THE EISENSTEIN IDEAL IN HIDA'S ORDINARY HECKE ALGEBRA

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Abstract. For $p \geq 5$ a prime and $i \not\equiv 0 \pmod{p-1}$ let $\mathbf{T}^{ord}_{par,(i)}$ denote Hida's ordinary Hecke algebra acting on the space of p-ordinary cusp forms of tame level one, arbitrary weight k, and tame nebentypus equal to $\omega^{(i-k)}$ (where ω is the Teichmuller character). Let $\mathcal{I}'_{Eis}$ denote the Eisenstein ideal in $\mathbf{T}^{ord}_{par,(i)}$, that is, the ideal generated by the elements $T_l - 1 - \langle l \rangle l^{-1}$ (for l a prime distinct from p) and the element $U_p - 1$. Let $L_p(\omega^i, 1-s)$ denote the Kubota-Leopoldt p-adic L-function. In this note we give a simple proof of the fact that there is a canonical isomorphism $\Lambda_{(i)}/L_p(\omega^i, 1-s) = \mathbf{T}^{ord}_{par,(i)}/\mathcal{I}'_{Eis}$ (a result originally established in [10]) and use this isomorphism to give a direct proof of the main result of [9]. Our proof of the isomorphism depends on constructing a certain element in the full Hecke algebra $\mathbf{T}_{(i)}$ which one can think of as being the "universal constant term" of a (not necessarily cuspidal) p-adic modular form.

Introduction

Let us fix a prime number $p \geq 5$. The object of this note is to study the relation between p-adic ordinary Eisenstein series and p-adic ordinary cusp forms.

Let $\mathbf{V}$ denote Katz's ring of generalized p-adic modular functions of level one (as constructed in [5,6,7,8]) and let $\mathbf{T}$ denote the ring of Hecke operators acting on $\mathbf{V}$ (as constructed in [1,2]). If $\Lambda := \mathbf{Z}_p[[\mathbf{Z}_p^\times]]$ denotes the completed group ring of $\mathbf{Z}_p^\times$ over $\mathbf{Z}_p$ then $\mathbf{T}$ is naturally a Λ algebra.

The Teichmuller character $\omega : \mathbf{F}_p^\times \to \mathbf{Z}_p^\times$ is uniquely determined as the inverse of the reduction map $\mathbf{Z}_p^\times \to \mathbf{F}_p^\times$. If we let Γ denote the kernel of the reduction map, and identify $\mathbf{F}_p^\times$ with its image under ω, then we may write $\mathbf{Z}_p^\times = \Gamma \times \mathbf{F}_p^\times$. The ring Λ is semi-local, and is equal to the product $\Lambda = \prod_{i \pmod{p-1}} \Lambda_{(i)}$, where $\Lambda_{(i)}$ is that local factor of Λ on which $\mathbf{F}_p^\times$ acts via $\omega^{(i)}$. If we forget the $\mathbf{F}_p^\times$ action, we find that $\Lambda_{(i)}$ is isomorphic to the completed group ring $\mathbf{Z}_p[[\Gamma]]$. We write $\mathbf{V}_{(i)} = \mathbf{V} \otimes_\Lambda \Lambda_{(i)}$ and $\mathbf{T}_{(i)} = \mathbf{T} \otimes_\Lambda \Lambda_{(i)}$, so that $\mathbf{T}_{(i)}$ is the ring of Hecke operators on $\mathbf{V}_{(i)}$.

Now follow [2,3] in cutting down to the ordinary part $\mathbf{T}^{ord}_{(i)}$ of $\mathbf{T}_{(i)}$. There is a natural morphism $\mathbf{T}^{ord}_{(i)} \to \Lambda_{(i)}$ obtained by considering the action of the Hecke operators on the Eisenstein series, whose kernel we denote by $\mathcal{I}_{Eis}$. Thus $\mathcal{I}_{Eis}$ is generated by the elements $T_l - \langle l \rangle l^{-1}$ (where l runs over all primes other than p) together with the element $U_p - 1$. We let $\mathbf{T}_{par,(i)}$ denote the quotient of $\mathbf{T}_{(i)}$ which acts on the subspace $\mathbf{V}_{par,(i)}$ of $\mathbf{V}_{(i)}$ consisting of cuspidal generalized modular functions, and let $\mathcal{I}'_{Eis}$ denote the image of $\mathcal{I}_{Eis}$ in $\mathbf{T}^{ord}_{par,(i)}$. Following the terminology of [10], we call $\mathcal{I}'_{Eis}$ the *Eisenstein ideal* in $\mathbf{T}^{ord}_{par,(i)}$.

