7. Theorem. *Let $\rho : G_E \rightarrow \mathrm{GL}_n(\mathcal{O})$ be a continuous Galois representation, and (ρ', N) the Frobenius-semisimplification of the corresponding Weil–Deligne representation. Then there is a ϖ -adically separated \mathcal{O} -lattice $\pi(\rho)^{\circ}$ in $\pi(\rho \otimes_{\mathcal{O}} \mathcal{K})$, unique*

up to homothety, such that $\pi(\rho)^\circ/\varpi\pi(\rho)^\circ$ is essentially AIG, and an embedding

$$\pi(\rho)^\circ/\varpi\pi(\rho)^\circ \rightarrow \pi(\bar{\rho}),$$

where $\bar{\rho} = \rho \otimes_{\mathcal{O}} K$. This embedding is an isomorphism if, and only if, the K -rank of \bar{N}^i equals the \mathcal{K} -rank of $(N \otimes_{\mathcal{O}} \mathcal{K})^i$ for all i .

Proof. Let $(\bar{\rho}', \bar{N})$ be the reduction mod ϖ of (ρ', N) . Then, by Lemmas 4.1.8 and 4.1.9, $(\bar{\rho}', \bar{N})^{\text{F-ss}}$ is the Frobenius-semisimplification of the Weil–Deligne representation attached to $\bar{\rho}$.

Over a finite extension \mathcal{K}' of \mathcal{K} , we may assume that ρ' splits as a direct sum of absolutely irreducible representations of W_E , and similarly for its restriction to I_E . The corresponding statements then hold for the semisimplification of $\bar{\rho}'$.

Let \mathcal{O}' be the integral closure of \mathcal{O} in \mathcal{K}' , and let K' be its residue field. Let \mathcal{S} and \mathcal{S}' be the segments associated to $(\rho', N) \otimes_{\mathcal{O}} \mathcal{O}'$ and $(\bar{\rho}', \bar{N})^{\text{F-ss}} \otimes_{K'} K'$. We have shown that $\mathcal{S} \preceq \mathcal{S}'$.

On the other hand, we have $\pi(\rho \otimes_{\mathcal{O}} \mathcal{K}')$ is isomorphic to $(|\cdot| \circ \det)^{-\frac{n}{2}} \pi(\mathcal{S})$; by Corollary 4.4.4 there is, up to homothety, a unique lattice $\pi(\rho)^\circ$ in $\pi(\rho \otimes_{\mathcal{O}} \mathcal{K})$ such that $\pi(\rho)^\circ/\varpi\pi(\rho)^\circ$ is essentially AIG; moreover one has an isomorphism

$$[\pi(\rho)^\circ/\varpi\pi(\rho)^\circ] \otimes_{K'} K' \xrightarrow{\sim} \pi(\bar{\mathcal{S}}).$$

As $\pi(\bar{\rho} \otimes_{K'} K')$ is isomorphic to $(|\cdot| \circ \det)^{-\frac{n}{2}} \pi(\mathcal{S}')$, and $\bar{\mathcal{S}} \preceq \mathcal{S}'$, we have an embedding of $[\pi(\rho)^\circ/\varpi\pi(\rho)^\circ] \otimes_{K'} K'$ in $\pi(\bar{\rho} \otimes_{K'} K')$. This embedding descends to K by Proposition 4.3.7.

Finally, this embedding is an isomorphism if, and only if, $\bar{\mathcal{S}}$ is equal to \mathcal{S}' . This is true if, and only if, the ranks of $N_{\bar{\mathcal{S}}, K}^i$ and $N_{\mathcal{S}', K}^i$ agree for all i ; it is easy to see this is equivalent to requiring that the ranks of $(N \otimes_{\mathcal{O}} K)^i$ and \bar{N}^i agree for all i . \square

4.5.8. Remark. An alternative approach to some of the above questions is given in [3], particularly Proposition 3.11. Chenevier constructs, for each Bernstein component \mathcal{B} of the category of smooth representations of $\text{GL}_n(E)$, a pseudocharacter of W_E valued in the algebra of functions on \mathcal{B} that “is compatible with the local Langlands correspondence”, in the sense that if one specializes this pseudocharacter at any irreducible representation of $\text{GL}_n(E)$ that lies in \mathcal{B} , one obtains the pseudocharacter of the semisimplification of the corresponding representation of W_E . From our perspective, this result allows us to deduce that the supercuspidal support of $\pi(\bar{\rho})$ is the reduction modulo ϖ of the supercuspidal support of $\pi(\rho)$, but it does not contain any information about the monodromy operator.

In cases where the embedding arising in the previous proposition is an isomorphism, we say that ρ is a *minimal lift* of $\bar{\rho}$. (Such lifts are lifts of $\bar{\rho}$ in which the ramification arising from the monodromy operator is as small as possible.) We will need this language in a broader context than that of representations over discrete valuation rings:

4.5.9. Definition. Let A be a reduced complete Noetherian local ring with finite residue field k of characteristic p , that is flat over $W(k)$, and let ρ be a continuous representation of G_E into $\text{GL}_n(A)$. Let (ρ', N) be the associated Weil–Deligne representation over $\text{GL}_n(A[\frac{1}{p}])$. If \mathfrak{p} is a characteristic zero prime of A , and \mathfrak{a} is a

prime of A whose closure contains \mathfrak{p} , we say $\rho_{\mathfrak{a}}$ is a minimal lift of $\rho_{\mathfrak{p}}$ if, for all i , the rank of $(N \otimes_A \kappa(\mathfrak{a}))^i$ is equal to the rank of $(N \otimes_A \kappa(\mathfrak{p}))^i$.

Note that, for any given \mathfrak{a} , the locus of \mathfrak{p} such that $\rho_{\mathfrak{a}}$ is a minimal lift of $\rho_{\mathfrak{p}}$ is Zariski open in the closure of \mathfrak{a} in $\text{Spec } A[\frac{1}{p}]$.

5. THE LOCAL LANGLANDS CORRESPONDENCE IN CHARACTERISTIC p

5.1. Definition and basic properties. We now construct an analogue of the Breuil-Schneider local Langlands correspondence for representations of G_E over finite fields of characteristic p . Such a correspondence should satisfy an analog of Theorem 4.5.7 for representations over discrete valuation rings of characteristic zero and residue characteristic p . Throughout this section we fix a finite field k of characteristic p , and let \mathcal{O} denote a complete discrete valuation ring of characteristic zero with field of fractions \mathcal{K} and finite residue field k' containing k .

Our starting point is the semisimple mod p local Langlands correspondence of Vigneras [20]. This is a map $\bar{\rho} \mapsto \bar{\pi}_{\text{ss}}(\bar{\rho})$ that associates to each n -dimensional irreducible representation $\bar{\rho} : W_E \rightarrow \text{GL}_n(\bar{\mathbb{F}}_p)$ an irreducible supercuspidal representation $\bar{\pi}_{\text{ss}}(\bar{\rho})$ over $\bar{\mathbb{F}}_p$. If q denotes the order of the residue field of E , and if k' is a finite field of characteristic p containing a square root of q , then this correspondence is defined over k' ; that is, if $\bar{\rho}$ is defined over k' , then $\bar{\pi}(\bar{\rho})$ descends uniquely to a representation over k' . Moreover, the correspondence is compatible with “reduction mod p ” in the following sense:

5.1.1. Theorem ([20, Thm. 1.6]). *Suppose that k' contains a square root of q . Let (ρ, N) be an n -dimensional Frobenius-semisimple Weil–Deligne representation of W_E over \mathcal{O} , and let π be the irreducible representation of $\text{GL}_n(E)$ over \mathcal{K} attached to $(\rho, N) \otimes_{\mathcal{O}} \mathcal{K}$ by the unitary local Langlands correspondence. Let $\bar{\rho} = \rho \otimes_{\mathcal{O}} \bar{\mathbb{F}}_p$, and let*

$$\bar{\rho}^{\text{ss}} = \bar{\rho}_1 \oplus \cdots \oplus \bar{\rho}_r$$

be a decomposition of $\bar{\rho}^{\text{ss}}$ into irreducible representations of W_E over $\bar{\mathbb{F}}_p$. Then π is \mathcal{O} -integral, and for any $\text{GL}_n(E)$ -stable \mathcal{O} -lattice L in π , and any Jordan–Hölder constituent $\bar{\pi}$ of $L \otimes_{\mathcal{O}} \bar{\mathbb{F}}_p$, one has:

$$\text{scs}(\bar{\pi}) = \{\bar{\pi}_{\text{ss}}(\bar{\rho}_1) \dots \bar{\pi}_{\text{ss}}(\bar{\rho}_r)\}.$$

5.1.2. Corollary. *Suppose that k' contains a square root of q . Let $\bar{\rho} : G_E \rightarrow \text{GL}_n(k)$ be a Galois representation, and let $\rho : G_E \rightarrow \text{GL}_n(\mathcal{O})$ be a lift of $\bar{\rho} \otimes_k k'$. Then $\pi(\rho \otimes_{\mathcal{O}} \mathcal{K})$ is \mathcal{O} -integral, and for any \mathcal{O} -lattice L in $\pi(\rho \otimes_{\mathcal{O}} \mathcal{K})$, the supercuspidal support of any Jordan–Hölder constituent $\bar{\pi}$ of $L \otimes_{\mathcal{O}} \bar{\mathbb{F}}_p$ depends only on $\bar{\rho}$.*

Proof. Let (ρ', N) be the Frobenius-semisimple Weil–Deligne representation over \mathcal{K} attached to ρ . Then ρ' is \mathcal{O} -integral and the semisimplification of its reduction mod p depends only on $\bar{\rho}$. By the definition of the Breuil-Schneider local Langlands correspondence, $\pi(\rho \otimes_{\mathcal{O}} \bar{\mathcal{K}})$ is (up to a twist by an integral character) a parabolic induction of representations that correspond (via unitary local Langlands) to irreducible summands of $\rho' \otimes_{\mathcal{O}} \bar{\mathcal{K}}$. These summands are integral, so $\pi(\rho \otimes_{\mathcal{O}} \bar{\mathcal{K}})$ is as well, and so is $\pi(\rho \otimes_{\mathcal{O}} \mathcal{K})$. Moreover, (up to a twist by $(|\cdot| \circ \det)^{\frac{n-1}{2}}$), every Jordan–Hölder constituent of $\pi(\rho \otimes_{\mathcal{O}} \bar{\mathcal{K}})$ corresponds via unitary local Langlands to a Weil–Deligne representation of the form $(\rho' \otimes_{\mathcal{O}} \bar{\mathcal{K}}, N')$ for some choice of monodromy operator N' .

Now if L is a lattice in $\pi(\rho \otimes_{\mathcal{O}} \mathcal{K})$, and $\bar{\pi}$ is a Jordan–Hölder constituent of $L \otimes_{\mathcal{O}} \overline{\mathbb{F}}_p$, then there exists a Jordan–Hölder constituent of $\pi(\rho \otimes_{\mathcal{O}} \overline{\mathcal{K}})$, and a lattice L' in this constituent, such that $\bar{\pi}$ is a Jordan–Hölder constituent of the mod p reduction of L' . The result thus follows from Theorem 5.1.1. \square

Let L be a lattice in $\pi(\rho \otimes_{\mathcal{O}} \mathcal{K})$, where $\rho : G_E \rightarrow \mathrm{GL}_n(\mathcal{O})$ is a lift of $\bar{\rho} \otimes_k k'$ for some $\bar{\rho} : G_E \rightarrow \mathrm{GL}_n(k)$. As $L \otimes_{\mathcal{O}} \overline{\mathbb{F}}_p$ has a unique generic Jordan–Hölder constituent, and up to isomorphism there is only one irreducible generic representation of G with given supercuspidal support, the generic Jordan–Hölder constituent of $L \otimes_{\mathcal{O}} \overline{\mathbb{F}}_p$ likewise depends only on $\bar{\rho}$.

