ALGEBRAIC GEOMETRY — FOURTH HOMEWORK (DUE FRIDAY FEB 7)

1. Recall that a topological space is called *irreducible* iff it cannot be written as the disjoint union of two proper closed subsets.

(a) Prove that a topological space X is irreducible iff any two non-empty open subsets of X have non-empty intersection.

(b) Prove that if a topological space X is the disjoint union of a finite number of irreducible closed subsets $X_1, \ldots, X_n$, none of which is contained in any of the others, and X_1 is a singleton (i.e. contains a single point x), then X_1 is a connected component of X (i.e. x is an isolated point of X).

(c) Give an example of a connected topological space which is the union of two irreducible closed sets, neither of which is contained in the other.

2. Let I be an ideal in $k[x_0, \ldots, x_n]$ (with k a field). Show that I is *homogeneous* (i.e. satisfies the property that if $f \in I$ and $f = f_0 + f_1 + \ldots + f_d$ is the decomposition of f into its graded pieces, then each $f_i \in I$) if and only if I is can be generated by a collection of homogeneous polynomials.

3. Let $Z \subset \mathbb{A}^n(\Omega)$ be a Zariski closed subset. Show that the projective closure of Z in $\mathbb{P}^n(\Omega)$ coincides with the Zariski closure of Z in $\mathbb{P}^n(\Omega)$.