The main result of this note is the following Theorem:

Theorem. *Suppose that $i \not\equiv 0 \pmod{p-1}$. Then the pullback of $\mathcal{I}'_{Eis}$ via the natural map $\Lambda_{(i)} \to \mathbf{T}^{ord}_{par,(i)}$ is equal to the principal ideal of $\Lambda_{(i)}$ generated by the Kubota-Leopoldt p-adic L-function $L_p(\omega^i, 1-s)$, and the consequent injection*

$$\Lambda_{(i)}/L_p(\omega^i, 1-s) \to \mathbf{T}^{ord}_{par,(i)}/\mathcal{I}'_{Eis}$$

is an isomorphism.

This has as a corollary the following result:

Corollary. *If $i \not\equiv 0 \pmod{p-1}$ then the element $\mathrm{Norm}_{\mathbf{T}^{ord}_{par,(i)}/\Lambda_{(i)}}(U_p - 1)$ of $\Lambda_{(i)}$ is divisible by $L_p(\omega^i, 1-s)$.*

This Theorem is proved (in a slightly different language) in [10]. The proof there is however rather indirect, depending on a "modulo p" version of the Kubota-Lang theory of modular units, and it occurs in conjunction with a subtle and sophisticated analysis of the arithmetic geometry of Jacobians of modular curves. The proof we will give is much simpler by comparison.

The Corollary was first proved in [9], where the proof again involves an analysis of the geometry of Jacobians of modular curves. In this note we point out that the Corollary is a direct consequence of the Theorem.

The proof of the Theorem that we give depends on constructing an element A_0 in $\mathbf{T}_{(i)}$ for each $i \not\equiv 0 \pmod{p-1}$ with the property that

$$a_0(f) = a_1(f|A_0)$$

for any generalized modular function f. (Here $a_n(f)$ denotes the n^{th} q-expansion coefficient of the generalized modular function f.) This construction generalizes [12, Théorème 9].

§1. The set-up

In this section we establish our notation and recall some results on p-adic modular forms. Unfortunately there seems to be no consensus in the literature as to the notation that should be used in this theory, and so we take some care to relate our notation to that of the various references we cite.

Fix a prime $p \geq 5$. We denote by $\mathbf{V}$ the ring of generalized p-adic modular functions of tame level one which are holomorphic at the cusps. This ring is constructed in [5,6,7,8] (see in particular [5, §4] for a discussion of the case of tame level one) in which it is denoted by $V_{\infty,\infty}$.

Let $\mathcal{M}_k$ denote the space of modular forms of level one and weight k defined over $\mathbf{Q}_p$ and let $\mathcal{M} = \bigoplus_k \mathcal{M}_k$ denote the graded ring of modular forms defined over $\mathbf{Q}_p$. The q-expansion map allows us to consider $\mathcal{M}$ as a subring of $\mathbf{Q}_p[[q]]$. We have the following elementary description of $\mathbf{V}$ [2,5]: $\mathbf{V}$ is equal to the p-adic completion of the intersection $\mathcal{M} \cap \mathbf{Z}_p[[q]]$, where this intersection is taken in $\mathbf{Q}_p[[q]]$.

We write $\Lambda := \mathbf{Z}_p[[\mathbf{Z}_p^\times]]$ for the completed group ring of $\mathbf{Z}_p^\times$ with coefficients in $\mathbf{Z}_p^\times$. The Teichmuller mapping $\omega : \mathbf{F}_p^\times \hookrightarrow \mathbf{Z}_p^\times$ induces the direct sum decomposition

$$\Lambda = \prod_{i \pmod{p-1}} \Lambda_{(i)},$$

where $\Lambda_{(i)}$ denotes a copy of the completed group ring $\mathbf{Z}_p[[\Gamma]]$ ($\Gamma = 1 + p\mathbf{Z}_p$) made into a Λ-algebra by having $\mathbf{F}_p^\times$ act via ω^i. For any Λ-module M we write

$$M_{(i)} := M \otimes_\Lambda \Lambda_{(i)}.$$

If $a \in \mathbf{Z}_p^\times$ then we use the notation $\langle a \rangle$ to denote the corresponding element of the group ring Λ, as well as its projection to any of the factors $\Lambda_{(i)}$ of Λ.

