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A construction of covers of arithmetic schemes

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Abstract

Let X be a regular arithmetic scheme, i.e. a regular integral separated scheme flat and of finite type over $\operatorname{Spec}\mathbb{Z}$. Assume that for all closed irreducible subschemes $C \subseteq X$ of dimension 1 with normalisation $\tilde{C}$ there are given open normal subgroups N_C of $\pi_1(\tilde{C})$, which fulfil the following compatibility condition: For all $\tilde{x} \in \tilde{C}_1 \times_X \tilde{C}_2$ the pre-images of N_{C_1} and N_{C_2} in $\pi_1(\tilde{x})$ coincide. If the indices of the N_C are bounded, then these data uniquely determine an open normal subgroup of $\pi_1(X)$, whose pre-image in $\pi_1(\tilde{C})$ is N_C for all C.

1. Introduction

This paper has been written in the attempt to provide a tool for higher-dimensional, non-abelian class field theory. It is a generalisation of results of [HW] for arithmetic surfaces. Its results support the idea that monodromy phenomena for arithmetic schemes are supported on the "1-skeleton" of the arithmetic scheme, i.e. the collection of curves on X and their incidence relations. This opens a possible pathway from 1-dimensional class field theory to a higher-dimensional generalisation.

Let X be an arithmetic scheme or a variety over a field K. We define a *covering problem* for X as the following data: For all irreducible curves $C \subseteq X$ with normalisation $\tilde{C}$ there is given an open normal subgroup $N_C \trianglelefteq \pi_1(\tilde{C})$. For all closed points $x \in X$ there is given an open normal subgroup $N_x \trianglelefteq \pi_1(x)$, such that for all C and x and all $\tilde{x} \in \tilde{C} \times_X x$ the pre-images of N_C and

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N_x in $\pi_1(\tilde{x})$ coincide (note: Whereas the map $\pi_1(\tilde{x}) \to \pi_1(\tilde{C})$ is only defined up to conjugation, the pre-image of a normal subgroup is well defined).

A *solution* of a covering problem is an open normal subgroup $N \trianglelefteq \pi_1(X)$ such that the N_C and the N_x are the pre-images of N. A covering problem is called *bounded*, if the indices of all N_C and N_x are bounded. A covering problem is called *abelian*, if all $\pi_1(\tilde{C})/N_C$ are abelian. Here is the most important result of the paper (see Theorems 25 and 26 for more general statements).

Theorem. *Let X be a regular arithmetic scheme. Then*

(1) *A covering problem for X has a solution iff it is bounded.*
(2) *If the covering problem is abelian then it is bounded.*
(3) *If a solution exists then it is unique.*

We consider also varieties over certain fields K. We always assume K is a perfect field and the p-Sylow subgroups of the absolute Galois group G_K are infinite for all primes p. We show the above theorem holds for a regular variety X over K under various additional assumptions on K, e.g. finite, Hilbertian or PAC, provided the covering problem is tame (i.e. the N_C define coverings which are tame, see Definition 16). If X is proper over a curve, then the assumption on tameness may be dropped. In case $\operatorname{char} K = 0$ the assumption on tameness is void.

The paper develops further the ideas from [HW]. The first change is in the very definition of covering problem. Here we use the normalisations of the curves on X instead of the possibly singular curves themselves. This makes it possible to get solutions in the sense of Definition 2 instead of merely weak solutions. Then the higher-dimensional case is included as opposed to arithmetic surfaces only. For this we use induction on the dimension. Now we get results on varieties over certain fields, too. In the case of abelian covering problems, now the boundedness condition on the indices is removed.

We use the Hilbert Irreducibility Theorem to prove a generalisation of the Approximation Lemma of Raskind [Ra, Lemma 6.21], which is needed for varieties over arbitrary base fields. [Wi2] contributes a Saito-type completely split result for regular subschemes of Henselian local rings extending Saito's original result which is only applicable in dimension 2. Grothendieck's theory of the specialisation of the fundamental group is used in the proof of Proposition 17. We need enough inert points, a result from [Wi1], for uniqueness and working only with weak solutions. The full strength of Čebotarev's density theorem is used in the proof of Proposition 28.

Notation. (a) We fix a perfect field K whose absolute Galois group G_K has infinite p-Sylow subgroups for all primes p.

(b) An *arithmetic scheme* is an integral, separated scheme, which is flat and of finite type over $\operatorname{Spec}\mathbb{Z}$. A *variety* is an integral, separated scheme, which is of finite type over K. If x is a point of X, then $\kappa(x)$ denotes the residue field.