We will also need to control the length of $L/\varpi L$, for lattices L of the sort appearing in the Corollary above. We first show:

5.1.3. Proposition. *Let $P = MU$ be a parabolic subgroup of $\mathrm{GL}_n(E)$, and let $\pi = \pi_1 \otimes \cdots \otimes \pi_r$ be an irreducible representation of M . There exists an integer c , depending only on n , such that the length of $\mathrm{Ind}_P^{\mathrm{GL}_n(E)} \pi_1 \otimes \cdots \otimes \pi_r$ is bounded above by c .*

Proof. This follows easily from [17], Proposition III.1.12. \square

5.1.4. Proposition. *Let $\bar{\rho} : G_E \rightarrow \mathrm{GL}_n(k)$ be a Galois representation, let $\rho : G_E \rightarrow \mathrm{GL}_n(\mathcal{O})$ be a lift of $\bar{\rho} \otimes_k k'$, and let L be a $\mathrm{GL}_n(E)$ -stable lattice in $\pi(\rho \otimes_{\mathcal{O}} \mathcal{K})$. There exists an integer c , depending only on n , such that the length of $L/\varpi L$ is bounded above by c .*

Proof. The length of $L/\varpi L$ is independent of L . As $\pi(\rho \otimes_{\mathcal{O}} \mathcal{K})$ is a parabolic induction of a tensor product of integral Steinberg representations, we can write

$$\pi(\rho \otimes_{\mathcal{O}} \mathcal{K}) = \mathrm{Ind}_P^{\mathrm{GL}_n(E)} \mathrm{St}_{\pi_1, n_1} \otimes \cdots \otimes \mathrm{St}_{\pi_i, n_i},$$

where the π_i are integral cuspidal representations of GL_n . For each i , St_{π_i, n_i} arises as the normalized parabolic induction of a tensor product of the form:

$$(| \circ \det |^{-\frac{n_i-1}{2}} \pi_i \otimes \cdots \otimes (| \circ \det |^{-\frac{n_i-1}{2}} \pi_i).$$

Thus there is a parabolic induction of a tensor product of irreducible, integral, cuspidal representations π'_j (all of which are twists of the π_i) that maps surjectively onto $\pi(\rho \otimes_{\mathcal{O}} \mathcal{K})$; if we choose a lattice L_j inside each of the π'_j , the parabolic induction of the tensor product of the L_j maps into a lattice L in $\pi(\rho \otimes_{\mathcal{O}} \mathcal{K})$. We then have a surjection of the parabolic induction of the tensor product of $L_j/\varpi L_j$ onto $L/\varpi L$. As each π'_j is cuspidal, so is $L_j/\varpi L_j$; as $(L_j/\varpi L_j)^{(n)}$ is one-dimensional we must have $L_j/\varpi L_j$ irreducible for all j . Thus the length of $L/\varpi L$ is bounded above by the maximum length of a parabolic induction of an irreducible representation of a Levi subgroup of $\mathrm{GL}_n(E)$, and the desired result follows by Proposition 5.1.3. \square

We can now prove the main result of this subsection.

5.1.5. Theorem. *There is a map $\bar{\rho} \mapsto \bar{\pi}(\bar{\rho})$ from the set of isomorphism classes of continuous representations $G_E \rightarrow \mathrm{GL}_n(k)$ to the set of isomorphism classes of finite length admissible smooth $\mathrm{GL}_n(E)$ -representations on k -vector spaces, uniquely determined by the following three conditions:*

- (1) *For any $\bar{\rho}$, the associated $\mathrm{GL}_n(E)$ -representation $\bar{\pi}(\bar{\rho})$ is essentially AIG.*

- (2) If \mathcal{K} is a finite extension of \mathbb{Q}_p , with ring of integers \mathcal{O} , uniformizer ϖ , and residue field k' containing k , $\rho : G_E \rightarrow \mathrm{GL}_n(\mathcal{O})$ is a continuous representation lifting $\bar{\rho} \otimes_k k'$, and L is a $\mathrm{GL}_n(E)$ -invariant \mathcal{O} -lattice in $\pi(\rho)$ such that $L/\varpi L$ is essentially AIG, then there is a $\mathrm{GL}_n(E)$ -equivariant embedding $L/\varpi L \hookrightarrow \bar{\pi}(\bar{\rho}) \otimes_k k'$. (Note that Theorem 3.3.2 shows that such an L always exists, and that it is unique up to homothety.)
- (3) The representation $\bar{\pi}(\bar{\rho})$ is minimal with respect to satisfying conditions (1) and (2), i.e. given any continuous representation $\bar{\rho} : G_E \rightarrow \mathrm{GL}_n(k)$, and any representation $\bar{\pi}$ of $\mathrm{GL}_n(E)$ satisfying these two conditions with respect to $\bar{\rho}$, there is a $\mathrm{GL}_n(E)$ -equivariant embedding $\bar{\pi}(\bar{\rho}) \hookrightarrow \bar{\pi}$.

Furthermore:

- (4) The formation of $\bar{\pi}(\bar{\rho})$ is compatible with extending scalars, i.e. given $\bar{\rho} : G_E \rightarrow \mathrm{GL}_n(k)$, and a finite extension k' of k , one has

$$\bar{\pi}(\bar{\rho} \otimes_k k') \cong \bar{\pi}(\bar{\rho}) \otimes_k k'.$$

- (5) The formation of $\bar{\pi}(\bar{\rho})$ is compatible with twists, i.e. given $\bar{\rho} : G_E \rightarrow \mathrm{GL}_n(k)$, and a continuous character $\bar{\chi} : G_E \rightarrow k^\times$, one has

$$\bar{\pi}(\bar{\rho} \otimes \bar{\chi}) = \bar{\pi}(\bar{\rho}) \otimes (\bar{\chi} \circ \det).$$

- (6) $\mathrm{End}_{\mathrm{GL}_n(E)}(\bar{\pi}(\bar{\rho})) = k$.

- (7) The representation $\bar{\pi}(\bar{\rho})$ has central character equal to $|\cdot|^{-\frac{n(n-1)}{2}}(\det \bar{\rho})$.

- (8) Suppose $(\bar{\rho} \otimes_k \bar{k})^{\mathrm{ss}}$ is the direct sum of irreducible representations $\bar{\rho}_1, \dots, \bar{\rho}_r$. Then every Jordan–Hölder constituent of $\bar{\pi}(\bar{\rho})$ has supercuspidal support equal to $|\cdot|^{-\frac{n-1}{2}} \{\bar{\pi}_{\mathrm{ss}}(\bar{\rho}_1), \dots, \bar{\pi}_{\mathrm{ss}}(\bar{\rho}_r)\}$

Proof. We first establish uniqueness: If $\bar{\pi}$ and $\bar{\pi}'$ are two finite length admissible smooth representations of $\mathrm{GL}_n(E)$ that satisfy properties (1), (2), and (3) with respect to $\bar{\rho}$, then by property (3) we have embeddings of $\bar{\pi}$ in $\bar{\pi}'$ and vice versa. As both $\bar{\pi}$ and $\bar{\pi}'$ have finite length these embeddings are isomorphisms.

We now turn to the construction of $\bar{\pi}(\bar{\rho})$. Let $\rho : G_E \rightarrow \mathrm{GL}_n(\mathcal{O})$ be a lift of $\bar{\rho} \otimes_k k'$, for some \mathcal{O}, k' as in property 2. (Choi ([4], Theorem 3.0.13) has shown that the characteristic zero fiber of the universal framed deformation ring of $\bar{\rho}$ is generically smooth of dimension n^2 , so such a lift always exists.) Suppose L is an \mathcal{O} -lattice in $\pi(\rho)$ such that $L/\varpi L$ is essentially AIG. The socle V of $L/\varpi L$ is absolutely irreducible and generic, and its supercuspidal support depends only on $\bar{\rho}$ and not the specific lift ρ chosen. As there is a unique generic representation with given supercuspidal support, V depends only on $\bar{\rho}$ up to isomorphism. In particular V is defined over k , as we can take \mathcal{O} to have residue field k .

Let $\mathrm{env}(V_{\bar{k}})$ be the essentially AIG envelope of $V \otimes_k \bar{k}$. For each lift ρ of $\bar{\rho}$, and each lattice L in $\pi(\bar{\rho})$ such that $L/\varpi L$ is essentially AIG, the socle of $(L/\varpi L) \otimes_k \bar{k}$ is isomorphic to $V \otimes_k \bar{k}$. Hence $(L/\varpi L) \otimes_k \bar{k}$ embeds uniquely (up to the action of \bar{k}^\times) in $\mathrm{env}(V_{\bar{k}})$. Let $\bar{\pi}(\bar{\rho})_{\bar{k}}$ be the sum, in $\mathrm{env}(V_{\bar{k}})$, of the images of $(L/\varpi L) \otimes_k \bar{k}$ in $\mathrm{env}(V_{\bar{k}})$ as ρ ranges over all lifts of $\bar{\rho}$.

By construction, $\mathrm{Gal}(\bar{k}/k)$ acts on $\mathrm{env}(V_{\bar{k}})$. This action preserves $\bar{\pi}(\bar{\rho})_{\bar{k}}$, as it permutes the images of $L/\varpi L$ for various \mathcal{O} and ρ . Thus $\bar{\pi}(\bar{\rho})_{\bar{k}}$ descends uniquely to a submodule $\bar{\pi}(\bar{\rho})$ of $\mathrm{env}(V)$. Clearly, $\bar{\pi}(\bar{\rho})$ satisfies properties (1) and (2). On the other hand, if $\bar{\pi}$ is any other representation satisfying properties (1) and (2), then the socle of $\bar{\pi}$ is isomorphic to V and hence $\mathrm{env}(V)$ contains a unique submodule

isomorphic to $\bar{\pi}$. As $\bar{\pi}$ satisfies property (2), $\bar{\pi} \otimes_k \bar{k}$ contains the images of $L/\varpi L$ in $\text{env}(V_{\bar{k}})$ for all lifts ρ of $\bar{\rho}$, and thus contains $\bar{\pi}(\bar{\rho}) \otimes_k \bar{k}$. It follows that $\bar{\pi}$ contains $\bar{\pi}(\bar{\rho})$, so $\bar{\pi}(\bar{\rho})$ satisfies property (3). Finally, $\bar{\pi}(\bar{\rho})$ is finite length by Proposition 5.1.4 and Corollary 3.2.17.

Now let k' be a finite extension of k . Then $\bar{\pi}(\bar{\rho}) \otimes_k k'$ clearly satisfies properties (1) and (2) with respect to $\bar{\rho} \otimes_k k'$, and thus admits an embedding of $\bar{\pi}(\bar{\rho} \otimes_k k')$ that is unique up to rescaling. The above construction shows that $\bar{\pi}(\bar{\rho}) \otimes_k \bar{k}$ and $\bar{\pi}(\bar{\rho} \otimes_k k') \otimes_k \bar{k}$ coincide as submodules of $\text{env}(V_{\bar{k}})$, so this embedding is an isomorphism.

Similarly, if $\bar{\chi}$ is a character of E^\times with values in k^\times , we can choose a lift χ of $\bar{\chi}$ to a character with values in $W(k)^\times$. Then if $\rho \otimes \chi$ is a lift of $\bar{\rho} \otimes (\bar{\chi} \circ \det)$ to a representation over \mathcal{O} , and $L \otimes \chi$ is a lattice in $\pi(\rho \otimes \chi)$ with $(L \otimes (\chi \circ \det))/\varpi(L \otimes (\chi \circ \det))$ essentially AIG, then L is a lattice in $\pi(\rho)$ with $L/\varpi L$ essentially AIG. Thus $L/\varpi L$ embeds in $\bar{\pi}(\bar{\rho}) \otimes_k k'$, so $(L/\varpi L) \otimes (\bar{\chi} \circ \det)$ embeds in $\bar{\pi}(\bar{\rho}) \otimes (\bar{\chi} \circ \det)$. Thus $\bar{\pi}(\bar{\rho}) \otimes (\bar{\chi} \circ \det)$ has property (2) and hence contains $\bar{\pi}(\bar{\rho} \otimes \bar{\chi})$. Conversely, replacing $\bar{\rho}$ with $\bar{\rho} \otimes \bar{\chi}$, we find that $\bar{\pi}(\bar{\rho} \otimes \bar{\chi}) \otimes (\bar{\chi}^{-1} \circ \det)$ contains $\bar{\pi}(\bar{\rho})$. Thus $\bar{\pi}(\bar{\rho})$ and $\bar{\pi}(\bar{\rho} \otimes \bar{\chi})$ have the same length, and so $\bar{\pi}(\bar{\rho} \otimes \bar{\chi})$ and $\bar{\pi}(\bar{\rho}) \otimes (\bar{\chi} \circ \det)$ are isomorphic.

The endomorphisms of $\bar{\pi}(\bar{\rho})$ are all scalar because $\bar{\pi}(\bar{\rho})$ is essentially AIG. In particular the center of $\text{GL}_n(E)$ acts on $\bar{\pi}(\bar{\rho})$ (and hence on all of its submodules) via a character. To compute this character, let ρ be any lift of $\bar{\rho}$, and let L be a lattice in $\pi(\rho)$ such that $L/\varpi L$ is essentially AIG. The center of $\text{GL}_n(E)$ acts on $\pi(\rho)$ via the character $|\cdot|^{\frac{n(n-1)}{2}} \det \rho$, and hence on $L/\varpi L$ via the character $|\cdot|^{\frac{n(n-1)}{2}} \det \bar{\rho}$. As $L/\varpi L$ embeds in $\bar{\pi}(\bar{\rho})$, this character is also the central character of $\bar{\pi}(\bar{\rho})$.

