Algebra 1 : First homework — due Monday, October 3

Do the following exercises from Fulton and Harris:

1.2, 1.11, 1.12, 1.13, 2.2, 2.3, 2.4, 2.5, 2.22, 2.25, 2.27

Also do the following exercises:

1. Let A be a ring (e.g. a group ring). (In this course, rings always are associative with 1.) Recall that we say that a submodule M' of an A-module M is a direct summand of M if there exists another A-submodule N' of M such that $M = M' \oplus M''$.

(a) Prove that M' is a direct summand of M if and only if there exists an A-linear projection from M to M', i.e. a morphism $\pi : M \to M'$ such that $\pi \circ \pi = \pi$.

(b) Prove that if $M = M' \oplus M''$, then there is an isomorphism $M/M' \cong M''$.

2. If M is a module over a ring A, and $\pi : M \to M''$ is a surjection of A-modules, prove that the kernel of π is a direct summand of M if and only if there exists a morphism of A-modules $\sigma : M'' \to M$ such that $\pi \circ \sigma = \mathrm{id}_{M''}$. (If these equivalent conditions hold, then we say that the surjection π *splits*, or *is split*, and we say that σ is a section to, or of, π.)

3. If M is a module over a ring A, we say that M is semisimple if every submodule of M is a direct summand of M.

(a) Show that M is semisimple if and only if every surjection of A-modules $M \to M''$ splits.

(b) Show that if M is semisimple, then any submodule or quotient of M is also semisimple (and hence that any *subquotient — i.e. quotient of a submodule* of M is semisimple).

4. Suppose that A is a k-algebra, for some field k, and that M is an A-module that is finite-dimensional as a k-vector space. Prove that M is semisimple if and only if M is isomorphic to a direct sum of finitely many simple A-modules.

Remark: It is true in general, i.e. for any module M over any ring A, that M is semisimple if only if M is a direct sum of (possibly infinitely many) simple A-modules.