(c) A *curve on X* is an integral closed subscheme of dimension 1.

(d) A *cover $Y \to X$* will always be the normalisation of a normal, integral scheme in a finite extension of its function field.

2. Covering problems and solutions

Remark 1. For a morphism $Y \to X$, the map $\pi_1(Y, y_0) \to \pi_1(X, x_0)$ is only defined after the choice of compatible base points. If x_0 is replaced by another base point x_0', there is an isomorphism $\pi_1(X, x_0) \cong \pi_1(X, x_0')$ which is determined up to an inner automorphism. This shows that the pre-image N_Y in $\pi_1(Y)$ of a normal subgroup $N_X \trianglelefteq \pi_1(X)$ is independent of the choice of base points. As only this operation will be performed on fundamental groups throughout the paper, base points for étale fundamental groups will be suppressed.

Definition 2. A *covering problem on* X consists of the following data. For each irreducible curve $C \subseteq X$ with normalisation $\tilde{C}$ there is given an open normal subgroup $N_C \trianglelefteq \pi_1(\tilde{C})$ and for each closed point $x \in X$ an open normal subgroup $N_x \trianglelefteq \pi_1(x)$ with the following compatibilities: For all $\tilde{x} \in \tilde{C} \times_X x$ the pre-images of N_C and N_x in $\pi_1(\tilde{x})$ coincide.

A *solution* of a covering problem is an open normal subgroup $N \trianglelefteq \pi_1(X)$ such that N_C is the pre-image of N in $\pi_1(\tilde{C})$ for all $C \subseteq X$ and N_x is the pre-image of N in $\pi_1(x)$ for all $x \in X$. A *weak solution* of a covering problem is an open normal subgroup $N \trianglelefteq \pi_1(X)$ such that N_x is the pre-image of N in $\pi_1(x)$ for all $x \in X$ (there is nothing said about the N_C).

A covering problem is called *trivial* if $N = \pi_1(X)$ is a solution, i.e. $N_C = \pi_1(\tilde{C})$ for all irreducible curves $C \subseteq X$ and $N_x = \pi_1(x)$ for all points x. It is called *weakly trivial* if $N = \pi_1(X)$ is a weak solution, i.e. $N_x = \pi_1(x)$ for all $x \in X$.

A covering problem is *bounded* if the N_C have bounded indices. In the case of a variety it is called *tame* if the N_C define covers of $\tilde{C}$ which are tamely ramified outside $\tilde{C}$ (see Definition 16). A covering problem is *abelian* if the $\pi_1(\tilde{C})/N_C$ are abelian.

Remark 3. Every open normal subgroup $N \trianglelefteq \pi_1(X)$ defines a covering problem P_N on X: Let the N_C and N_x be the pre-images of N. P_N has N as a solution.

Remark 4. The strategy of proof for the main result, the existence of solutions, consists of three parts:

(1) Proposition 17 provides an *upper bound* for the solution: Birationally étale locally the covering problem becomes trivial.
(2) Proposition 24 yields a weak solution in the situation that is provided by (1).
(3) Weak solutions are solutions (Lemma 14).

Remark 5. On a regular scheme X of dimension $\geqslant 1$ a covering problem can be presented in a slightly different manner. Assume for all irreducible curves $C \subseteq X$ there are given open normal subgroups $N_C \trianglelefteq \pi_1(\tilde{C})$, which fulfil the following compatibility: For any two curves $C_1, C_2 \subseteq X$ and any $\tilde{x} \in \tilde{C}_1 \times_X \tilde{C}_2$ the pre-images of N_{C_1} and N_{C_2} in $\pi_1(\tilde{x})$ agree.

To these data there is associated a covering problem in the above sense. Let $x \in X$ be a closed point and C a curve which contains x as a regular point. This exists as X is regular. Define $N_x \trianglelefteq \pi_1(x)$ as the pre-image of N_C. The normal subgroup N_x does not depend on the choice of C: Let C_1, C_2 be two curves which contain x as a regular point. There is a unique $\tilde{x} \in \tilde{C}_1 \times_X \tilde{C}_2$ above x. The point $\tilde{x}$ can be identified with x. The pre-images of N_{C_1} and N_{C_2} in $\pi_1(x)$ agree because of the assumed compatibility.