As $\bar{\pi}(\bar{\rho})$ is essentially AIG, every Jordan–Hölder constituent of $\bar{\pi}(\bar{\rho})$ has the same supercuspidal support. To determine this supercuspidal support, let ρ be any lift of $\bar{\rho}$, and let L be a lattice in $\pi(\rho)$ such that $L/\varpi L$ is essentially AIG. The representations $|\cdot|^{-\frac{n-1}{2}} \pi(\rho)$ and ρ then correspond under unitary local Langlands, and so, by Theorem 5.1.1, the supercuspidal support of any Jordan–Hölder constituent of $|\cdot|^{-\frac{n-1}{2}} L/\varpi L$ is equal to $\{\bar{\pi}_{\text{ss}}(\bar{\rho}_1), \dots, \bar{\pi}_{\text{ss}}(\bar{\rho}_r)\}$. \square

5.2. The local Langlands correspondence for GL_2 in characteristic p . For $\text{GL}_2(E)$, at least in odd characteristic, the correspondence $\bar{\rho} \mapsto \bar{\pi}(\bar{\rho})$ can be made fairly concrete. We give a complete picture (for p odd) in [12]; the approach described there runs into difficulties when $p = 2$ because the essentially AIG envelope of a cuspidal (but not supercuspidal) representation is rather complicated in this case. The first thing to observe is:

5.2.1. Proposition. *Let $\bar{\rho} : G_E \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ be a representation, and suppose that $\bar{\rho}^{\text{ss}}$ is not a twist of $1 \oplus |\cdot|$. Then $\bar{\pi}(\bar{\rho})$ is the unique representation of $\text{GL}_2(E)$ whose supercuspidal support is given by part (8) of Theorem 5.1.5.*

Proof. Part (8) of Theorem 5.1.5, together with our hypothesis on $\bar{\rho}$ implies that the supercuspidal support of any Jordan–Hölder constituent of $\bar{\pi}(\bar{\rho})$ is either a single supercuspidal representation of $\text{GL}_2(E)$, or a pair of characters of $\text{GL}_1(E)$ that do not differ by a factor of $|\cdot|$. In either case, there is, up to isomorphism, a *unique* irreducible representation $\bar{\pi}$ of GL_2 that has that particular supercuspidal support; in particular $\bar{\pi}$ is generic. Thus every Jordan–Hölder constituent of $\text{env}(\bar{\pi})$ is isomorphic to $\bar{\pi}$; as $\bar{\pi}$ is generic and $\text{env}(\bar{\pi})$ is essentially AIG there is only one

such Jordan–Hölder constituent. In particular $\bar{\pi} = \text{env}(\bar{\pi})$. and so As $\pi(\bar{\rho})$ is contained in $\text{env}(\bar{\pi})$, we must have $\pi(\bar{\rho}) = \bar{\pi}$. \square

When $\bar{\rho}^{\text{ss}}$ is a twist of $1 \oplus \begin{smallmatrix} | \\ | \end{smallmatrix}$, the situation is more complicated, as $\pi(\bar{\rho})$ will typically not be irreducible. As the correspondence $\bar{\rho} \mapsto \pi(\bar{\rho})$ is compatible with twists, it suffices to describe $\pi(\bar{\rho})$ when $\bar{\rho}^{\text{ss}} = 1 \oplus \begin{smallmatrix} | \\ | \end{smallmatrix}$. In this case $\pi(\bar{\rho})$ has supercuspidal support $\{1, \overline{\begin{smallmatrix} | \\ | \end{smallmatrix}}\}$. The details of this will be carried out in [12]; here we content ourselves with summarizing the results.

First, assume that the order q of the residue field of E is not congruent to ± 1 modulo p . (This is the so-called *banal* situation.) Here there are two irreducible representations of G with supercuspidal support $\{1, \overline{\begin{smallmatrix} | \\ | \end{smallmatrix}}\}$: the character $\overline{\begin{smallmatrix} | \\ | \end{smallmatrix}} \circ \det$ and the twisted Steinberg representation $\text{St} \otimes (\overline{\begin{smallmatrix} | \\ | \end{smallmatrix}} \circ \det)$. The latter representation is generic, and its envelope is the unique nonsplit extension of $\overline{\begin{smallmatrix} | \\ | \end{smallmatrix}} \circ \det$ by $\text{St} \otimes (\overline{\begin{smallmatrix} | \\ | \end{smallmatrix}} \circ \det)$.

On the Galois side there is, up to isomorphism, a unique nonsplit $\bar{\rho}$ whose semisimplification is $1 \oplus \begin{smallmatrix} | \\ | \end{smallmatrix}$. Then $\pi(\bar{\rho})$ is equal to $\text{St} \otimes (\overline{\begin{smallmatrix} | \\ | \end{smallmatrix}} \circ \det)$ if $\bar{\rho}$ is nonsplit, and to the unique nonsplit extension of $\overline{\begin{smallmatrix} | \\ | \end{smallmatrix}} \circ \det$ by $\text{St} \otimes (\overline{\begin{smallmatrix} | \\ | \end{smallmatrix}} \circ \det)$ if $\bar{\rho}$ is split.

Next, assume that p is odd and q is congruent to -1 modulo p . In this case there are three irreducible representations of G with supercuspidal support $\{1, \overline{\begin{smallmatrix} | \\ | \end{smallmatrix}}\}$: the trivial character, the character $\overline{\begin{smallmatrix} | \\ | \end{smallmatrix}} \circ \det$, and a cuspidal generic representation that Vigneras denotes by $\pi(1)$ (see [17, II.2.5] for a discussion of this). Up to isomorphism, there is a unique nonsplit extension of the trivial character by $\pi(1)$ and similarly a unique nonsplit extension of $\overline{\begin{smallmatrix} | \\ | \end{smallmatrix}} \circ \det$ by $\pi(1)$. The envelope $\text{env}(\pi(1))$ is the unique extension of $1 \oplus (\overline{\begin{smallmatrix} | \\ | \end{smallmatrix}} \circ \det)$ by $\pi(1)$ that contains both of these nonsplit extensions as submodules.

In this case $\pi(\bar{\rho}) = \text{env}(\pi(1))$ if $\bar{\rho}$ is split. If $\bar{\rho}$ is not split, it is either an extension of $\overline{\begin{smallmatrix} | \\ | \end{smallmatrix}}$ by 1 or an extension of 1 by $\overline{\begin{smallmatrix} | \\ | \end{smallmatrix}}$. In the first case, $\pi(\bar{\rho})$ is the nonsplit extension of $(\overline{\begin{smallmatrix} | \\ | \end{smallmatrix}} \circ \det)$ by $\pi(1)$; in the second case $\pi(\bar{\rho})$ is the nonsplit extension of the trivial character by $\pi(1)$.

Finally, assume that p is odd and q is congruent to 1 modulo p . In this case $\overline{\begin{smallmatrix} | \\ | \end{smallmatrix}}$ is the trivial character. The only irreducible representations of G with supercuspidal support $\{1, 1\}$ in this case are the Steinberg representation St and the trivial representation. Moreover, $\text{Ext}^1(1, \text{St})$ is two-dimensional, and naturally isomorphic to $H^1(G_E, 1)$. The envelope $\text{env}(\text{St})$ is isomorphic to the universal extension of 1 by St , and thus has length three.

In this case $\pi(\bar{\rho})$ is equal to $\text{env}(\text{St})$ if $\bar{\rho}$ is split. On the other hand, the nonsplit $\bar{\rho}$ with trivial semisimplification are in bijection with the one-dimensional subspaces of $H^1(G_E, 1)$, and hence in natural bijection with the one-dimensional subspaces of $\text{Ext}^1(1, \text{St})$. These in turn correspond to the nonsplit extensions of 1 by St . For any nonsplit $\bar{\rho}$, $\pi(\bar{\rho})$ is the corresponding extension of 1 by St .

6. THE LOCAL LANGLANDS CORRESPONDENCE IN FAMILIES

6.1. The set-up. Throughout this section we will be considering representations over rings A satisfying the following condition:

6.1.1. Condition. A is a complete reduced Noetherian local ring, with finite residue field k of characteristic p , which is flat over the ring of Witt vectors $W(k)$.

We will typically write \mathfrak{m} for the maximal ideal of A . Note that the condition of being flat over $W(k)$ is equivalent to A being p -torsion free, or again (since A is reduced), to each minimal prime of A being of residue characteristic 0. We will write $\kappa(\mathfrak{p})$ to denote the residue field of a prime ideal \mathfrak{p} of A ; thus $\kappa(\mathfrak{p})$ is the fraction field of the complete local domain A/\mathfrak{p} . We write $\mathcal{K}(A) := \prod_{\mathfrak{a} \text{ minimal}} \kappa(\mathfrak{a})$ (where, as indicated, the product is taken over the finitely many minimal primes of A) for the total quotient ring of A . Since A is reduced, the natural map $A \rightarrow \mathcal{K}(A)$ is an embedding.

6.2. Statement of the correspondence and related results. Now let E be a number field. Let v be a non-archimedean place of E , and let $\rho : G_{E_v} \rightarrow \mathrm{GL}_n(A)$ be a continuous representation (when the target is equipped with its \mathfrak{m} -adic topology). For each prime ideal \mathfrak{p} of A , let $\rho_{\mathfrak{p}} : G_{E_v} \rightarrow \mathrm{GL}_n(\kappa(\mathfrak{p}))$ denote the representation obtained from ρ by extending scalars from A to $\kappa(\mathfrak{p})$. In the particular case of the maximal ideal, we also write $\bar{\rho} := \rho_{\mathfrak{m}}$. If \mathfrak{p} is a prime of A with residue characteristic zero, we write $\tilde{\pi}(\rho_{\mathfrak{p}})$ for the smooth $\kappa(\mathfrak{p})$ -dual of the representation $\pi(\rho_{\mathfrak{p}})$ defined in Definition 4.2.1.

In the situations that we will consider below, we will have a finite S of non-archimedean places of E , all prime to p , and for each $v \in S$ we will have a continuous representation $\rho_v : G_{E_v} \rightarrow \mathrm{GL}_n(A)$.

We are now ready to describe the local Langlands correspondence for local Galois representations over A .

6.2.1. Theorem. *Let S denote a finite set of non-archimedean places of E , none of which lie over p , and suppose for each $v \in S$ that we are given a representation $\rho_v : G_{E_v} \rightarrow \mathrm{GL}_n(A)$. If we write $G := \prod_{v \in S} \mathrm{GL}_n(E_v)$, then there is (up to isomorphism) at most one admissible smooth representation V of G over A satisfying the following conditions:*

- (1) V is A -torsion free (i.e. all associated primes of V are minimal primes of A , or equivalently, the natural map $V \rightarrow \mathcal{K}(A) \otimes_A V$ is an embedding).
- (2) For each minimal prime \mathfrak{a} of A , there is a G -equivariant isomorphism

$$\bigotimes_{v \in S} \tilde{\pi}(\rho_{v, \mathfrak{a}}) \xrightarrow{\sim} \kappa(\mathfrak{a}) \otimes_A V.$$

- (3) The G -cosocle $\mathrm{cosoc}(V/\mathfrak{m}V)$ of $V/\mathfrak{m}V$ is absolutely irreducible and generic, while the kernel of the natural surjection $V/\mathfrak{m}V \rightarrow \mathrm{cosoc}(V/\mathfrak{m}V)$ contains no generic subrepresentations. (In other words, the smooth dual of $V/\mathfrak{m}V$ is essentially AIG.)

Any such V satisfies the following additional conditions:

- (4) V is cyclic as an $A[G]$ -module.
- (5) $\mathrm{End}_{A[G]}(V) = A$.

We postpone the proof of the theorem to the following subsection.

6.2.2. Definition. If in the context of the preceding theorem an $A[G]$ -module V satisfying conditions (1), (2), and (3) exists, then we write $\tilde{\pi}(\{\rho_v\}_{v \in S}) := V$. (This is justified by the uniqueness statement of the theorem.) If S consists of a single place v then we write $\tilde{\pi}(\rho_v)$ rather than $\tilde{\pi}(\{\rho_v\}_{v \in S})$.

6.2.3. Remark. We don't consider here the problem of proving in general that a representation V satisfying conditions (1), (2) and (3) of Theorem 6.2.1 exists, although we conjecture that it does. (This is Conjecture 1.4.1 of the introduction.) When $n = 2$ and p is odd, or when p is a banal prime, this conjecture is a result of the second author [11], [10].