We define an action of $\mathbf{Z}_p^\times$ on $\mathcal{M}$ by defining the action on each direct summand $\mathcal{M}_k$ via the formula $f|\langle\gamma\rangle = \gamma^k f$. One can show that with respect to this action $\mathcal{M} \cap \mathbf{Z}_p[[q]]$ is a $\mathbf{Z}_p^\times$ submodule of $\mathcal{M}$ [5]. Thus there is induced an action of $\mathbf{Z}_p^\times$ on $\mathbf{V}$ [5] via which we may regard $\mathbf{V}$ as a Λ-module and $\mathrm{End}(\mathbf{V})$ as a Λ-algebra.

We let $\mathbf{T}$ denote the completion in the compact-open topology of $\mathrm{End}(\mathbf{V})$ of the Λ-subalgebra generated by the Hecke operators T_l ($l \neq p$) and U_p. (In this choice of notation we follow [1]. This Hecke algebra is denoted by a script 'H' in [2]. Note that in the reference [11] the symbol $\mathbf{T}$ is used to denote a Hecke algebra which we would denote by $\prod_{i \not\equiv 2 \pmod{p-1}} \mathbf{T}^{ord}_{par,(i)}$ in the notation to be introduced below. See [1, §II.1.2, §III.1.2] for an explanation of why the compact open topology is the appropriate topology to consider.) It will sometimes be convenient to write $T_p := U_p$. If n is any positive integer, then the usual formulae allow us to write the Hecke operator T_n as a polynomial in the T_l (l ranging over all primes dividing n) with coefficients in Λ. To be precise:

(1) $T_1 =$ the identity endomorphism; if m and n are coprime then $T_{mn} = T_m T_n$;
(2) if l is a prime distinct from p then for any integer $k \geq 2$

$$T_{l^k} = T_l T_{l^{k-1}} - \langle l \rangle l^{-1} T_{l^{k-2}};$$

(3) for any positive integer k, $T_{p^k} = T_p^k$.

Note that the particular case $T_{l^2} = T_l^2 + \langle l \rangle l^{-1}$ of (3) yields the alternative description of $\mathbf{T}$ as the completion of the $\mathbf{Z}_p$-subalgebra of $\mathrm{End}(\mathbf{V})$ generated by all the Hecke operators T_n.

For each $i \pmod{p-1}$ the subspace $\mathbf{V}_{(i)}$ of $\mathbf{V}$ is $\mathbf{T}$-invariant, and $\mathbf{T}$ acts on $\mathbf{V}_{(i)}$ through its quotient $\mathbf{T}_{(i)}$. If T is any operator in $\mathbf{T}$, we will denote by the same letter its image in $\mathbf{T}_{(i)}$ (for any (i)). This will apply in particular when $T = T_n$ for some integer n.

We define the subspace $\mathbf{V}_{par}$ of $\mathbf{V}$ by working with the spaces $\mathcal{S}_k$ of cusp forms of level one and weight k in place of the spaces $\mathcal{M}_k$. It is easily seen that $f \in \mathbf{V}$ is an element of the subspace $\mathbf{V}_{par}$ if and only if for every $a \in \mathbf{Z}_p^\times$ the element $f|\langle a \rangle$ of $\mathbf{V}$ has vanishing constant term in its q-expansion. The subspace $\mathbf{V}_{par}$ of $\mathbf{V}$ is $\mathbf{T}$-invariant, and we denote by $\mathbf{T}_{par}$ that quotient of $\mathbf{T}$ which acts faithfully on $\mathbf{V}_{par}$. (Note that this quotient is denoted by $\mathbf{T}_0$ in [1] and by a script 'h' in [2,3].)

§2. The universal constant term

We need the following variant of Katz's Key Lemma [6]:

Proposition 1. *If $f \in \mathbf{V}_{(i)}[1/p]$ for some $i \not\equiv 0 \pmod{p-1}$, and $a_n(f) \in \mathbf{Z}_p$ for every $n \geq 1$, then $a_0(f) \in \mathbf{Z}_p$, and thus $f \in \mathbf{V}$.*

Proof. Suppose that $a_0(f) \notin \mathbf{Z}_p$. Let $\nu > 0$ be the p-adic valuation of $1/a_0(f)$, so that $p^\nu f \in \mathbf{V}_{(i)}$ and $p^\nu a_0(f) \in \mathbf{Z}_p^\times$. Then

$$p^\nu f - p^\nu a_0(f) \equiv 0 \pmod{p^\nu}$$

by virtue of our assumption about $a_n(f)$ for $n > 0$.