To show that the normal subgroups N_C and N_x define a covering problem in the above sense, let $\tilde{x} \in \tilde{C} \times_X x$. Choose a curve C' which contains x as a regular point. $\tilde{x}$ can be viewed as a point of $\tilde{C} \times_X \tilde{C}'$ above x. The pre-images of N_C and $N_{C'}$ in $\pi_1(\tilde{x})$ agree. But the pre-image of $N_{C'}$ equals the pre-image of N_x because of the factorisation $\tilde{x} \to x \to C'$. This shows that the pre-images of N_C and N_x in $\pi_1(\tilde{x})$ agree.

Remark 6 *(Functorial behaviour of covering problems).* Let $X' \to X$ be a morphism of arithmetic schemes or varieties. Let P be a covering problem on X. There is a covering problem on X' which is induced by P, it is called the *induced covering problem*: Let $x' \in X'$ be closed. Let x be the image of x' in X. Define $N_{x'} \trianglelefteq \pi_1(x')$ as the pre-image of N_x. Let $C' \subseteq X'$ be an irreducible curve. The image of C' on X either is an irreducible curve C or a closed point x. Define $N_{C'} \trianglelefteq \pi_1(\tilde{C}')$ as the pre-image of N_C or of N_x, respectively. We have to show that these open normal subgroups define a covering problem on X'. Let $\tilde{x}' \in \tilde{C}' \times_{X'} x'$. The following two commutative diagrams transform into commutative diagrams of fundamental groups, if compatible base points are chosen, i.e. the images of a geometric point over $\tilde{x}'$. They imply that the pre-images of $N_{C'}$ and $N_{x'}$ in $\pi_1(\tilde{x}')$ are equal:

$$\begin{array}{ccccc} x' & \longleftarrow & \tilde{x}' & \longrightarrow & \tilde{C}' \\ \downarrow & & \downarrow & & \downarrow \\ x & \longleftarrow & \tilde{x} & \longrightarrow & \tilde{C} \end{array} \qquad \begin{array}{ccccc} x' & \longleftarrow & \tilde{x}' & \longrightarrow & \tilde{C}' \\ \downarrow & & \downarrow & & \downarrow \\ x & \longleftarrow & x & \longrightarrow & x. \end{array} \tag{1}$$

Here $\tilde{x}$ is the image of $\tilde{x}'$ in $\tilde{C} \times_X x$.

Definition 7. Let X be a normal scheme. Let $\pi_1^{\mathrm{cs}}(X)$ denote the *completely split fundamental group* of X, i.e. that quotient of the fundamental group, which classifies finite, étale covers, in which all closed points split completely. X is said to *have enough inert points*, if for every morphism $Y \to X$ of finite type, where Y is normal, we have $\pi_1^{\mathrm{cs}}(Y) = 1$.

Remark 8. To establish this property for some X it is sufficient to look at the maximal abelian quotient of $\pi_1^{\mathrm{cs}}(Y)$, for all Y as in the definition. If X is the spectrum of a field it is sufficient to look at curves over X. This can be shown by using the Approximation Lemma 20.

Lemma 9. *Finite fields, Hilbertian fields and PAC fields, which have infinite p-Sylow subgroups in their absolute Galois group for all primes p, have enough inert points.* $\operatorname{Spec}\mathbb{Z}$ *has enough inert points.*

Proof. The first sentence is [Wi1, Remark 3]. The second statement follows from the Čebotarev density theorem (together with the Approximation Lemma 20), or its successors ([La, p. 393], [Ra, Lemma 1.7]). □

Lemma 10 *(Splitting Principle). Let $x \in X$ be a point and $N_1, N_2 \trianglelefteq \pi_1(X)$ two normal subgroups such that the pre-images of N_1 and N_2 in $\pi_1(x)$ coincide. Let $Y_1, Y_2 \to X$ be the two étale covers belonging to N_1, N_2. In the cover $Y_1 \times_X Y_2 \to Y_1$ all the points $y \in Y_1$ over x split completely.*

Proof. $\pi_1(y)$ can be viewed as the pre-image of N_1 in $\pi_1(x)$. Let $z \in Y_1 \times_X Y_2$ be a point over y. $\pi_1(z)$ can be viewed as the pre-image of $N_1 \cap N_2$ in $\pi_1(x)$. Therefore $\pi_1(y) = \pi_1(z)$ and y splits completely as stated. □

Lemma 11. *Let P_1, P_2 be two covering problems on X. The intersection of the respective normal subgroups defines a covering problem $P_1 \cap P_2$. This defines a partial ordering on the set of covering problems on X, namely $P_1 \leqslant P_2 :\Leftrightarrow P_1 = P_1 \cap P_2$.*

Now let X be normal and let N_1, N_2 be solutions for P_1, P_2, respectively. If $P_1 \leqslant P_2$, then $N_1 \subseteq N_2$. In particular every covering problem has at most one solution.

