ALGEBRAIC GEOMETRY — SEVENTH HOMEWORK (DUE FRIDAY MARCH 14)

Please complete all the questions. (All rings are commutative with 1, and all ring homomorphisms preserve 1.)

1. Let A be a ring.

(a) If I and J are two ideals of A, prove that the natural map

$$A/(I \cap J) \to A/I \times A/J$$

is an injection, and that it is surjective if and only if $I + J = A$. (Bonus question: give a nice description of the cokernel in general.)

(b) If I and J are ideals of A for which $I + J = A$, prove that $I \cap J = IJ$.

(c) If I, J_1, and J_2 are ideals of A, prove that the following are equivalent: (i) $I + J_1 = I + J_2 = A$; (ii) $I + J_1 \cap J_2 = A$; (iii) $I + J_1 J_2 = A$.

(d) If $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ are mutually distinct maximal ideals of A, prove that $\mathfrak{m}_1 \cdots \mathfrak{m}_n$ (the product of the $\mathfrak{m}_i$) equals $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ (the intersection of the $\mathfrak{m}_i$), and that the natural map

$$A/(\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n) \to A/\mathfrak{m}_1 \times \cdots \times A/\mathfrak{m}_n$$

is an isomorphism.

2. Let A be a ring, and let I and J be a pair of ideals in A such $I + J = A$ and $IJ = 0$.

(a) Prove that $I^2 = I$ and $J^2 = J$. (So our hypotheses on I and J are ideal-theoretic analogues of asking that they be orthogonal projectors on a Hilbert space.)

(b) If M is an A-module, and if we define

$$M[I] = \{m \in M \mid am = 0 \text{ for all } a \in I\},$$

and similarly

$$M[J] = \{m \in M \mid am = 0 \text{ for all } a \in I\},$$

prove that $M = M[I] \oplus M[J]$. (This is analogous to decomposing a Hilbert space using a pair of orthogonal projectors.)

3. If A is a ring and $\mathfrak{p}$ is a prime ideal of A, prove that the localization $A_{\mathfrak{p}}$ is *not* the zero ring. (This is implicit in Lemma 14.4.3 of the notes, but prove it directly and explicitly. Lemma 14.3.5 of the notes could be a useful tool.)

4. Compute the following tensor products. [Hint: describe the module being tensored via generators and relations.]

(a) $\mathbb{Z}[i] \otimes_{\mathbb{Z}[2i]} \mathbb{Z}[i]$.
(b) $\mathbb{C}[t] \otimes_{\mathbb{C}[t^2,t^3]} \mathbb{C}[t]$.

5. Let X be a quasi-projective algebraic set, and let Z be a closed subset of X. Let $\iota : Z \hookrightarrow X$ denote the inclusion. If $\varphi : Y \to Z$ is a morphism from another quasi-projective algebraic set Y, prove (a) that φ is projective if and only if the composite $\iota\varphi$ is projective; (b) that φ is finite if and only if the composite $\iota\varphi$ is finite.

6. For each of the following morphisms of algebraic sets, identify whether or not it is a finite morphism. For each question, draw a picture illustrating the geometric situation under discussion.

(a) The inclusion of $\mathbb{A}^2(\Omega) \setminus \{0\}$ into $\mathbb{A}^2(\Omega)$.

(b) The morphism from $\mathbb{A}^1(\Omega)$ to the curve in $\mathbb{A}^2$ with equation $y^2 = x^3$, given by $t \mapsto (t^2, t^3)$.

(c) The natural projection from the blow up of $\mathbb{A}^2(\Omega)$ at the origin to $\mathbb{A}^2(\Omega)$ (as discussed in exercise 3 of HW sheet 5).

(d) The morphism from $\mathbb{A}^2(\Omega)$ to the cone $X \subset \mathbb{A}^3(\Omega)$ with equation $u^2 = v^2 + w^2$, defined by $u = (x^2 + y^2)/2, v = xy, w = (x^2 - y^2)/2$.