In the global applications considered in the work of the first author [6, 7], and in subsequent applications, the problem that we will confront will rather be that of having a smooth G -representation at hand (for a certain ring A), which we wish to show satisfies the conditions to be $\tilde{\pi}(\{\rho_v\}_{v \in S})$ for an appropriate ρ . Thus one of our goals in the following subsection is to establish a workable criterion for recognizing $\tilde{\pi}(\{\rho_v\}_{v \in S})$ (namely Theorem 6.2.15 below).

The following result shows that the existence of $\tilde{\pi}(\{\rho_v\}_{v \in S})$ is equivalent to the existence of the collection of representations $\tilde{\pi}(\rho_v)$, and explains the relation between them. We postpone its proof to the following subsection.

6.2.4. Proposition. *In the context of Theorem 6.2.1, the $A[G]$ -module $\tilde{\pi}(\{\rho_v\}_{v \in S})$ exists if and only if each of the individual $A[\mathrm{GL}_n(E_v)]$ -modules $\tilde{\pi}(\rho_v)$ exist. Furthermore, $\tilde{\pi}(\{\rho_v\}_{v \in S})$ is isomorphic to the maximal torsion free quotient of the tensor product (taken over A) $\bigotimes_{v \in S} \tilde{\pi}(\rho_v)$.*

The following two theorems, whose proofs we again postpone, describe the sense in which the representation $\tilde{\pi}(\{\rho_v\})$ interpolates the Breuil-Schneider modified local Langlands correspondence over $\mathrm{Spec} A[\frac{1}{p}]$.

6.2.5. Theorem. *Let \mathfrak{p} be a prime of $A[\frac{1}{p}]$, and suppose that \mathfrak{p} lies on exactly one irreducible component of $\mathrm{Spec} A[\frac{1}{p}]$. Then, assuming that $\tilde{\pi}(\{\rho_v\}_{v \in S})$ exists, there is a $\kappa(\mathfrak{p})$ -linear G -equivariant surjection*

$$\bigotimes_{v \in S} \tilde{\pi}(\rho_{v, \mathfrak{p}}) \rightarrow \kappa(\mathfrak{p}) \otimes_A \tilde{\pi}(\{\rho_v\}_{v \in S}).$$

Moreover, if there exists a minimal prime \mathfrak{a} of A such that $\rho_{\mathfrak{a}}$ is a minimal lift of $\rho_{\mathfrak{p}}$, then this surjection is an isomorphism.

It seems likely that the above result holds even when \mathfrak{p} is contained in multiple irreducible components of $\mathrm{Spec} A[\frac{1}{p}]$. Nonetheless we are at present only able to prove a somewhat weaker statement:

6.2.6. Theorem. *Assume that $\tilde{\pi}(\{\rho_v\}_{v \in S})$ exists, let \mathfrak{p} be a prime of $\mathrm{Spec} A[\frac{1}{p}]$, let $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ be the minimal primes of A contained in \mathfrak{p} , for each $i = 1, \dots, r$ let V_i be the maximal A -torsion free quotient of $\tilde{\pi}(\{\rho_v\}_{v \in S}) \otimes_A A/\mathfrak{a}_i$, and denote by W the image of the diagonal map*

$$\kappa(\mathfrak{p}) \otimes_A \tilde{\pi}(\{\rho_v\}_{v \in S}) \rightarrow \prod_i \kappa(\mathfrak{p}) \otimes_{A/\mathfrak{a}_i} V_i.$$

Then there is a $\kappa(\mathfrak{p})$ -linear G -equivariant surjection

$$\bigotimes_{v \in S} \tilde{\pi}(\rho_{v, \mathfrak{p}}) \rightarrow W.$$

Moreover, if there exists a minimal prime \mathfrak{a} of A such that $\rho_{\mathfrak{a}}$ is a minimal lift of $\rho_{\mathfrak{p}}$, then this surjection is an isomorphism.

6.2.7. Conjecture. Under the hypotheses of Theorem 6.2.6, the map

$$\kappa(\mathfrak{p}) \otimes_A \tilde{\pi}(\{\rho_v\}_{v \in S}) \rightarrow W$$

is an isomorphism. In particular the conclusion of Theorem 6.2.5 holds for all \mathfrak{p} .

Although this conjecture seems difficult to establish in general, we have the following result for small n , which we prove in the next section.

6.2.8. Proposition. Conjecture 6.2.7 holds when $n = 2$ or $n = 3$.

In a similar vein, we conjecture:

6.2.9. Conjecture. Assuming that $\tilde{\pi}(\{\rho_v\}_{v \in S})$ exists, there is a G -equivariant k -linear surjection:

$$\bigotimes_{v \in S} \tilde{\pi}(\bar{\rho}) \rightarrow k \otimes_A \tilde{\pi}(\{\rho_v\}_{v \in S}).$$

One can also describe the behavior of $\tilde{\pi}(\{\rho_v\}_{v \in S})$ under base change. Suppose that B is another ring satisfying Condition 6.1.1, and that $f : A \rightarrow B$ is a local homomorphism. If we are given a Galois representation $\rho_v : G_{E_v} \rightarrow \mathrm{GL}_n(A)$ where v does not lie over p , then we may then apply the preceding considerations to the Galois representations $B \otimes_A \rho_v$. The following proposition relates $\tilde{\pi}(\rho_v)$ and $\tilde{\pi}(B \otimes_A \rho_v)$. (We again postpone the proof to the following subsection.)

6.2.10. Proposition. Suppose that for every minimal prime of $\mathrm{Spec} B$, its image \mathfrak{p} in $\mathrm{Spec} A$ is contained in a minimal prime \mathfrak{a} of A such that $\rho_{v, \mathfrak{a}}$ is a minimal lift of $\rho_{v, \mathfrak{p}}$. (For example, suppose that each component of $\mathrm{Spec} B$ dominates a component of $\mathrm{Spec} A$.) Then if $\tilde{\pi}(\{\rho_v\}_{v \in S})$ exists, so does $\tilde{\pi}(\{B \otimes_A \rho_v\}_{v \in S})$, and there is a natural surjection: $B \otimes_A \tilde{\pi}(\{\rho_v\}_{v \in S}) \rightarrow \tilde{\pi}(\{B \otimes_A \rho_v\}_{v \in S})$.

We now give some examples illustrating Definition 6.2.2.

6.2.11. Example. Suppose that $A = \mathcal{O}$ is the ring of integers in a finite extension \mathcal{K} of \mathbb{Q}_p . If $\rho : G_{E_v} \rightarrow \mathrm{GL}_n(\mathcal{O})$ is continuous (for some place v of E that does not lie over p), write $\rho_{\mathcal{K}} := \mathcal{K} \otimes_{\mathcal{O}} \rho$. Then $\tilde{\pi}(\rho)$ exists, and is the smooth contragredient to the lattice $\pi(\rho_{\mathcal{K}})^{\circ}$ of Theorem 3.3.2.

6.2.12. Remark. Suppose given A as in Theorem 6.2.1, and a continuous representation $\rho : G_{\mathbb{Q}_\ell} \rightarrow \mathrm{GL}_2(A)$ for some $\ell \neq p$. Consider a point $\mathfrak{p} \in \mathrm{Spec} A[\frac{1}{p}]$. If $\rho_{\mathfrak{p}}$ is not of the form $\chi \oplus | \cdot | \chi$ for some character χ of $G_{\mathbb{Q}_\ell}$, then for *any* minimal prime \mathfrak{a} of A , containing \mathfrak{p} , $\rho_{\mathfrak{a}}$ is necessarily a minimal lift of $\rho_{\mathfrak{p}}$, and so (assuming that $V := \tilde{\pi}(\rho)$ exists), the surjection of Theorem 6.2.5 is an isomorphism; that is, $V_{\mathfrak{p}}$ is isomorphic to $\tilde{\pi}(\rho_{\mathfrak{p}})$. On the other hand if $\rho_{\mathfrak{p}}$ does have the form $\chi \oplus | \cdot | \chi$, then there exist non-minimal lifts of $\rho_{\mathfrak{p}}$, and so $V_{\mathfrak{p}}$ need not *a priori* be isomorphic to $\tilde{\pi}(\rho_{\mathfrak{p}})$. We now give an example showing that it can indeed happen that $V_{\mathfrak{p}}$ is not isomorphic to $\tilde{\pi}(\rho_{\mathfrak{p}})$.

6.2.13. Example. Suppose that ℓ and p are distinct, and that $\ell \not\equiv 1 \pmod{p}$. If A is as in Theorem 6.2.1, then $\mathrm{Ext}_{\mathbb{Z}_p[G_{\mathbb{Q}_\ell}]}^1(| \cdot |, 1)$ is free of rank 1 over A . Let c denote a generator of this Ext^1 -module, and for any $a \in A$, let $\rho_a : G_{\mathbb{Q}_\ell} \rightarrow \mathrm{GL}_2(A)$ be the rank two representation underlying $a \cdot c$. One checks that if a is a regular

element, then $V := \tilde{\pi}(\rho_a)$ exists, and in fact is isomorphic to St_A (the Steinberg representation of $\text{GL}_2(\mathbb{Q}_\ell)$ with coefficients in A); in particular, it is independent of the regular element a . Note that if $a \in \mathfrak{p} \in \text{Spec } A[\frac{1}{p}]$ (i.e. the regular function associated to a vanishes at \mathfrak{p}), then $\rho_{a,\mathfrak{p}} := \kappa(\mathfrak{p}) \otimes \rho_a$ is split and hence unramified, and thus $\tilde{\pi}(\rho_{a,\mathfrak{p}})$ is a non-split extension of Steinberg by trivial. In particular, at such a point \mathfrak{p} , $V_{\mathfrak{p}}$ fails to be isomorphic to $\tilde{\pi}(\rho_{a,\mathfrak{p}})$.

6.2.14. Example. Suppose that ℓ and p are distinct, p is odd, and $\ell \not\equiv \pm 1 \pmod{p}$. Let A be the ring $W(k)[[a, b]]/ab$. Fix a Frobenius element Fr of $G_{\mathbb{Q}_\ell}$ and a generator σ of the p -power inertia in $G_{\mathbb{Q}_\ell}$. Let ρ be the representation of $G_{\mathbb{Q}_\ell}$ such that ρ is trivial on the prime-to- p inertia in $G_{\mathbb{Q}_\ell}$, and such that:

$$\rho(\text{Fr}) = \begin{pmatrix} q(1+a) & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho(\sigma) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

It is not hard to show in this situation that $V := \tilde{\pi}(\rho)$ exists. On the other hand, V is principal series at the minimal prime ideal $\langle b \rangle$ of A , and Steinberg along the minimal prime ideal $\langle a \rangle$ of A . In particular the Iwahori invariants of V have different ranks along these two branches, showing that V cannot be free as an A -module.

We conclude with a “recognition theorem” that is useful for verifying that a given $A[G]$ -module is isomorphic to $\tilde{\pi}(\{\rho_v\}_{v \in S})$. As with the other results of this section, we defer its proof to the next subsection.

6.2.15. Theorem. *Let V be an admissible smooth $A[G]$ -module, such that the smooth dual of $V/\mathfrak{m}V$ is essentially AIG, and suppose that there exists a Zariski dense subset Σ of $\text{Spec } A[\frac{1}{p}]$ such that:*

- (1) *For all $v \in S$, and all \mathfrak{p} in Σ , there exists a minimal prime \mathfrak{a} of A such that $\rho_{v,\mathfrak{a}}$ is a minimal lift of $\rho_{v,\mathfrak{p}}$.*
- (2) *For each point \mathfrak{p} of Σ there exists an isomorphism:*

$$\kappa(\mathfrak{p}) \otimes_A V \xrightarrow{\sim} \bigotimes_{v \in S} \tilde{\pi}(\rho_{v,\mathfrak{p}}).$$

- (3) *The diagonal map:*

$$V \rightarrow \prod_{\mathfrak{p} \in \Sigma} \bigotimes_{v \in S} \tilde{\pi}(\rho_{v,\mathfrak{p}}).$$

is an injection.

Then V satisfies conditions (1), (2), and (3) of Theorem 6.2.1; that is, $\tilde{\pi}(\{\rho_v\}_{v \in S})$ exists and is isomorphic to V .

6.3. The proofs of Theorem 6.2.1 and some related results. We develop a series of deductions involving the various conditions of Theorem 6.2.1. These will be used not only to prove Theorem 6.2.1, and the other outstanding results from the preceding subsection, but also to provide a criterion for verifying the conditions of Theorem 6.2.1, which will be useful in applications.

