Algebra 1 : sample questions for $\mathfrak{sl}_3$

1. Work out the details of the discussion at the top of p. 180, in chapter 13, and show that $\mathrm{Sym}^n V$ (where V is the standard 3-dimensional representation of $\mathfrak{sl}_3$) is irreducible, and similarly with $\mathrm{Sym}^n V^*$. As part of this, you should determine all the weights appearing in these reps., and show that each weight space is one-dimensional (or, as people say, that each weight appears with multiplicity one).

2. Read the discussion at the end of §13.1 to see that the kernel of the natural map $\mathrm{Sym}^2 V \otimes V^* \to V$ is the irrep. of highest weight $2L_1 - L_3$. (This is the irrep. called $\Gamma_{2,1}$ in the text.)

3. Let W denote the adjoint rep. and let V be the standard rep. Decompose $V \otimes W$ into irreps. [Hint: Remember that $W \oplus \mathrm{triv} = V \otimes V^*$, so that $(V \otimes W) \oplus V = V \otimes V^* \otimes V = V \otimes V \otimes V^*$, and now use the fact that $V \otimes V = \mathrm{Sym}^2 V \oplus \wedge^2 V$, and the fact that $\wedge^2 V = V^*$, along with the previous exercise, and the known decomposition of $V^* \otimes V^*$.]

4. This is just a remark: Claim 13.4 of the text describes the general decomposition of tensor products of symmetric powers of V and V^*. I won't expect you to remember this in the exam, but it is a useful fact.

5. Reading §13.5 and doing the exercises there will be more than enough practice. Note that the example at the beginning of that section (decomposing $V \otimes \Gamma_{2,1}$) uses Claim 13.4, but the second example (decomposing $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$) does not, and not all of the exercises do; e.g. exercise 13.9 is linear algebra related to the second example, and for exercise 13.10, you can just compute the weights of $\wedge^2(\mathrm{Sym}^2 V)$ directly, and compare them with the weights of $\Gamma_{2,1}$, which you know from ex. 2 above. Ex. 3.11 you can also do just by computing weights, and comparing them with highest weight reps. that you know.