Proof. $N_1 \cap N_2$ is a solution of $P_1 \cap P_2$. As $P_1 = P_1 \cap P_2$ this implies that N_1 and $N_1 \cap N_2$ are two solutions for P_1. Let $Y_{1,2} \to Y_1 \to X$ the associated étale coverings. The Approximation Lemma 20 gives an irreducible curve $C \subseteq X$ whose generic point is inert in $Y_{1,2} \to X$. Then it is inert in $Y_1 \to X$. We have $\pi_1(\tilde{C})/N_C \cong \pi_1(X)/(N_1 \cap N_2) \cong \pi_1(X)/N_1$. This shows $N_1 \cap N_2 = N_1$, i.e. $N_1 \leqslant N_2$. □

Remark 12. If one is only interested in X which have enough inert points, the fact that $Y_{1,2} \to Y_1$ is completely split directly implies $N_1 \cap N_2 = N_1$. Then it is not necessary to use Lemma 20.

Remark 13. Let N be the solution of a covering problem on a normal scheme X. The Approximation Lemma 20 gives a curve C on X such that $\pi_1(X)/N \cong \pi_1(\tilde{C})/N_C$. This shows that every statement which is true for all $\pi_1(\tilde{C})/N_C$ holds for $\pi_1(X)/N$ as well.

Lemma 14. *Assume that X is a normal connected scheme which has enough inert points. Let P be a covering problem on X. Then every weak solution of P is a solution.*

Proof. *Uniqueness of weak solution if* $\dim(X) = 1$: Let P be a covering problem for X and N, $N' \trianglelefteq \pi_1(X)$ two weak solutions. These define étale covers $Y, Y' \to X$. In the cover $Y \times_X Y' \to Y$ all closed points split completely (splitting principle, Lemma 10). Hence there is an X-morphism $Y \to Y'$. This implies $N \subseteq N'$. By symmetry $N = N'$.

Every weak solution is a solution: Let N be a weak solution of P. For any irreducible curve $C \subseteq X$ let N'_C be the pre-image of N in $\pi_1(\tilde{C})$. For all $\tilde{x} \in \tilde{C}$ the pre-image of N'_C in $\pi_1(\tilde{x})$ equals the pre-image of N_x, since N is a weak solution. But the pre-image of N_C in $\pi_1(\tilde{x})$ equals the pre-image of N_x as well, by definition of covering problem. As the normal subgroups N_C and N'_C have the same pre-image in $\pi_1(\tilde{x})$, the first part of this proof for the induced covering problem on $\tilde{C}$ implies that $N_C = N'_C$, which means that N is a solution of P. □

3. Triviality of induced covering problems

Lemma 15. *Let X be an arithmetic scheme or a variety over K. Let* $\dim X \geqslant 1$, *in case of an arithmetic scheme let* $\dim X \geqslant 2$. *Then X allows étale locally the structure of a fibration $X \subseteq \bar{X} \to W$ into smooth projective curves with boundary $\bar{X} \setminus X$ a disjoint union of sections $s_i : W \to \bar{X}$ and a section $s : W \to X$. The scheme W can be chosen to be regular. In the case that X is proper over a curve I may choose $X = \bar{X}$ to be proper over W.*

Proof. Fibration élementaire [SGA4, XI, 3.3] shows that after replacing X by an open subscheme there is a fibration $X \subseteq \bar{X} \to W$ into smooth projective curves with boundary $\bar{X} \setminus X$ finite étale over W. Replace W by a finite étale cover to achieve the requirements.

Now assume $X \to C$ is a proper morphism to a curve. Let $X' \to \operatorname{Spec} \kappa(C)$ be the generic fiber. The first part of the lemma gives a rational map $X' \to W'$ over $\operatorname{Spec} \kappa(C)$ which satisfies the required conditions. Since X' is proper over $\kappa(C)$ and the relative dimension is 1, the rational map extends to a proper morphism on an open subset of X' and W'. This can be extended to a proper morphism $X \to W$ over an open subscheme of C, such that the required conditions are fulfilled. □

Definition 16. Let C be a geometric curve. An étale cover $D \to C$ is called *tamely ramified*, if its extension to the smooth completions $\bar{D} \to \bar{C}$ is tamely ramified, i.e. the ramification indices are not divisible by the characteristic of K (recall K is assumed to be perfect).