Let a be a generator of the cyclic group $\mathbf{F}_p^\times \hookrightarrow \mathbf{Z}_p^\times$. Since $f \in \mathbf{V}_{(i)}$, $p^\nu f|\langle a\rangle = a^i p^\nu f$, while since $p^\nu a_0(f)$ is a constant, $p^\nu a_0(f)|\langle a\rangle = p^\nu a_0(f)$. Thus

$$(1-a^i)p^\nu a_0(f) \equiv p^\nu a_0(f) - a^i p^\nu f \equiv (p^\nu a_0(f) - p^\nu f)|\langle a\rangle \equiv 0 \pmod{p^\nu}.$$

Since $i \not\equiv 0 \pmod{p-1}$ our choice of a guarantees that $(1-a^i) \not\equiv 0 \pmod{p}$. This implies that $p^\nu a_0(f) \equiv 0 \pmod{p^\nu}$, a contradiction to our assumption that $a_0(f) \notin \mathbf{Z}_p$. Thus we conclude that in fact $a_0(f) \in \mathbf{Z}_p$, and the q-expansion principle now shows that $f \in \mathbf{V}_{(i)}$. □

From this we deduce the following Corollary:

Corollary 2. *For any p-adically complete $\mathbf{Z}_p$-algebra R we have an isomorphism of continuous R-modules, functorial in R:*

$$\mathbf{V}_{(i)} \hat{\otimes}_{\mathbf{Z}_p} R \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(\mathbf{T}_{(i)}, R),$$

given via the pairing

$$(f, T) \to a_1(f|T),$$

provided $i \not\equiv 0 \pmod{p-1}$. (Here $\mathbf{T}_{(i)}$ has its compact-open topology, R has its p-adic topology, and $\mathrm{Hom}_{\mathrm{cont}}(\mathbf{T}_{(i)}, R)$ has the topology of uniform convergence.)

Proof. This result can be proved using the methods of [1, §III.1.2] (see also [2, §2]) since the preceding Corollary assures us that the constant terms of the elements of $\mathbf{V}_{(i)}$ don't cause any trouble. (See [1, §III.1.3] for a discussion of this trouble in the case when one allows $i \equiv 0 \pmod{p-1}$.) □

The morphism $f \to a_0(f)$ yields a morphism

$$\mathbf{V}_{(i)} \hat{\otimes} R \to R,$$

functorial in p-adically complete $\mathbf{Z}_p$-algebras R. Thus we get a morphism

$$\mathrm{Hom}_{\mathrm{cont}}(\mathbf{T}_{(i)}, R) \to R,$$

functorial in R. Since $\mathbf{T}_{(i)}$, as a topological ring, is the inverse limit of p-adically complete rings (for example, the classical Hecke algebras at finite level and weight; see [1, §II.1.2] or [2, §2]) this morphism corresponds to an element $A_{0,(i)} \in \mathbf{T}_{(i)}$, the "universal constant term" of the elements of $\mathbf{V}_{(i)}$. By construction, if $f \in \mathbf{V}_{(i)}$, then $a_1(f|A_{0,(i)}) = A_{0,(i)}(f)$.

Lemma 3. *The following conditions are equivalent for any $f \in \mathbf{V}_{(i)}$:*
(i) $f|TA_{(i)} = 0$ for all $T \in \mathbf{T}_{(i)}$;
(ii) $f \in V_{par,(i)}$.

Proof. If $f|TA_{(i)} = 0$ for all $T \in \mathbf{T}_{(i)}$ then in particular for any $a \in \mathbf{Z}_p^\times$ we compute that

$$a_0(f|\langle a\rangle) = a_1(f|\langle a\rangle A_{0,(i)}) = 0,$$

and thus $f \in \mathbf{V}_{par,(i)}$. Hence *(i)* implies *(ii)*.

