ALGEBRAIC GEOMETRY — FIRST HOMEWORK (DUE FRIDAY JAN 17)

Please complete all the questions. For each question (even question 2, if you can see how) please provide examples/graphs/pictures illustrating the ideas behind the question and your answer.

1. Suppose that the field k is algebraically closed. Prove that an affine conic (i.e. a degree 2 curve in the affine plane) is smooth at every point if and only if the conic is either irreducible, or the union of two distinct parallel lines.

2. Prove that if $f \in k[x_1, \ldots, x_n]$ is homogeneous of degree n, then $\sum_{i=1}^n x_i \dfrac{\partial f}{\partial x_i} = nf$.

3. Suppose that f and g are non-zero homogeneous polynomials in two variables, and that f and g are of different degrees. Show that if the curve $f + g = 0$ has a singular point other than the point $(0, 0)$, then it is reducible (i.e. $f + g$ is a reducible polynomial). [Hint: Remember the formula from the previous exercise.]

4. Suppose that f is an irreducible cubic polynomial in two variables. Prove that the affine curve $f = 0$ has at most one singular point. [Hint: Use the previous exercise.]

5. Let $f \in \mathbb{R}[x, y]$ be a non-constant polynomial, and suppose that $(x_0, y_0) \in \mathbb{R}^2$ lies on the curve C cut out by $f = 0$. Show that (x_0, y_0) is an *isolated point* in the set of real points of C if and only if the tangent cone to f at (x_0, y_0) has no real linear factors.

6. Let $f \in \mathbb{C}[x, y]$ be a non-constant polynomial, cutting out the plane curve C. Prove that the set of complex points of C is *not* a compact topological space.

7. Let $f \in \mathbb{R}[x, y]$ be a non-constant polynomial, cutting out the plane curve C. Prove that if C has no real points at infinity, then the set of real points of C is a compact topological space. Is the converse true?

8. Let $f \in \mathbb{R}[x, y]$ be a non-constant polynomial, cutting out the plane curve C, and suppose that P is a real point at infinity of C, at which the intersection multiplicity of C and the line at infinity equals 1.

(a) Explain why C is smooth at P.

(b) Let ℓ be the tangent line to P at C. Explain why ℓ is *not* the line at infinity.

(c) Prove that the set of real points of C has as an asymptote along the line ℓ.