Proposition 17. *Let X be an arithmetic scheme or a variety over K. Let P be a covering problem on X. Assume either P is bounded or all $\pi_1(\tilde{C})/N_C$ are abelian and have bounded exponent. If X is a variety assume that P is tame, unless X is proper over a curve. Then the induced problem on Y becomes weakly trivial for a suitable étale morphism $Y \to X$.*

Proof. We prove by induction on the dimension of X. If $\dim(X) \leqslant 1$ the claim is obvious. So assume $\dim(X) \geqslant 2$.

We may assume X has the form of an elementary fibration as in Lemma 15.

Because of the induction hypothesis and after replacing W by an ètale cover we may assume that the covering problem which is induced on $s : W \subseteq X$ is weakly trivial. If X is proper over a curve, then according to the above choices $X \to W$ is proper. As the assumption on tame ramification will only be applied to the fibres of $X \to W$, this is satisfied in this case as well.

Let $E = \kappa(X)$ and $F = \kappa(W)$. Let F_{alg} denote the algebraic closure of F. Let M be the common bound on the indices of the N_C, or the common bound on the exponents, respectively. We will show that there is a finite Galois extension $E'|E$, such that $E'F_{\mathrm{alg}}|EF_{\mathrm{alg}}$ is the compositum L of all extensions of EF_{alg} of degree $\leqslant M$, which are tamely ramified over $X \times_W F$, respectively of all such abelian extensions of exponent $\leqslant M$. For this it is enough to show that $\pi_1^{\mathrm{tame}}(X \times_W F_{\mathrm{alg}})$ is finitely generated. Then it has only a finite number of quotients of degree $\leqslant M$ (respectively of abelian quotients of exponent $\leqslant M$) and the intersection of the respective kernels has finite index. But according to Grothendieck [SGA1, XIII, Corollaire 2.12] the group $\pi_1^{\mathrm{tame}}(X \times_W \kappa(W)_{\mathrm{alg}})$ is finitely generated.

Let $w \in W$ be closed. The geometric fundamental group for the fibre C of $X \to W$ over w is

$$\pi_1^{\mathrm{tame}}\big(C \times_{\kappa(w)} \kappa(w)_{\mathrm{alg}}\big), \quad \text{a quotient of } \pi_1^{\mathrm{tame}}\big(X \times_W \kappa(W)_{\mathrm{alg}}\big) \tag{2}$$

according to Grothendieck's theory of the specialisation of the fundamental group [SGA1, Exposé X] and its generalisation to the tame case [SGA1, Exposé XIII].

We define X' as the normalisation of X in $E'|E$. After finite extension of F and shrinking W, $X' \to X$ is étale. Consider the commutative diagram

$$\begin{array}{ccc} \pi_1^{\mathrm{tame}}(C \times_{\kappa(w)} \kappa(w)_{\mathrm{alg}}) & \twoheadleftarrow & \pi_1^{\mathrm{tame}}(X \times_W \kappa(W)_{\mathrm{alg}}) \\ & \searrow & \downarrow \\ & & \mathrm{Gal}(E'|E). \end{array} \tag{3}$$

The kernel of $\pi_1^{\mathrm{tame}}(X \times_W \kappa(W)_{\mathrm{alg}}) \to \mathrm{Gal}(E'|E)$ is the intersection of all normal subgroups of index $\leqslant M$ (respectively with abelian quotient of exponent $\leqslant M$). Hence the kernel of $\pi_1(C \times_{\kappa(w)} \kappa(w)_{\mathrm{alg}}) \to \mathrm{Gal}(E'|E)$ is contained in the intersection of all normal subgroups of index $\leqslant M$ (respectively with abelian quotient of exponent $\leqslant M$).

For all $w \in W$ with fibre C the normal subgroup N_C has index less or equal to M (respectively abelian quotient of bounded exponent). Hence N_C contains the kernel of $\pi_1^{\mathrm{tame}}(C \times_{\kappa(w)}$

$\kappa(w)_{\mathrm{alg}}) \to \mathrm{Gal}(E'|E)$. After replacing X by X' this kernel becomes $\pi_1^{\mathrm{tame}}(C \times_{\kappa(w)} \kappa(w)_{\mathrm{alg}})$. This shows that we can assume that N_C describes an extension of the field of constants.