Conversely, if f is an element of $\mathbf{V}_{par,(i)}$, so is $f|T_n$ for every n, and so

$$a_n(f|A_{0,(i)}) = a_1(f|A_{0,(i)}|T_n) = a_1(f|T_n|A_{0,(i)}) = a_0(f|T_n) = 0$$

for every $n \geq 1$, showing that $f|A_0 = 0$. Since $\mathbf{V}_{par,(i)}$ is closed under the action of $\mathbf{T}_{(i)}$ we conclude more generally that $f|TA_{0,(i)} = 0$ for any $T \in \mathbf{T}_{(i)}$. Hence *(ii)* implies *(i)*. $\square$

The preceding Lemma shows that the principal ideal generated by $A_{0,(i)}$ is in the kernel of the surjection $\mathbf{T}_{(i)} \to \mathbf{T}_{par,(i)}$, and so there is an induced surjection $\mathbf{T}_{(i)}/A_{0,(i)} \to \mathbf{T}_{par,(i)}$.

Proposition 4. *The morphism $\mathbf{T}_{(i)}/A_{0,(i)} \to \mathbf{T}_{par,(i)}$ is an isomorphism.*

Proof. Consider the following commutative diagram, for any p-adically complete $\mathbf{Z}_p$-algebra R:

$$\begin{tikzcd} & \mathrm{Hom}_{\mathrm{cont}}(\mathbf{T}_{par,(i)}, R) \arrow[d] \\ \mathbf{V}_{par,(i)} \hat{\otimes}_{\mathbf{Z}_p} R \arrow[ur, "\sim"] \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_{\mathrm{cont}}(\mathbf{T}_{(i)}/A_{0,(i)}, R) \arrow[d] \\ \mathbf{V}_{(i)} \hat{\otimes}_{\mathbf{Z}_p} R \arrow[r, "\sim"] & \mathrm{Hom}_{\mathrm{cont}}(\mathbf{T}_{(i)}, R) \end{tikzcd}$$

In this diagram the left hand vertical arrow is the completion of the tensor product of the inclusion of $\mathbf{V}_{par,(i)}$ in $\mathbf{V}_{(i)}$ with the identity morphism of R and the right hand vertical arrows are the injections determined by the surjections

$$\mathbf{T}_{(i)} \to \mathbf{T}_{(i)}/A_{0,(i)} \to \mathbf{T}_{par,(i)}.$$

The lower horizontal arrow is the isomorphism of Corollary 2 and the diagonal arrow is the isomorphism of [1, Corollary III.1.3]. Lemma 3 shows that the upper horizontal arrow is also an isomorphism. Thus the morphism

$$\mathrm{Hom}_{\mathrm{cont}}(\mathbf{T}_{par,(i)}, R) \to \mathrm{Hom}_{\mathrm{cont}}(\mathbf{T}_{(i)}/A_{0,(i)}, R)$$

is an isomorphism for every p-adically complete $\mathbf{Z}_p$-algebra R. Since any principal ideal is closed in $\mathbf{T}_{(i)}$ we conclude that the morphism

$$\mathbf{T}_{(i)}/A_{0,(i)} \to \mathbf{T}_{par,(i)}$$

is an isomorphism. $\square$

§3. Ordinary parts

In this section we cut down to the ordinary part $\mathbf{T}^{ord}$ of $\mathbf{T}$, as in [2,3,9,11]. (Note that we follow the conventions of [1] in using a superscript *ord* to denote taking the ordinary part of any $\mathbf{T}$-module. The convention of [2,3] is to use either a subscript or a superscript 0, while the convention of [11] is to use a subscript U. Note however that in this last reference $\mathbf{T}$ (with no subscript) is used to denote the ordinary part of the Hecke algebra.) The factorization $\mathbf{T} = \prod_i \mathbf{T}_{(i)}$ yields a factorization $\mathbf{T}^{ord} = \prod_i \mathbf{T}^{ord}_{(i)}$. If T is any element of $\mathbf{T}_{(i)}$ we will use the same symbol to denote its image in the ring $\mathbf{T}^{ord}_{(i)}$. We will apply this convention in particular to the operators T_n (n an integer) and $A_{0,(i)}$.

For any $i \pmod{p-1}$ we denote by $\mathcal{I}_{Eis,(i)}$ the ideal in $\mathbf{T}^{ord}_{(i)}$ generated by the elements $T_l - 1 - \langle l \rangle l^{-1}$ (where l ranges over all primes distinct from p) together with the element $U_p - 1$. The natural morphism $\Lambda_{(i)} \to \mathbf{T}^{ord}_{(i)}$ induces an isomorphism

$$\Lambda_{(i)} \xrightarrow{\sim} \mathbf{T}/\mathcal{I}_{Eis,(i)}.$$

The known formulae for the constant terms of Eisenstein series of level one and weight k show that if $i \not\equiv 0 \pmod{p-1}$ then the image of $A_{0,(i)}$ in $\mathbf{T}/\mathcal{I}_{Eis,(i)} = \Lambda_{(i)}$ is equal to the Kubota-Leopoldt p-adic L-function $L_p(\omega^i, 1-s)$. (For the construction of this L-function, see [4,6]. For the connection with Eisenstein series from the p-adic point of view, see [6,12].)