If A is a local ring with residue field K , and V is a representation of F over A , we let \bar{V} denote the representation $V \otimes_A K$.

6.3.1. Lemma. *Let A be a Noetherian $W(k)$ -algebra that is a local ring with residue field K . If V is an admissible smooth representation of G over A , then $\overline{V}^{(n)}$ is 1-dimensional over K if and only if $V^{(n)}$ is a cyclic A -module.*

Proof. Theorem 3.1.14 shows that if $\overline{V}^{(n)}$ is one-dimensional then $V^{(n)}$ is a finitely generated A -module, and so the lemma follows from Nakayama's lemma together with the isomorphism $V^{(n)} \otimes_A K \xrightarrow{\sim} \overline{V}^{(n)}$. \square

6.3.2. Lemma. *Let A be a Noetherian $W(k)$ -algebra that is a local ring with residue field K in which all ℓ_i are invertible. If V is an admissible smooth representation of G over A , then the following are equivalent:*

- (1) *For any non-zero quotient $K[G]$ -module \overline{W} of \overline{V} , one has $\overline{W}^{(n)} \neq 0$.*
- (2) *For any non-zero quotient $A[G]$ -module W of V , one has $W^{(n)} \neq 0$.*
- (3) *$\mathfrak{J}(\overline{V})$ generates \overline{V} over $K[G]$.*
- (4) *$\mathfrak{J}(V)$ generates V over $A[G]$.*

Proof. It is clear that (2) implies (1), as any quotient of \overline{V} is also a quotient of V . Suppose that (1) holds and that W is a quotient of V with $W^{(n)} = 0$. Then $\overline{W}^{(n)} = 0$, and so (1) implies that $\overline{W} = 0$. Then $W = 0$ by Nakayama's Lemma, so (1) implies (2).

If \overline{W} is a quotient of \overline{V} , then we have that $\overline{W}^{(n)}$ is a quotient of $\overline{V}^{(n)}$ since the derivative functor is exact. If we let \overline{U} denote the $K[G]$ -submodule of \overline{V} generated by $\mathfrak{J}(\overline{V})$, we see that $\overline{W}^{(n)}$ vanishes if and only if \overline{W} is a quotient of $\overline{V}/\overline{U}$. Thus (1) and (3) are equivalent.

Clearly (4) implies (3), since $\mathfrak{J}(\overline{V})$ is the image of $\mathfrak{J}(V)$ in \overline{V} . Conversely, suppose $\mathfrak{J}(\overline{V})$ generates \overline{V} over $K[G]$. Since $\mathfrak{J}(V)$ maps surjectively onto $\mathfrak{J}(\overline{V})$, Lemma 2.1.7 implies that $\mathfrak{J}(V)$ generates V over $A[G]$. \square

6.3.3. Lemma. *Let A be a local ring satisfying Condition 6.1.1. If V is an admissible smooth representation of G over A , and if $V_{\mathfrak{a}}^{(n)}$ is nonzero for each minimal prime \mathfrak{a} of A , then if $V^{(n)}$ is a cyclic A -module, it is in fact free of rank 1 over A .*

Proof. Since $(V^{(n)})_{\mathfrak{a}} = (V_{\mathfrak{a}})^{(n)}$, we have that $(V^{(n)})_{\mathfrak{a}}$ is nonzero for all \mathfrak{a} . Our hypotheses on A imply that A injects into the product of the fields $A_{\mathfrak{a}}$; it follows that the annihilator of $V^{(n)}$ in A is the zero ideal. The lemma follows. \square

6.3.4. Proposition. *Let A be a local ring satisfying Condition 6.1.1, let V be an admissible smooth representation of G over A , and suppose that $(V_{\mathfrak{a}})^{(n)}$ is nonzero for each minimal prime \mathfrak{a} of A , that $\overline{V}^{(n)}$ is one-dimensional, and that for any non-zero quotient $k[G]$ -module \overline{W} of \overline{V} , one has $\overline{W}^{(n)} \neq 0$. Then:*

- (1) *$V^{(n)}$ is free of rank 1 over A .*
- (2) *$\mathfrak{J}(V)$ generates V over $A[G]$.*
- (3) *$\text{End}_{A[G]}(V) = A$.*

Proof. The first claim follows immediately from Lemmas 6.3.1 and 6.3.3. The second is a consequence of Lemma 6.3.2.

By Proposition 3.1.16, the natural map $A \rightarrow \text{End}_{A[P_n]}(\mathfrak{J}(V))$ is an isomorphism. This latter map factors as the composition

$$A \rightarrow \text{End}_{A[G]}(V) \rightarrow \text{End}_{A[P_n]}(\mathfrak{J}(V)),$$

and restriction of endomorphisms from V to $\mathfrak{J}(V)$ is injective because $\mathfrak{J}(V)$ generates V . Thus $\text{End}_{A[G]}(V) = A$. \square

The following result gives some equivalent formulations of the hypotheses on \bar{V} appearing in the preceding proposition.

6.3.5. Lemma. *Let K be a field in which all ℓ_i are invertible. If \bar{V} is an admissible smooth representation of G over K , then the following are equivalent:*

- (1) $\bar{V}^{(n)}$ is one-dimensional, and for any non-zero quotient $K[G]$ -module \bar{W} of \bar{V} , one has $\bar{W}^{(n)} \neq 0$ (and hence $\bar{V}^{(n)}$ is isomorphic to $\bar{W}^{(n)}$, so that $\bar{W}^{(n)}$ is again one-dimensional).
- (2) $\bar{V}^{(n)}$ is one-dimensional, and $\mathfrak{J}(\bar{V})$ generates \bar{V} over $K[G]$.
- (3) \bar{V} is of finite length (and hence has a cosocle), $\text{cosoc}(\bar{V})$ is absolutely irreducible, and $\bar{V}^{(n)}$ is isomorphic to $(\text{cosoc}(\bar{V}))^{(n)}$, with both being non-zero.
- (4) The smooth K -dual \bar{V}^\vee of \bar{V} is essentially AIG.

Proof. The equivalence of conditions (1) and (2) follows from Lemma 6.3.2 (applied with $A = K$ and $V = \bar{V}$).

If condition (2) holds, then \bar{V} is finitely generated over $K[G]$, and hence of finite length, by [17], II.5.10. Write $\text{cosoc}(\bar{V}) = \bigoplus_j \bar{W}_j$, where each \bar{W}_j is irreducible. Condition (1) (which also holds, since it is equivalent to condition (2), as we have already observed) shows that $\bar{W}_j^{(n)}$ is one-dimensional for each j . Since the composition:

$$\bar{V}^{(n)} \rightarrow (\text{cosoc}(\bar{V}))^{(n)} \xrightarrow{\sim} \bigoplus_j \bar{W}_j^{(n)}$$

is surjective (the derivative functor is exact), we see that in fact there is only one summand, and hence that $\text{cosoc}(\bar{V})$ is irreducible. Proposition 6.3.4 (applied with $A = K$ and $V = \bar{W}$) then implies that $\text{End}_G(\text{cosoc}(\bar{V})) = K$, and hence that $\text{cosoc}(\bar{V})$ is in fact absolutely irreducible. Thus (2) implies (3).

If condition (3) holds, then by assumption $(\text{cosoc}(\bar{V}))^{(n)}$ is nonzero. It is therefore one-dimensional by Theorem 3.1.15. Thus $\bar{V}^{(n)}$ is one-dimensional, giving the first half of condition (1). The second half of condition (1) follows from the fact that $\text{cosoc}(\bar{V})$ is irreducible, and satisfies $(\text{cosoc}(\bar{V}))^{(n)} \neq 0$. Thus (3) implies (1).

Now consider the smooth dual \bar{V}^\vee . If \bar{V} is finite length, the socle of \bar{V}^\vee is the smooth dual of the cosocle of \bar{V} . In particular $\text{soc}(\bar{V}^\vee)$ is absolutely irreducible and generic if and only if $\text{cosoc}(\bar{V})$ is. Moreover, the map $(\text{soc}(\bar{V}^\vee))^{(n)} \rightarrow (\bar{V}^\vee)^{(n)}$ is dual to the map $\bar{V}^{(n)} \rightarrow (\text{cosoc}(\bar{V}))^{(n)}$ so that one is an isomorphism if and only if the other is. Thus (3) is equivalent to (4). \square

6.3.6. Lemma. *If A is a reduced Noetherian $W(k)$ -algebra, and if V is an admissible $A[G]$ -module, then the following are equivalent:*

- (1) V is A -torsion free, i.e. every associated prime of V is a minimal prime of A .
- (2) The natural map $V \rightarrow V \otimes_A \mathcal{K}(A)$ is an injection, where $\mathcal{K}(A)$ is the product over minimal primes \mathfrak{a} of A of the fields $A_{\mathfrak{a}}$.

(3) *The natural map*

$$V \rightarrow \prod_{\mathfrak{a}} (V/\mathfrak{a}V)^{\text{tf}}$$

is injective, where \mathfrak{a} runs over the minimal primes of A and $(V/\mathfrak{a}V)^{\text{tf}}$ is the maximal A/\mathfrak{a} -torsion free quotient of $V/\mathfrak{a}V$.

If these equivalent conditions hold, then for any Zariski dense set of primes Σ of $\text{Spec } A$, the map

$$V \rightarrow \prod_{\mathfrak{p} \in \Sigma} V \otimes_A \kappa(\mathfrak{p})$$

is injective.

Proof. If one replaces V with a finitely generated A -module M in conditions (1), (2) and (3), then the equivalence of these conditions is standard commutative algebra. On the other hand, if U is a sufficiently small compact open subgroup of G , then V^U is a direct summand of V that is finitely generated as an A -module, and V is the union of the V^U .

On the other hand, it is easy to see that each of conditions (1), (2) and (3) holds for V if, and only if, it holds with V^U in place of V for each sufficiently small compact open subgroup U of V . As (1), (2) and (3) are equivalent conditions on each such V^U , they are equivalent conditions on V .

Now let Σ be a Zariski dense set of primes of A , suppose that the equivalent conditions (1), (2), and (3) hold, and suppose that x is an element of V that maps to zero in $V \otimes_A \kappa(\mathfrak{p})$ for all $\mathfrak{p} \in \Sigma$. Choose a compact open subgroup U of G fixing x ; then V^U is finitely generated over A , and x maps to zero in $V^U \otimes_A \kappa(\mathfrak{p})$ for all $\mathfrak{p} \in \Sigma$. It follows that the support of x (considered as an element of V^U) is a closed subset of $\text{Spec } A$ contained in the complement of Σ . In particular that the annihilator of x is not contained in any minimal prime of A , contradicting condition (1). \square

6.3.7. Lemma. *Let A be a reduced Noetherian $W(k)$ -algebra, and let V_1 and V_2 be two admissible smooth $A[G]$ -modules such that:*

- (1) *For each i , $V_i^{(n)}$ is free of rank one over A .*
- (2) *For each i , V_i is generated by $\mathfrak{J}(V_i)$ as an $A[G]$ -module.*
- (3) *There exists a Zariski dense set Σ of primes of A such that for all $\mathfrak{p} \in \Sigma$, $(V_1)_{\mathfrak{p}}$ is isomorphic to $(V_2)_{\mathfrak{p}}$, as $A_{\mathfrak{p}}[G]$ -modules.*
- (4) *The natural map:*

$$V_i \rightarrow \prod_{\mathfrak{p} \in \Sigma} V_i \otimes_A \kappa(\mathfrak{p})$$

is injective for each i . (This is automatic if V_i is A -torsion free.)

Then there is an A -linear G -equivariant isomorphism $V_1 \xrightarrow{\sim} V_2$.

Proof. Let \mathcal{K}' be the product over $\mathfrak{p} \in \Sigma$ of the residue fields $\kappa(\mathfrak{p})$. Condition (3) gives us an isomorphism

$$\prod_{\mathfrak{p}} V_1 \otimes_A \kappa(\mathfrak{p}) \xrightarrow{\sim} \prod_{\mathfrak{p}} V_2 \otimes_A \kappa(\mathfrak{p}).$$

There are natural injections:

$$V_i \otimes_A \mathcal{K}' \rightarrow \prod_{\mathfrak{p}} V_i \otimes_A \kappa(\mathfrak{p})$$

for each i ; these are not in general isomorphisms. However, for sufficiently small compact open subgroups U of G , V_i^U is a direct summand of V_i that is finitely generated as an A -module. For each such U , the induced map:

$$V_i^U \otimes_A \mathcal{K}' \rightarrow \left(\prod_{\mathfrak{p}} V_i \otimes_A \kappa(\mathfrak{p}) \right)^U$$

is an isomorphism. Passing to the limit on both sides, we obtain an isomorphism of $V_i \otimes_A \mathcal{K}'$ with the space of smooth elements of $\prod_{\mathfrak{p}} V_i \otimes_A \kappa(\mathfrak{p})$. Thus the isomorphism:

$$\prod_{\mathfrak{p}} V_1 \otimes_A \kappa(\mathfrak{p}) \xrightarrow{\sim} \prod_{\mathfrak{p}} V_2 \otimes_A \kappa(\mathfrak{p})$$

induces an isomorphism of $V_1 \otimes_A \mathcal{K}'$ with $V_2 \otimes_A \mathcal{K}'$.