As the covering problem is weakly trivial on $W \subseteq X$, for all closed points $w \in W$ we have $N_w = \pi_1(w)$ and w is rational on its special fibre C. This shows that the extension of the base field $\kappa(w)$ of C given by N_C must be trivial, since it induces a trivial extension of $\kappa(w)$. We get $\pi_1(C) = N_C$ for all fibres. This implies $\pi_1(x) = N_x$ for all closed points $x \in X$, hence the covering problem is weakly trivial. □

4. Extension of weak solutions

Lemma 18. *Let A be a regular, Henselian, excellent, Noetherian local domain of dimension $d \geqslant 1$. Let $B|A$ be a nontrivial, local, normal extension. Then the set of prime ideals $\mathfrak{p}$ of A of height $d-1$, which do not split completely in $B|A$ is dense in* $\mathrm{Spec}(A)$.

Proof. As A is regular [Wi2, Proposition 5] shows that it is enough to find one prime of height $d-1$ which does not split completely. Assume there is none such. Then the extension $B|A$ is unramified outside the maximal ideal of A. The purity of the branch locus [SGA1, X.3.1] states that the branch locus has pure codimension 1. As the maximal ideal has codimension at least 2, the extension $B|A$ must be étale. Let $\mathfrak{p}$ be a prime ideal of height $d-1$, whose closure Z in $\mathrm{Spec}(A)$ is regular. The induced cover of Z is étale, hence regular. But it is connected, as B is local. Thus it is irreducible and $\mathfrak{p}$ is inert in $B|A$. This is a contradiction to the assumption. □

Proposition 19. *Let $X' \subseteq X$ be an open inclusion of regular arithmetic schemes or regular varieties. Let P be a covering problem on X. If the induced covering problem P' on X' has a weak solution N', then there is a unique weak solution N of P on X which induces the given weak solution N'.*

Proof. First assume that X is a (regular) curve C. The weak solution N' of the induced covering problem on X' defines an étale cover $Y' \to X'$. Let $Y \to X$ be the normalisation of X in the function field of Y', which is a (ramified) cover of regular curves. The normal subgroup $N_C \lhd \pi_1(X)$ defines an étale cover $W \to X$. That N' is a weak solution of the induced problem means that the pre-images of N' and N_C in $\pi_1(x)$ are equal for all $x \in X'$. In the cover $Y \times_X W \to W$ all the closed points over X' split completely (splitting principle, Lemma 10). According to [Wi1, Theorem 1] all closed points of W split completely, whence $Y \to X$ is étale and defines an open normal subgroup $N \trianglelefteq \pi_1(X)$. Furthermore this shows that the pre-image of N in $\pi_1(x)$ for all $x \in X \setminus X'$ contains the pre-image N_x of N_C. In the cover $Y \times_X W \to Y$ all the closed points over X' split completely (splitting principle, Lemma 10). [Wi1, Theorem 1] shows that all closed points of Y split completely. Therefore N_x equals the pre-image of N in $\pi_1(x)$ for all $x \in X \setminus X'$. This shows that N is a weak solution, which induces the given solution N' on X'.

Now we prove the general case. The weak solution $N' \trianglelefteq \pi_1(X')$ of P' on X' defines an étale cover $Y' \to X'$. Let Y be the normalisation of X in the function field of Y'. We have to prove $Y \to X$ is étale and defines a weak solution of P. Let $x \in X \setminus X'$ and b a branch of an irreducible curve $C \subseteq X$ in the point x, such that C meets X'. The branch b defines a point $\tilde{x} \in \tilde{C}$ over x. Let $\kappa(b)$ denote the henselisation of $\kappa(C)$ in $\tilde{x} \in \tilde{C}$. The field $\kappa(b)$ is a discretely valued Henselian field with residue class field $\kappa(\tilde{x})$. This defines a homomorphism $G_{\kappa(b)} \to \pi_1(\tilde{x}) \to \pi_1(x)$. There is a morphism $\mathrm{Spec}(\kappa(b)) \to X'$. This defines a homomorphism $G_{\kappa(b)} \to \pi_1(X')$.