We let $\mathbf{T}^{ord}_{par,(i)}$ denote the ordinary part of $\mathbf{T}_{par,(i)}$. Proposition 4 shows that $\mathbf{T}^{ord}_{par,(i)} = \mathbf{T}^{ord}_{(i)}/A_{0,(i)}$. We let $\mathcal{I}'_{Eis,(i)}$ denote the image of the ideal $\mathcal{I}_{Eis}$ in $\mathbf{T}^{ord}_{par,(i)}$.

Theorem 5. *Suppose that $i \not\equiv 0 \pmod{p-1}$. Then the pullback of $\mathcal{I}'_{Eis,(i)}$ via the natural map $\Lambda_{(i)} \to \mathbf{T}^{ord}_{par,(i)}$ is equal to the principal ideal of $\Lambda_{(i)}$ generated by the Kubota-Leopoldt p-adic L-function $L_p(\omega^i, 1-s)$, and the consequent injection*

$$\Lambda_{(i)}/L_p(\omega^i, 1-s) \to \mathbf{T}^{ord}_{par,(i)}/\mathcal{I}'_{Eis,(i)}$$

is an isomorphism.

Proof. This follows from a consideration of the following commutative diagram:

$$\begin{tikzcd}
& \mathbf{T}^{ord}_{(i)} \arrow[d] \arrow[dr] & \\
\Lambda_{(i)} \arrow[ur] \arrow[r, "\sim"] \arrow[d] & \mathbf{T}^{ord}_{(i)}/\mathcal{I}_{Eis,(i)} \arrow[d] & \mathbf{T}^{ord}_{(i)}/A_{0,(i)} = \mathbf{T}^{ord}_{par,(i)} \arrow[dl] \\
\Lambda_{(i)}/L_p(\omega^i, 1-s) \arrow[r, "\sim"] & \mathbf{T}^{ord}_{(i)}/(\mathcal{I}_{Eis}, A_{0,(i)}) = \mathbf{T}^{ord}_{par,(i)}/\mathcal{I}'_{Eis,(i)}. &
\end{tikzcd}$$

□

In [2] it is shown that $\mathbf{T}^{ord}_{par,(i)}$ is finite flat over $\Lambda_{(i)}$ for every $i \pmod{p-1}$. (An alternative proof is given in [9] for the case $i \neq 2$.) Thus we can define the norm map

$$\mathrm{Norm}_{\mathbf{T}^{par}_{(i)}/\Lambda_{(i)}} : \mathbf{T}^{ord}_{par,(i)} \to \Lambda_{(i)}.$$

Corollary 6. *If $i \not\equiv 0 \pmod{p-1}$ then the element $\mathrm{Norm}_{\mathbf{T}^{ord}_{par,(i)}/\Lambda_{(i)}}(U_p - 1)$ of $\Lambda_{(i)}$ is divisible by $L_p(\omega^i, 1-s)$.*

Proof. If $\mathbf{T}^{ord}_{par,(i)} = 0$ there is nothing to prove. Thus we suppose that $\mathbf{T}^{ord}_{par,(i)} \neq 0$, so that we may regard Λ as a subring of $\mathbf{T}^{ord}_{par,(i)}$.

Write $N := \mathrm{Norm}_{\mathbf{T}^{par}_{(i)}/\Lambda_{(i)}}(U_p - 1)$. Then we have an inclusion of ideals

$$N\Lambda_{(i)} \subset (U_p - 1)\mathbf{T}^{ord}_{par,(i)} \cap \Lambda_{(i)} \subset \mathcal{I}'_{Eis,(i)} \cap \Lambda_{(i)} = L_p(\omega^i, 1-s)\Lambda_{(i)}$$

(where the first inclusion follows immediately from the definition of the norm and the final equality is the result of Theorem 5). $\square$

