ALGEBRAIC GEOMETRY — THIRD HOMEWORK (DUE FRIDAY JAN 31)

There are 6 questions, plus a bonus question. The bonus question is open-ended, and is intended just to get you thinking about an interesting and non-obvious geometric phenomenon: given 8 (general) points in the plane, they determine a 9th point.

1. From the previous homework, you know that the space of cubic projective curves is a copy of $\mathbb{P}^9$.

(a) Show that the space of cubics passing through some fixed point P is a hyperplane in $\mathbb{P}^9$.

(b) Show that the space of cubics passing through two distinct points P and Q is a codimension two linear subspace in $\mathbb{P}^9$.

(c) Continue this analysis for larger numbers of points. In particular, show that the space of cubics passing through 8 distinct points in "general position" is a line in $\mathbb{P}^9$, and give a precise meaning to "general position".

Bonus question: *Preliminary discussion.* Bézout's Theorem (which we will prove later in the course) shows that two plane cubics in general position with regard to one another, and hence any two members of the pencil that they span, intersect in 9 points. The previous question shows that in fact this pencil is already determined by 8 of the 9 intersection points.

From this you can deduce two things:

(a) 9 points in general position *cannot* arise as the intersection of two cubic projective curves.

(b) 8 points in general position determine a 9th point (namely the 9th common point of intersection of the pencil of cubics determined by the 8 given points).

We can regard this process as defining a kind of 8-ary operation

$$(\mathbb{P}^2)^8 \dashrightarrow \mathbb{P}^2.$$

(We use a broken arrow because this operation is not really defined on *all* 8-tuples of points in $\mathbb{P}^2$, but only on those in general position. It is an example of what is called a *rational map* in algebraic geometry.)

Actual bonus question: Can you say anything intelligent about this map? Is is possible to compute a formula for it? Can you say anything about what this formula would look like (e.g. its degree)?

2. If k is an infinite perfect field (e.g. any field of char. zero, or any algebraically closed field) and if $f \in k[x, y]$ is non-constant irreducible, cutting out the affine plane curve C, then show that $C(k)$ contains a smooth point. [Hint: To begin with, think of $f(x, y)$ as an irreducible element $k(y)[x]$, and consider its derivative f_x with respect to x. Use known facts about irreducible polynomials in one variable and their derivatives to make some deductions.]

3. (a) Give an example of a non-perfect field k and an irreducible polynomial $f \in k[x, y]$ so that $C(k)$ contains no smooth point. (Here C is the curve with equation f.)

(b) Show, for any f as in (a), that $C(l)$ contains no smooth point for every field extension l of k.

4. If k is an algebraically closed field and $f \in k[x, y]$ is non-constant irreducible, show that you can make an affine linear change of coordinates so that f can be written in the form $y^d + f_{d-1}(x)y^{d-1} + \cdots + f_0$, *and* such that f_y is non-zero. Conclude that for all but finitely many choices of $x \in k$, the d solutions to $f(x, y) = 0$ (for this fixed value of x) are *distinct*. [Hint: Use the argument from class about projecting from a point at infinity not lying on the projective closure of C, but be a little bit more careful so that you control the y-derivative as well.]

5. If k is an algebraically closed field and $f \in k[x, y]$ is non-constant irreducible, cutting out the curve C, and if $g \in k[x, y]$ is non-constant, cutting out the curve D, and if $C(k) \subseteq D(k)$, then prove that f divides g in $k[x, y]$. [Hint: Change coordinates as in the previous question, and then use the division algorithm to write $g = qf + r$ in $k[x, y]$, where the degree in y of r is less than the degree in y of f. Now deduce that r must actually vanish.]

6. If k is an algebraically closed field, and $f, g \in k[x, y]$ are non-constant polynomials cutting out curves C and D, show that $C(k) \subseteq D(k)$ iff every irreducible factor of f is an irreducible factor of g.