Claim. *The pre-image of $N' \trianglelefteq \pi_1(X')$ in $G_{\kappa(b)}$ equals the pre-image of $N_x \trianglelefteq \pi_1(x)$.*

Let $\tilde{C}' := \tilde{C} \times_X X'$. There are commutative diagrams

$$\begin{array}{ccccc} \operatorname{Spec}(\kappa(b)) & \longrightarrow & \tilde{C}' & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}(\mathcal{O}^{\mathrm{hen}}_{\tilde{C},\tilde{x}}) & \longrightarrow & \tilde{C} & \longrightarrow & X \end{array} \qquad \begin{array}{ccccc} G_{\kappa(b)} & \longrightarrow & \pi_1(\tilde{C}') & \longrightarrow & \pi_1(X') \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1(\tilde{x}) & \longrightarrow & \pi_1(\tilde{C}) & \longrightarrow & \pi_1(X). \end{array} \tag{4}$$

The pre-image of N' in $\pi_1(\tilde{C}')$ defines a weak solution of P restricted to $\tilde{C}'$. The first part of the proof for the curve $\tilde{C}$ implies that there is a weak solution for P restricted to $\tilde{C}$ which we call $\tilde{N}_C \trianglelefteq \pi_1(\tilde{C})$ extending the weak solution on $\tilde{C}'$: The pre-image of $\tilde{N}_C$ in $\pi_1(\tilde{C}')$ equals the pre-image of N'. Furthermore the pre-image of N_x in $\pi_1(\tilde{x})$ equals the pre-image of $\tilde{N}_C$ by definition of induced covering problem. Now the commutativity of the diagram implies the claim.

The prime ideals of height $d-1$ of the Henselian local ring $\mathcal{O}^{\mathrm{hen}}_{X,x}$ correspond to the branches b of curves C in x. The residue field of b is the Henselian field $\kappa(b)$ defined above. Let D be a local extension of $\mathcal{O}^{\mathrm{hen}}_{X,x}$ defined by $Y \to X$, namely the localisation of $\mathcal{O}^{\mathrm{hen}}_{X,x} \times_X Y$ at a maximal ideal. The prime ideals b of $\mathcal{O}^{\mathrm{hen}}_{X,x}$ which belong to branches of irreducible curves which meet X' are those contained in the open subset $\operatorname{Spec}(\mathcal{O}^{\mathrm{hen}}_{X,x}) \times_X X'$. For those b the extension of $\kappa(b)$ defined by $D|\mathcal{O}^{\mathrm{hen}}_{X,x}$ equals the extension given by $N_x \trianglelefteq \pi_1(x)$ as was proved above. Let $A|\mathcal{O}^{\mathrm{hen}}_{X,x}$ be the étale cover defined by the separable extension of its residue field $\kappa(x)$ given by N_x. Let $B := A \otimes_{\mathcal{O}^{\mathrm{hen}}_{X,x}} D$. This is a locally normal ring. In $B|A$ all primes of height $d-1$ contained in a nonempty open subset split completely (splitting principle, Lemma 10). Lemma 18 implies that B is a direct sum of copies of A, hence $D \subseteq A$ is étale over $\mathcal{O}^{\mathrm{hen}}_{X,x}$. But in $B|D$ all the prime ideals of height $d-1$ contained in a nonempty open subset split completely (splitting principle, Lemma 10). As above this implies that B is a direct sum of copies of D, hence $A = D$.

This shows that $Y \to X$ is étale and defines a normal subgroup $N \trianglelefteq \pi_1(X)$ whose pre-image in $\pi_1(x)$ is N_x. So N is a weak solution for P which induces N'. □

5. The Approximation Lemma

We show that there are enough curves on a regular arithmetic scheme or a regular variety. This generalises a lemma of Raskind [Ra, Lemma 6.21]. Here we include refinement (b) and the case of infinite K.

Lemma 20 *(Approximation Lemma). Let $f : X \to W$ be a smooth, quasi-projective morphism, where W is a regular curve over K or a regular arithmetic scheme of dimension* 1. *Let X be connected. Let $x_1, \ldots, x_m \in X$ be closed points that have distinct images in W. Then the following hold*:

(a) *There is an irreducible curve $C \subseteq X$ which contains each x_i as a regular point.*
(b) *If $Y \to X$ is a connected and étale cover then C can be chosen such that the generic point of C is inert in $Y \to X$.*

Proof. Let d be the fibre dimension of $f : X \to W$. If $d = 0$ the claim is obvious. So assume $d \geqslant 1$. Put $w_i = f(x_i)$. First we prove a reduction step: We may assume that $\kappa(x_i) = \kappa(w_i)$, if in turn, we prove the additional claim that we may prescribe $x_{m+1}, \dots, x_l$ which do not lie on C, where the total number of x_i in the fibre over $w \in W$ is bounded by the degree of $\kappa(w)$ over its prime field, if $\kappa(w)$ is finite.

