ALGEBRAIC GEOMETRY — THIRD HOMEWORK (DUE WEDNESDAY FEB 12)

Please complete all the questions.

1. (a) Consider the nodal cubic curve $XY = (Y - X)^3$ over a field k. As in the last homework set, show that the smooth points on this curve are parameterized by the lines $y = tx$ of slope $t \neq 0, \infty$ passing through the origin. Show that the "chord–tangent" law defines a group structure on the non-singular points of (the projectivization of) C, with the unique point at infinity as the idenity, and that the preceding bijection identifies this group with the multiplicative group $k^\times$.

(b) Consdider the cuspidal cubic curve $X^2 = Y^3$. Analogously to (a), show that the smooth points on this curve are parameterized by the lines $y = tx$ of slope $t \neq \infty$. Show that the "chord–tangent" law defines a group structure on the non-singular points of (the projectivization of) C, with the unique point at infinity as the idenity, and that the preceding bijection identifies this group with the additive group $(k, +)$.

2. Prove a precise statement of the form "If 9 points in $\mathbb{P}^2(k)$ are in general position, then there is a unique cubic curve passing through them, which is furthermore irreducible", i.e. replace "general position" by a precise collection of conditions (of the form "no 3 of which are colinear", or something similar), and prove your result by arguing with linear subspaces in the $\mathbb{P}^9$ of cubics.

3. Let k be an algebraically closed field not of characteristic 2, let $f(x) \in k[x]$ be a cubic in x with distinct roots in k, and consider the projective cubic curve C whose affine equation is $y^2 = f(x)$.

(a) Show that C has a unique point at infinity, which we denote O.

(b) Show that C is smooth at all of its points.

(c) Show that the point O at infinity is an inflection point.

(d) Use the chord-tangent law to make $C(k)$ into a group, taking O to be the identity. Show that the negative (with respect to the group structure) of a point $(x, y) \in C(k)$ is equal to $(x, -y)$.

(e) Show that C has 3 points of exact order 2. (Hint: consider solving the equation $-P = P$ in $C(k)$.)

4. We let k be the field $\mathbb{C}$ of complex numbers, let

$$f(x) = x^3 - x^2 - 4x + 1 \in \mathbb{C}[x],$$

and let C be as in the previous question.

(a) Explicitly compute the six intersection points of C with the conic $x^2 + y^2 = 1$.

(b) Add up these six points using the group law on C, and verify explicitly that they sum uto O.