Moreover, by Condition (4), V_i embeds in $V_i \otimes_A \mathcal{K}'$ for each i . We may thus regard V_1 and V_2 as submodules of $V_1 \otimes_A \mathcal{K}'$.

By (1), $(V_1 \otimes_A \mathcal{K}')^{(n)}$ is free of rank one over \mathcal{K}' , and $V_i^{(n)}$ is a free A -submodule of $(V_1 \otimes_A \mathcal{K}')^{(n)}$ for each i . There thus exists an element c of $(\mathcal{K}')^\times$ such that $cV_2^{(n)}$ and $V_1^{(n)}$ coincide as submodules of $(V_1 \otimes_A \mathcal{K}')^{(n)}$. It follows that $c\mathfrak{J}(V_2)$ and $\mathfrak{J}(V_1)$ coincide as submodules of $\mathfrak{J}(V_1 \otimes_A \mathcal{K}')$. Since V_1 and V_2 are generated by $\mathfrak{J}(V_1)$ and $\mathfrak{J}(V_2)$ over $A[G]$, we must have $V_1 = cV_2$; in particular V_1 and V_2 are isomorphic. \square

We can now prove the uniqueness claim of Theorem 6.2.1.

6.3.8. Proposition. *Let A be a local ring satisfying Condition 6.1.1, and let V_1 and V_2 be two admissible smooth $A[G]$ -modules. Suppose that:*

- (1) *The V_i are A -torsion free.*
- (2) *For each minimal prime \mathfrak{a} of A , $(V_i)_{\mathfrak{a}}^{(n)}$ is nonzero.*
- (3) *For each i , \bar{V}_i satisfies the equivalent conditions of Lemma 6.3.5.*
- (4) *For each minimal prime \mathfrak{a} of A , there is a G -equivariant isomorphism $(V_1)_{\mathfrak{a}} \xrightarrow{\sim} (V_2)_{\mathfrak{a}}$.*

Then there is an A -linear G -equivariant isomorphism $V_1 \cong V_2$ (which, by part (3) of Proposition 6.3.4, is uniquely determined up to multiplication by an element of A^\times).

Proof. By part (1) of Proposition 6.3.4, we have that $V_i^{(n)}$ is free of rank 1 over A for each i . As the minimal primes of A are dense in $\text{Spec } A$, it thus follows by Lemma 6.3.7 that V_1 is isomorphic over $A[G]$ to V_2 . \square

The purpose of our next collection of results, which are rather technical, is to allow us to make a tensor factorization in the context of Theorem 6.2.1, and hence work with one E_v at a time.

6.3.9. Proposition. *Let K be a field in which all ℓ_i are invertible. If \bar{V} is an admissible smooth representation of G over K satisfying the equivalent conditions of Lemma 6.3.5, there exist admissible smooth representations \bar{V}_v of G_v ($v \in S$), each individually satisfying the equivalent conditions of Lemma 6.3.5 (with G replaced by G_v), together with a G -equivariant surjection $\bigotimes_v \bar{V}_v \rightarrow \bar{V}$.*

Proof. We proceed by induction on the cardinality s of S . In the case when $s = 1$ there is nothing to prove, and so we assume that $s > 1$, and write $S = v_1, \dots, v_s$, $G' = G_{v_2} \times \dots \times G_{v_s}$, so that $G = G_{v_1} \times G'$. Since $\text{cosoc}(\bar{V})$ is absolutely irreducible,

there is an isomorphism $\text{cosoc}(\bar{V}) \xrightarrow{\sim} \bar{\pi}_{v_1} \otimes \bar{\pi}'$, where $\bar{\pi}_{v_1}$ (resp. $\bar{\pi}'$) is a generic absolutely irreducible representation of G_{v_1} (resp. G').

Since \bar{V} is of finite length, we may and do choose a quotient \bar{W} of \bar{V} which is maximal with respect to the following property: there is a surjective map

$$\phi : \bar{V}_{v_1} \otimes \bar{V}' \rightarrow \bar{W},$$

where \bar{V}_{v_1} and \bar{V}' each satisfy the equivalent conditions of Lemma 6.3.5 (with respect to G_{v_1} and G' respectively). Since $\text{cosoc}(\bar{V})$ satisfies these conditions, we see that $\bar{W} \neq 0$. Thus $\text{cosoc}(\bar{V})$ is a quotient of \bar{W} , and hence $\bar{V}^{(n)}$ is isomorphic to $\bar{W}^{(n)}$, as both are one-dimensional.

Let \bar{U} be the kernel of the quotient map $\bar{V} \rightarrow \bar{W}$, and suppose that \bar{U} is non-zero. Extending scalars if necessary, we then may find a non-zero absolutely irreducible quotient $\bar{\theta}_{v_1} \otimes \bar{\theta}'$ of \bar{U} , where $\bar{\theta}_{v_1}$ (resp. $\bar{\theta}'$) is an absolutely irreducible representation of G_{v_1} (resp. G'). If we let \bar{T} denote the kernel of the quotient map $\bar{U} \rightarrow \bar{\theta}_{v_1} \otimes \bar{\theta}'$, and if we write $\bar{X} := \bar{V}/\bar{T}$, then there is a short exact sequence

$$(6.1) \quad 0 \rightarrow \bar{\theta}_{v_1} \otimes \bar{\theta}' \rightarrow \bar{X} \rightarrow \bar{W} \rightarrow 0,$$

which we may pullback via ϕ to obtain a short exact sequence

$$(6.2) \quad 0 \rightarrow \bar{\theta}_{v_1} \otimes \bar{\theta}' \rightarrow \bar{Y} \rightarrow \bar{V}_{v_1} \otimes \bar{V}' \rightarrow 0.$$

Applying the n -th derivative functor to (6.1), (and recalling that the surjections $\bar{V}^{(n)} \rightarrow \bar{X}^{(n)} \rightarrow \bar{W}^{(n)}$ are in fact isomorphisms,) we obtain an isomorphism:

$$\bar{\theta}_{v_1}^{(n)} \otimes_K (\bar{\theta}')^{(n)} \xrightarrow{\sim} (\bar{\theta}_{v_1} \otimes_K \bar{\theta}')^{(n)} = 0.$$

Hence either $\bar{\theta}_{v_1}^{(n)} = 0$ or $(\bar{\theta}')^{(n)} = 0$. Also, we conclude that $\bar{\theta}_{v_1} \otimes \bar{\theta}$ cannot be a quotient of \bar{V} , and hence cannot be a quotient of \bar{X} . Thus (6.1) is non-split, and hence (6.2) is also non-split (since ϕ is surjective).

The non-split short exact sequence (6.2) corresponds to a non-trivial element of $\text{Ext}_G^1(\bar{V}_{v_1} \otimes \bar{V}', \bar{\theta}_{v_1} \otimes \bar{\theta}')$, which by the Künneth formula admits the description

$$\begin{aligned} & \text{Ext}_G^1(\bar{V}_{v_1} \otimes \bar{V}', \bar{\theta}_{v_1} \otimes \bar{\theta}') \\ & \xrightarrow{\sim} \text{Hom}_{G_{v_1}}(\bar{V}_{v_1}, \bar{\theta}_{v_1}) \otimes \text{Ext}_{G'}^1(\bar{V}', \bar{\theta}') \oplus \text{Ext}_{G_{v_1}}^1(\bar{V}_{v_1}, \bar{\theta}_{v_1}) \otimes \text{Hom}_{G'}(\bar{V}', \bar{\theta}'). \end{aligned}$$

Now by assumption, the n th derivative (as a $\text{GL}_n(E_{v_1})$ -module) of any non-zero quotient of \bar{V}_{v_1} is a non-zero space, while the n th derivative (as a G' -module) of any non-zero quotient of \bar{V}' is a non-zero space. Thus if $(\bar{\theta}_{v_1})^{(n)} = 0$, then $\text{Hom}_{G_1}(\bar{V}_{v_1}, \bar{\theta}_{v_1}) = 0$, and thus \bar{Y} corresponds to a non-trivial element of the tensor product $\text{Ext}_{G_1}^1(\bar{V}_{v_1}, \bar{\theta}_{v_1}) \otimes \text{Hom}_{G'}(\bar{V}', \bar{\theta}')$. Concretely, this means we may form a non-trivial extension E_1 of \bar{V}_{v_1} by $\bar{\theta}_{v_1}$, and find a non-zero map $\psi : \bar{V}' \rightarrow \bar{\theta}'$ (which is then surjective, since $\bar{\theta}'$ is irreducible), so that \bar{Y} is obtained as the pushforward of $E_1 \otimes \bar{V}'$ via the map

$$\text{id} \otimes \psi : \bar{\theta}_{v_1} \otimes \bar{V}' \rightarrow \bar{\theta}_{v_1} \otimes \bar{\theta}'.$$

Thus \bar{Y} , and hence \bar{X} , is a quotient of $E_1 \otimes \bar{V}'$, contradicting the maximality of \bar{W} . If instead we had $(\bar{\theta}')^{(n)} = 0$, then we would similarly conclude that \bar{X} may

be written as a quotient of $\bar{V}_1 \otimes E_2$, for some non-trivial extension E_2 of \bar{V}' by $\bar{\theta}'$, again contradicting the maximality of \bar{W} .

From these contradictions we conclude that in fact $\bar{U} = 0$, and thus that $\bar{V} = \bar{W}$. Thus we may write \bar{V} as a quotient of $\bar{V}_{v_1} \otimes \bar{V}'$ as above. Applying the inductive hypothesis to \bar{V}' , the proposition follows. \square

6.3.10. Corollary. *If V is an admissible smooth representation of G over a local ring A satisfying Condition 6.1.1, such that $\bar{V} := V/\mathfrak{m}V$ satisfies the equivalent conditions of Lemma 6.3.5, and if $S' \subset S$ is any subset, then the $G_{S'}$ -representation $V^{(n), S \setminus S'} / \mathfrak{m}V^{(n), S \setminus S'}$ satisfies the conditions of Lemma 6.3.5 (with respect to $A[G_{S'}]$).*

Proof. Choose a surjection $\bigotimes_{v \in S} \bar{V}_v \rightarrow \bar{V}$ satisfying the conditions of the preceding proposition. Since $\bar{V}_v^{(n)}$ is one-dimensional for each v (and so in particular for each $v \in S \setminus S'$), applying the exact functor $\bar{V} \mapsto \bar{V}^{(n), S \setminus S'}$ yields a surjection

$$\bigotimes_{v \in S'} \bar{V}_v \xrightarrow{\sim} \left(\bigotimes_{v \in S} \bar{V}_v \right)^{(n), S \setminus S'} \rightarrow \bar{V}^{(n), S \setminus S'} \xrightarrow{\sim} V^{(n), S \setminus S'} / \mathfrak{m}V^{(n), S \setminus S'}.$$

The lemma follows. \square

We now return to the setting of the previous subsection. That is, for each v is S we are given a representation $\rho_v : G_{E_v} \rightarrow \mathrm{GL}_n(A)$. The above results allow us to establish Theorem 6.2.1 and Proposition 6.2.4 more or less immediately.

Proof of Theorem 6.2.1. Suppose we have V_1, V_2 satisfying conditions (1), (2), and (3) of Theorem 6.2.1. Then for all minimal primes \mathfrak{a} of A , we have a $\kappa(\mathfrak{a})$ -linear G -equivariant isomorphism $(V_1)_{\mathfrak{a}} \xrightarrow{\sim} (V_2)_{\mathfrak{a}}$. Thus V_1 and V_2 satisfy all of the hypotheses of Proposition 6.3.8, and are therefore isomorphic. Moreover, $V_1^{(n)}$ is cyclic as an A -module by Lemma 6.3.1. Let x be any element of V_1 whose image in $V_1^{(n)}$ generates $V_1^{(n)}$ as an A -module, and let W be the $A[G]$ -submodule of V_1 generated by x . Then the image of $W^{(n)}$ in $V_1^{(n)}$ contains the image of x , and hence is all of $V_1^{(n)}$. It follows that $\mathfrak{J}(W)$ is equal to all of $\mathfrak{J}(V_1)$, and hence, by Proposition 6.3.4, part (2), we must have $W = V_1$. Thus V_1 is cyclic as an $A[G]$ -module. Finally, $\mathrm{End}_{A[G]}(V_1)$ is isomorphic to A by Proposition 6.3.4, part (3). \square

Proof of Proposition 6.2.4. Suppose that for each v , we have a representation $\tilde{\pi}(\rho_v)$ satisfying conditions (1), (2), and (3) of Theorem 6.2.1 for ρ_v . Then it is clear that the maximal A -torsion free part of the tensor product over all v of $\tilde{\pi}(\rho_v)$ satisfies the conditions of Theorem 6.2.1 for the collection $\{\rho_v\}$.