We can improve the result of the Corollary a little: we may complete $\mathbf{T}^{ord}_{par,(i)}$ at the ideal $\mathcal{I}'_{Eis,(i)}$ to obtain a local ring $\mathbf{T}^{ord,Eis}_{par,(i)}$. There is a consequent factorization of $\mathbf{T}^{ord}_{par,(i)}$ as a product

$$\mathbf{T}^{ord}_{par,(i)} = \mathbf{T}^{ord,Eis}_{par,(i)} \times \mathbf{T}^{ord,0}_{par,(i)}.$$

Then one sees immediately that the above corollary remains true with $\mathbf{T}^{ord,Eis}_{par,(i)}$ in place of $\mathbf{T}^{ord}_{par,(i)}$:

Corollary 7. *If $i \not\equiv 0 \pmod{p-1}$ then the element $\mathrm{Norm}_{\mathbf{T}^{ord}_{par,(i)}/\Lambda_{(i)}}(U_p - 1)$ of $\Lambda_{(i)}$ is divisible by $L_p(\omega^i, 1-s)$.*

We now explain the connection between this result and the result of [9]. In that reference a certain Hecke ring denoted $\mathbf{T}(\chi)$ is considered, where χ is a power of the Teichmuller character. If $\chi = \omega^i$ then the ring $\mathbf{T}(\chi)$ of [9] is equal to the completion (returning to the notation of this paper) of the ring $\mathbf{T}^{ord}_{par,(i+2)}$ at the principal ideal generated by $(U_p - 1)$. Let us write $\mathbf{T}^{ord,U_p\equiv 1}_{par,(i+2)}$ for this completion. Then the following diagram commutes:

$$\begin{tikzcd} \Lambda_{(i)} \arrow[r, "\langle a\rangle \mapsto a^{-2}\langle a\rangle"] \arrow[d] & \Lambda_{(i+2)} \arrow[d] \\ \mathbf{T}(\chi) \arrow[r, equal] & \mathbf{T}^{ord,U_p\equiv 1}_{par,(i)}. \end{tikzcd}$$

In this diagram the left vertical arrow gives the Λ-algebra structure on $\mathbf{T}(\chi)$ which is considered in [9], the right vertical arrow gives the Λ-algebra structure considered on $\mathbf{T}^{ord,U_p\equiv 1}_{par,(i)}$ in this note, and the upper horizontal arrow is the isomorphism $\Lambda_{(i)} \to \Lambda_{(i+2)}$ given by the formula $\langle a\rangle \mapsto a^{-2}\langle a\rangle$ for elements $a \in \Gamma$. (The reason for the shifting by two is that in [9,11] all actions of Γ are considered with regard to weight two modular forms, so that the Γ-action on weight two modular forms with trivial nebentypus is taken to be trivial, whereas we have followed [2,5] in defining this action to be given by the formula $f|\langle a\rangle := a^2 f$.)

On [9, p. 520] a local factor $\mathbf{T}(\chi)^{Eis}$ of $\mathbf{T}(\chi)$ is defined, and the identification $\mathbf{T}(\chi) = \mathbf{T}^{par,U_p\equiv 1}_{par,(i+2)}$ induces an identification $\mathbf{T}(\chi)^{Eis} = \mathbf{T}^{ord,Eis}_{par,(i+2)}$. Also note that under the isomorphism induced by $\langle a\rangle \mapsto a^{-2}\langle a\rangle$ the element $L_p(\chi\omega^2, -1-s)$ of $\Lambda_{(i)}$ is taken to the element $L_p(\omega^{i+2}, 1-s)$ of $\Lambda_{(i+2)}$.

Thus Corollary 7 is equivalent to the following result (in the notation of [9]): the element $\mathrm{Norm}_{\mathbf{T}(\chi)/\Lambda_{(i)}}(U_p - 1)$ of $\Lambda_{(i)}$ is divisble by $L_p(\chi\omega^2, -1-s)$. In [9] this norm is denoted by $D_p(\chi, s)^{Eis}$. Thus we have provided a proof of the statement "$L_p(\chi\omega^2, -1-s)$ divides $D_p(\chi, s)^{Eis}$" which occurs at the bottom of [9, p. 520]. (In [9] the further restriction $i \neq 0$ is imposed. However, this is no real restriction, since $L_p(\omega^2, -1-s)$ is a unit in $\mathbf{Z}_p[[\Gamma]]$ when $p \geq 5$, as follows from the fact that $\zeta(-1) = -1/12$.)

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