Conversely, suppose we have a representation $\tilde{\pi}(\{\rho_v\}_{v \in S})$ satisfying the hypotheses of Theorem 6.2.1 for the collection $\{\rho_v\}$. Then for any minimal prime \mathfrak{a} of A , we have an isomorphism:

$$\tilde{\pi}(\{\rho_v\}_{v \in S}) \otimes_A \kappa(\mathfrak{a}) \xrightarrow{\sim} \bigotimes_{v \in S} \tilde{\pi}(\rho_{v, \mathfrak{a}}).$$

Fixing a place v , and taking derivatives at all $v' \neq v$, we obtain an isomorphism:

$$\left(\tilde{\pi}(\{\rho_v\}_{v \in S}) \right)_{\mathfrak{a}}^{(n), S \setminus \{v\}} \xrightarrow{\sim} \tilde{\pi}(\rho_{v, \mathfrak{a}}).$$

Moreover $(\tilde{\pi}(\{\rho_v\}_{v \in S}))^{(n), S \setminus \{v\}}$ is A -torsion free, and (by Corollary 6.3.10) satisfies condition (3) of Theorem 6.2.1. Thus $\tilde{\pi}(\{\rho_v\}_{v \in S})^{(n), S \setminus \{v\}}$ is isomorphic to $\tilde{\pi}(\rho_v)$ (so in particular the latter exists). \square

We now turn to Theorems 6.2.5 and 6.2.6. Once we have established these, Proposition 6.2.10 will be an easy consequence. We first need the following lemma:

6.3.11. Lemma. *Let A be a normal \mathbb{Q}_p -algebra that is an integral domain with field of fractions \mathcal{K} , and let (ρ', N) be a Frobenius-semisimple Weil–Deligne representation over A that splits (over \mathcal{K}) as a direct sum of absolutely indecomposable Weil–Deligne representations $\mathrm{Sp}_{\rho_i, N_i}$. There exist characters $\chi_i : W_E \rightarrow A^\times$ such that $\rho_i \otimes_K \chi_i$ is defined over a finite extension \mathcal{K}_0 of \mathbb{Q}_p contained in A .*

Proof. By Lemma 4.1.5 we know that such characters χ_i exist with values in \mathcal{K}^\times ; it suffices to show that they take values in A^\times . Let \mathcal{O} be the localization of A at a height one prime. Then $\rho' \otimes_A \mathcal{K}$ is \mathcal{O} -integral, so each ρ_i is \mathcal{O} -integral as well. Thus $\det \rho_i$ is a character with values in \mathcal{O}^\times . Since this is true for all \mathcal{O} , $\det \rho_i$ takes values in A^\times . Moreover, $(\det \rho_i) \otimes_{\mathcal{K}} \chi_i$ takes values in \mathcal{K}_0^\times , and \mathcal{K}_0 is contained in A , so some power of χ_i takes values in A^\times . But then χ_i must take values in A^\times as well since A is normal. \square

6.3.12. Lemma. *Let A be a local ring satisfying Condition 6.1.1, let $\{\rho_v\}$ be a collection of representations $G_{E_v} \rightarrow \mathrm{GL}_n(A)$, and suppose that $\tilde{\pi}(\{\rho_v\}_{v \in S})$ exists. Then, for each minimal prime \mathfrak{a} of A , the representation $\tilde{\pi}(\{\rho_v \otimes_A A/\mathfrak{a}\}_{v \in S})$ exists, and is isomorphic to the maximal A/\mathfrak{a} -torsion free part of $\tilde{\pi}(\{\rho_v\}_{v \in S}) \otimes_A A/\mathfrak{a}$.*

Proof. It is straightforward to see that $\tilde{\pi}(\{\rho_v\}_{v \in S}) \otimes_A A/\mathfrak{a}$ satisfies conditions (2) and (3) of Theorem 6.2.1, so its maximal A/\mathfrak{a} -torsion free quotient does as well. This quotient also satisfies condition (1) of Theorem 6.2.1 by construction. \square

Proof of Theorem 6.2.5. By Proposition 6.2.4, it suffices to consider the case when S has only one element v . By Lemma 6.3.12 we may assume A is a domain with field of fractions \mathcal{K} . Fix an algebraic extension \mathcal{K}' of \mathcal{K} such that \mathcal{K}' contains a square root of ℓ , where ℓ is the residue characteristic of v , and such that the Frobenius-semisimple Weil–Deligne representation associated to $\rho_v \otimes_A \mathcal{K}'$ splits as a direct sum of absolutely indecomposable Weil–Deligne representations $\mathrm{Sp}_{\rho_i, N_i}$ over \mathcal{K}' . Let \mathcal{K}_0 be the maximal subfield of \mathcal{K}' that is algebraic over \mathbb{Q}_p .

Let A' be the integral closure of $A_{\mathfrak{p}}$ in \mathcal{K}' , and let \mathfrak{p}' be a prime ideal of A' over \mathfrak{p} . Then, by Lemma 6.3.11, there exist characters χ_i , with values in $(A'_{\mathfrak{p}'})^\times$, such that for each i , $\rho_i \otimes \chi_i^{-1}$ is defined over \mathcal{K}_0 .

Let π_i be the admissible representation of $\mathrm{GL}_n(E_v)$ over \mathcal{K}_0 that corresponds to $\rho_i \otimes \chi_i$ under the unitary local Langlands correspondence. Then for each i , $(\mathrm{St}_{\pi_i, N_i} \otimes_{\mathcal{K}_0} \mathcal{K}') \otimes (\chi_i \circ \det)$ corresponds to $\mathrm{Sp}_{\rho_i, N_i}$ under unitary local Langlands. Without loss of generality, we assume that the representations π_i are ordered so that for all $i < j$, $(\mathrm{St}_{\pi_i, N_i} \otimes_{\mathcal{K}_0} \kappa(\mathfrak{p}')) \otimes \chi_i$ does not precede $(\mathrm{St}_{\pi_j, N_j} \otimes_{\mathcal{K}_0} \kappa(\mathfrak{p}')) \otimes \chi_j$. (It is then also true that $(\mathrm{St}_{\pi_i, N_i} \otimes_{\mathcal{K}_0} \mathcal{K}') \otimes \chi_i$ does not precede $(\mathrm{St}_{\pi_j, N_j} \otimes_{\mathcal{K}_0} \mathcal{K}') \otimes \chi_j$.) Let M be the smooth $A'_{\mathfrak{p}'}$ -linear dual of the module:

$$\left(| \circ \det \right)^{-\frac{n-1}{2}} \mathrm{Ind}_Q^{\mathrm{GL}_n(E_v)} \bigotimes_i \left[(\mathrm{St}_{\pi_i, N_i} \otimes_{\mathcal{K}_0} A'_{\mathfrak{p}'}) \otimes \chi_i \right],$$

where Q is a suitable block upper triangular parabolic subgroup of $\mathrm{GL}_n(E_v)$. Then, by construction, $M \otimes_{A'_{\mathfrak{p}'}} \mathcal{K}'$ is isomorphic to $\tilde{\pi}(\rho \otimes_A \mathcal{K}')$. Moreover, because of our assumptions on the ordering of the π_i , the smooth $\kappa(\mathfrak{p}')$ -dual of $M/\mathfrak{p}'M$ is essentially AIG by Corollary 4.3.3, and hence $\mathfrak{J}(M)$ generates M as an $A'_{\mathfrak{p}'}[G]$ -module. Moreover $M^{(n)}$ is free of rank one over $A'_{\mathfrak{p}'}$ by Corollary 3.1.13. Finally, M is $A'_{\mathfrak{p}'}$ -torsion free by construction (in fact, M is free over $A'_{\mathfrak{p}'}$.) Thus by Lemma 6.3.7, M is isomorphic to $\tilde{\pi}(\rho_v) \otimes_A A'_{\mathfrak{p}'}$.

The injection of Theorem 4.5.7 yields a surjection:

$$\tilde{\pi}(\rho_v \otimes_A \kappa(\mathfrak{p}')) \rightarrow M \otimes_{A'_{\mathfrak{p}'}} \kappa(\mathfrak{p}')$$

that is an isomorphism if, and only if, $\rho_v \otimes_A K$ is a minimal lift of $\rho_v \otimes_A \kappa(\mathfrak{p})$. Taking smooth duals and applying Lemma 3.2.5, we see that this surjection descends to the desired surjection

$$\tilde{\pi}(\rho_{v,\mathfrak{p}}) \rightarrow \tilde{\pi}(\rho_v) \otimes_A \kappa(\mathfrak{p}).$$

□

Proof of Theorem 6.2.6. As above, it suffices by Proposition 6.2.4 to consider the case when S has only one element v . By Lemma 6.3.12, for each minimal prime \mathfrak{a} of A containing \mathfrak{p} we have a surjection:

$$\tilde{\pi}(\rho_v) \otimes_A \kappa(\mathfrak{p}) \rightarrow \tilde{\pi}(\rho_v \otimes_A A/\mathfrak{a}) \otimes_A \kappa(\mathfrak{p}).$$

Then W is the image of $\tilde{\pi}(\rho_v) \otimes_A \kappa(\mathfrak{p})$ in the product

$$\prod_{\mathfrak{a}} \tilde{\pi}(\rho_v \otimes_A A/\mathfrak{a}) \otimes_A \kappa(\mathfrak{p}).$$

By Theorem 6.2.5 we also have surjections:

$$f_{\mathfrak{a}} : \tilde{\pi}(\rho_{v,\mathfrak{p}}) \rightarrow \tilde{\pi}(\rho_v \otimes_A A/\mathfrak{a}) \otimes_A \kappa(\mathfrak{p})$$

for all minimal primes \mathfrak{a} of A contained in \mathfrak{p} . This gives a diagonal map:

$$\tilde{\pi}(\rho_{v,\mathfrak{p}}) \rightarrow \prod_{\mathfrak{a}} \tilde{\pi}(\rho_v \otimes_A A/\mathfrak{a}) \otimes_A \kappa(\mathfrak{p}).$$

Let W' be the image of this map. It suffices to show that W' is isomorphic to W .

The spaces $W^{(n)}$ and $(W')^{(n)}$ are one-dimensional $\kappa(\mathfrak{p})$ -subspaces of

$$\prod_{\mathfrak{a}} [\tilde{\pi}(\rho_v \otimes_A A/\mathfrak{a}) \otimes_A \kappa(\mathfrak{p})]^{(n)}$$

that project isomorphically onto each factor. There thus exists for each \mathfrak{a} a scalar $c_{\mathfrak{a}}$ in $\kappa(\mathfrak{p})^{\times}$ such that $c(W')^{(n)}$ coincides with $W^{(n)}$ as subspaces of

$$\prod_{\mathfrak{a}} [\tilde{\pi}(\rho_v \otimes_A A/\mathfrak{a}) \otimes_A \kappa(\mathfrak{p})]^{(n)},$$

where c is the automorphism of this product given by multiplication by $c_{\mathfrak{a}}$ on the factor corresponding to \mathfrak{a} .

As the K -duals of $W \otimes_{\kappa(\mathfrak{p})} K$ and $W' \otimes_{\kappa(\mathfrak{p})} K$ are essentially AIG, this implies that W and cW' coincide, and thus W and W' are isomorphic. □

Proof of Proposition 6.2.8. As usual, we invoke Proposition 6.2.4 to reduce to the case where S has a single element v . As in the proof of Theorem 6.2.6, let W be

the image of $\kappa(\mathfrak{p}) \otimes_A \tilde{\pi}(\rho_v)$ under the diagonal map

$$\kappa(\mathfrak{p}) \otimes_A \tilde{\pi}(\rho_v) \rightarrow \prod_i \kappa(\mathfrak{p}) \otimes_{A/\mathfrak{a}_i} V_i,$$

where $\mathfrak{a}_1, \dots, \mathfrak{a}_i$ are the minimal primes of A contained in \mathfrak{p} and, for each i , V_i is the maximal A -torsion free quotient of $\tilde{\pi}(\rho_v) \otimes_A A/\mathfrak{a}_i$.

As $\tilde{\pi}(\rho_v)$ embeds in the product of the V_i , every Jordan–Hölder constituent of $\kappa(\mathfrak{p}) \otimes_A \tilde{\pi}(\rho_v)$ is isomorphic to a Jordan–Hölder constituent of $\kappa(\mathfrak{p}) \otimes_{A/\mathfrak{a}_i} V_i$ for some i , and hence to a Jordan–Hölder constituent of W . In particular, every Jordan–Hölder constituent of the kernel of the map

$$\kappa(\mathfrak{p}) \otimes_A \tilde{\pi}(\rho_v) \rightarrow W$$

is a Jordan–Hölder constituent of $\kappa(\mathfrak{p}) \otimes_A \tilde{\pi}(\rho_v)$ that appears with multiplicity at least two. Since the smooth dual of $\kappa(\mathfrak{p}) \otimes_A \tilde{\pi}(\rho_v)$ is essentially AIG, Corollary 4.3.11 above shows that no such Jordan–Hölder constituent can exist when $n = 2$ or 3 . \square

Proof of Proposition 6.2.10. By Proposition 6.2.4 we may assume S consists of a single element. For any minimal prime \mathfrak{b} of B , we have by Theorem 6.2.6 a surjection:

$$\tilde{\pi}(\rho_v) \otimes_A \kappa(f^{-1}(\mathfrak{b})) \rightarrow \tilde{\pi}(\rho_{v, f^{-1}(\mathfrak{b})})$$

and hence (after a base change) a surjection:

$$\tilde{\pi}(\rho_v) \otimes_A \kappa(\mathfrak{b}) \rightarrow \tilde{\pi}(\rho_v \otimes_A \kappa(\mathfrak{b})).$$

Let V be the image of the composed map:

$$\tilde{\pi}(\rho_v) \otimes_A B \rightarrow \prod_{\mathfrak{b}} \tilde{\pi}(\rho_v) \otimes_A \kappa(\mathfrak{b}) \rightarrow \prod_{\mathfrak{b}} \tilde{\pi}(\rho_v \otimes_A \kappa(\mathfrak{b})).$$

One easily verifies that V satisfies conditions (1), (2) and (3) of Theorem 6.2.1 for the representation $\rho_v \otimes_A B$ over B . \square

We now turn to the proof of Theorem 6.2.15. This will require several preliminary lemmas.

6.3.13. Lemma. *Suppose that Theorem 6.2.15 holds when S has only one element. Then Theorem 6.2.15 holds for an arbitrary finite set S .*

Proof. Suppose we have established Theorem 6.2.15 in the case in which S has only one element. We can then establish the general case as follows: suppose V satisfies the conditions of Theorem 6.2.15 for the collection $\{\rho_v\}$. If we fix a place $v \in S$, then $V^{(n), S \setminus \{v\}}$ satisfies the conditions of Theorem 6.2.15 for the representation ρ_v . Thus $V^{(n), S \setminus \{v\}}$ is isomorphic to $\tilde{\pi}(\rho_v)$. It follows by Proposition 6.2.4 that $\tilde{\pi}(\{\rho_v\}_{v \in S})$ exists and is isomorphic to the maximal torsion-free quotient of the tensor product of the representations $V^{(n), S \setminus \{v\}}$.

For any prime \mathfrak{p} of A lying over a prime of Σ , we have an isomorphism:

$$V \otimes_A \kappa(\mathfrak{p}) \xrightarrow{\sim} \bigotimes_{v \in S} V^{(n), S \setminus \{v\}} \otimes_A \kappa(\mathfrak{p}).$$

It thus follows by Lemma 6.3.7 that V and $\tilde{\pi}(\{\rho_v\}_{v \in S})$ are isomorphic, as required. \square

6.3.14. Proposition. *Let A be a local ring satisfying Condition 6.1.1, let ρ_v be an n -dimensional representation of G_{E_v} over A , and let V be an admissible $A[G]$ -module such that:*

- (1) V is torsion free over A .
- (2) The smooth dual of $V/\mathfrak{m}V$ is essentially AIG.
- (3) There exists a Zariski dense set of primes Σ in $\text{Spec } A[\frac{1}{p}]$ such that for each prime $\mathfrak{p} \in \Sigma$, $V \otimes_A \kappa(\mathfrak{p})$ is isomorphic to $\tilde{\pi}(\rho_{v,\mathfrak{p}})$,

Then V satisfies conditions (1), (2), and (3) of Theorem 6.2.1 with respect to ρ_v ; that is, $\tilde{\pi}(\rho_v)$ exists and is isomorphic to V .

Proof. Note that conditions (1) and (3) of Theorem 6.2.1 are immediate from the hypotheses. It thus suffices to construct, for each minimal prime \mathfrak{a} of A , an isomorphism $V_{\mathfrak{a}} \xrightarrow{\sim} \tilde{\pi}(\rho_{v,\mathfrak{a}})$.

Fix a minimal prime \mathfrak{a} of A . The map $V \rightarrow (V/\mathfrak{a}V)^{\text{tf}}$ becomes an isomorphism after localizing at any prime \mathfrak{p} of A that contains \mathfrak{a} but no other minimal prime of A . Thus, replacing V with $(V/\mathfrak{a}V)^{\text{tf}}$, A with A/\mathfrak{a} , and Σ with the set of \mathfrak{p} in Σ that contain \mathfrak{a} but no other minimal prime of A , we reduce to the case where A is a domain with field of fractions \mathcal{K} .

Let \mathcal{K}' be an algebraic extension of \mathcal{K} such that \mathcal{K}' contains a square root of ℓ , where ℓ is the residue characteristic of v , and such that the Frobenius-semisimple Weil–Deligne representation associated to $\rho_v \otimes_A \mathcal{K}'$ splits as a direct sum of absolutely indecomposable Weil–Deligne representations Sp_{ρ_i, N_i} over \mathcal{K}' . Let \mathcal{K}_0 be the maximal subfield of \mathcal{K}' that is algebraic over \mathbb{Q}_p .

Let A' be the integral closure of $A[\frac{1}{p}]$ in \mathcal{K}' . By Lemma 6.3.11, there exist characters χ_i , with values in $(A')^\times$, such that for each i , $\rho_i \otimes \chi_i$ is defined over \mathcal{K}_0 .

Let π_i be the admissible representation of $\text{GL}_n(E_v)$ over K_0 that corresponds to $\rho_i \otimes \chi_i$ under the unitary local Langlands correspondence. Then for each i , $(\text{St}_{\pi_i, N_i} \otimes_{\mathcal{K}_0} \mathcal{K}') \otimes (\chi_i \circ \det)$ corresponds to Sp_{ρ_i, N_i} under unitary local Langlands. Without loss of generality, we assume that the representations π_i are ordered so that for all $i < j$, $(\text{St}_{\pi_i, N_i} \otimes_{\mathcal{K}_0} \mathcal{K}') \otimes (\chi_i \circ \det)$ does not precede $(\text{St}_{\pi_j, N_j} \otimes_{\mathcal{K}_0} \mathcal{K}') \otimes (\chi_j \circ \det)$. Then, for all \mathfrak{p} in an open dense subset U_1 of $\text{Spec } A'$, and all $i < j$, $(\text{St}_{\pi_i, N_i} \otimes_{\mathcal{K}_0} \kappa(\mathfrak{p})) \otimes (\chi_i \circ \det)$ does not precede $(\text{St}_{\pi_j, N_j} \otimes_{\mathcal{K}_0} \kappa(\mathfrak{p})) \otimes (\chi_j \circ \det)$.

Let M be the smooth A' -linear dual of the module:

$$\left(| \circ \det \right)^{-\frac{n-1}{2}} \text{Ind}_Q^{\text{GL}_n(E_v)} \bigotimes_i \left[(\text{St}_{\pi_i, n_i} \otimes_{\mathcal{K}_0} A') \otimes \chi_i \right],$$

where Q is a suitable block upper triangular parabolic subgroup of $\text{GL}_n(E_v)$. Let U_2 be the open dense subset of $\text{Spec } A'$ consisting of those \mathfrak{p}' such that $\rho_v \otimes_A \mathcal{K}'$ is a minimal lift of $\rho_v \otimes_A \kappa(\mathfrak{p}')$. Then for all $\mathfrak{p}' \in U_2$, $M \otimes_{A'} \kappa(\mathfrak{p}')$ is isomorphic to $\tilde{\pi}(\rho_v \otimes_A \kappa(\mathfrak{p}'))$.

Now let M' be the $A'[G]$ -submodule of M generated by $\mathfrak{J}(M)$. Then $(M')^{(n)}$ is isomorphic to $M^{(n)}$, and the latter is locally free of rank one over A' by Corollary 3.1.13. The module M' is A' -torsion free, as it is contained in the free A' -module M . Moreover, for all $\mathfrak{p}' \in U_1 \cap U_2$, we have isomorphisms:

$$M' \otimes_{A'} \kappa(\mathfrak{p}') \xrightarrow{\sim} M \otimes_{A'} \kappa(\mathfrak{p}') \xrightarrow{\sim} \tilde{\pi}(\rho_v \otimes_A \kappa(\mathfrak{p}')).$$

Set

$$M'' = M' \otimes_{A'} \left((M')^{(n)} \right)^{-1}.$$

Then $(M'')^{(n)}$ is free of rank one over A' .

Let Σ' be the set of primes of A' lying over primes in Σ . Then $\Sigma' \cap U_1 \cap U_2$ is dense in $\text{Spec } A'$, and we have isomorphisms: $V \otimes_A \kappa(\mathfrak{p}') \xrightarrow{\sim} M'' \otimes_{A'} \kappa(\mathfrak{p}')$ for all $\mathfrak{p}' \in \Sigma' \cap U_1 \cap U_2$. It follows by Lemma 6.3.7 that $V \otimes_A A'$ is isomorphic to M'' . In particular $V \otimes_A \mathcal{K}'$ is isomorphic to $\tilde{\pi}(\rho_v \otimes_A \mathcal{K}')$, and hence $V \otimes_A \mathcal{K}$ is isomorphic to $\tilde{\pi}(\rho_v \otimes_A \mathcal{K})$, as required. \square

Proof of Theorem 6.2.15. By Lemma 6.3.13 it suffices to consider the case where S has a single element. Let V^{tf} be the maximal A -torsion free quotient of V , and let V^{tor} be the kernel of the map $V \rightarrow V^{\text{tf}}$. Then $V^{\text{tor}}/\mathfrak{m}V^{\text{tor}}$ embeds in $V/\mathfrak{m}V$, and the latter has finite length by Lemma 6.3.5. It follows that $V^{\text{tor}}/\mathfrak{m}V^{\text{tor}}$ is finitely generated as an $A[G]$ -module, and (since V is \mathfrak{m} -adically separated), thus V^{tor} is finitely generated as well. In particular its support is a closed subset Z of $\text{Spec } A$. Let U be the complement of Z in $\text{Spec } A$. Then for all \mathfrak{p} in $\Sigma \cap U$, we have isomorphisms:

$$\tilde{\pi}(\rho_{v,\mathfrak{p}}) \xrightarrow{\sim} V \otimes_A \kappa(\mathfrak{p}) \xrightarrow{\sim} V^{\text{tf}} \otimes_A \kappa(\mathfrak{p}).$$

It follows by Proposition 6.3.14 that V^{tf} satisfies conditions (1), (2), and (3) of Theorem 6.2.1; in particular $\tilde{\pi}(\rho_v)$ exists and is isomorphic to V^{tf} . Thus, by Theorem 6.2.5, $V^{\text{tf}} \otimes_A \kappa(\mathfrak{p})$ is isomorphic to $\tilde{\pi}(\rho_v \otimes_A \kappa(\mathfrak{p}))$ for all \mathfrak{p} for which there exists a minimal prime \mathfrak{a} of A such that $\rho_{v,\mathfrak{a}}$ is a minimal lift of $\rho_{v,\mathfrak{p}}$. In particular this holds for all \mathfrak{p} in Σ . Thus we have an isomorphism:

$$V \otimes_A \kappa(\mathfrak{p}) \xrightarrow{\sim} V^{\text{tf}} \otimes_A \kappa(\mathfrak{p})$$

for all $\mathfrak{p} \in \Sigma$; by Lemma 6.3.7 it follows that V is isomorphic to V^{tf} , and hence that V satisfies conditions (1), (2), and (3) of Theorem 6.2.1. \square

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