Lemma 21. *There exists a finite cover $W' \to W$, étale in a neighbourhood of the w_i, such that there exist points $w'_i \in W'$ over w_i such that $\kappa(x_i) = \kappa(w'_i)$. Furthermore $Y' = Y \times_W W'$ is connected.*

Proof. Let $p_i(t) \in \kappa(w_i)[t]$ be a separable polynomial, one of whose roots generates the field extension $\kappa(x_i)|\kappa(w_i)$ for $i = 1, \dots, m$. We may choose the p_i such that they have a common degree. According to the approximation theorem for the finite set of valuations of F defined by the w_i there is a polynomial $p(t) \in F[t]$, whose image in $F_{w_i}[t]$ has integer coefficients and reduces to the $p_i(t)$. We may further assume that the image of $p(t)$ in $F_{w_0}[t]$ (for some $w_0 \neq w_i$) induces a completely ramified extension. Then $p(t)$ is irreducible. Adjoining a root of p defines a cover $W' \to W$ which is étale over the w_i. For each i let $w'_i \in W'$ be a point over w_i with $\kappa(w'_i) = \kappa(x_i)$. Furthermore p can be chosen such that $Y \times_W W' \to Y$ is connected, i.e. $Y \to X$ and $X \times_W W' \to X$ are linearly disjoint. This will be the case, if the point w_0 which is completely ramified in $W' \to W$ has nonempty pre-image on X. □

After replacing W by a neighbourhood of the w_i I may assume $W' \to W$ is finite étale.

Let $X' = X \times_W W'$ and $Y' = Y \times_W W'$. Choose for every $i = 1, \dots, m$ a point x'_i on X' over x_i and w'_i such that $\kappa(x'_i) = \kappa(w'_i) = \kappa(x_i)$. In the fibre over a point $w' \in W'$ there are at most $[\kappa(w') : \kappa(w)]$ pre-images of a point $x \in X$ on X': The exact number is the number of factors of $\kappa(w') \otimes_{\kappa(w)} \kappa(x)$.

Assume the Approximation Lemma applies to $Y' \to X'$ and $X' \to W'$: Let there be an irreducible curve C' on X', which contains the x'_i as regular points and does not contain the other pre-images of x_i on X', such that the generic point of C' is inert in $Y' \to X'$. Then the image $C \subseteq X$ of $C' \subseteq X'$ satisfies the requirements of the Approximation Lemma: The generic point of C is inert in $Y \to X$ since it is inert in the base change $Y' \to X'$. I claim x_i is a regular point on C:

The étale cover $X' \to X$ induces a finite étale cover of the normalisations $\tilde{C}' \to \tilde{C}$. Here $\tilde{C}'$ is a connected component of $\tilde{C} \times_X X' \to \tilde{C}$. If C had $\geqslant 2$ branches in x_i, there would be $\geqslant 2$ corresponding points of $\tilde{C}$, hence $\geqslant 2$ pre-images on $\tilde{C}'$. These correspond to the branches of C' in the pre-images of x_i. But this number is 1, because C' has one branch in x'_i and no branch in the other pre-images. The single branch of C in x_i must be regular, since its pre-image in X' is regular, and $X' \to X$ is étale at x_i.

Now prove the lemma under the assumption $\kappa(x_i) = \kappa(w_i)$.

Let $X \subseteq \mathbb{P}^N_W$ be a locally closed immersion.

Lemma 22. *Assume $N > d$. After shrinking W there exists a section $s : W \to \mathbb{P}^N_W$ such that the projection*

$$\pi : \mathbb{P}^N_W \setminus s(W) \to \mathbb{P}^{N-1}_W$$

fulfils the following:

(1) $\pi|_X$ *is étale onto its image in a neighbourhood of the* x_i.
(2) *The images of the* x_i *under* π *are distinct.*

Proof. We first show that there exists a point $P_w \in \mathbb{P}^N_{\kappa(w)}$ which achieves such a projection for each of the finitely many fibres. Then the section can be found as to reduce to each P_w in the fibre over w for the finitely many fibres which contain one of the x_i's.

A point $P_w \in \mathbb{P}^N_{\kappa(w)}$ achieves 1, 2 above iff

(1) the projection is étale at x iff P_w does not lie on any of the tangent spaces of $X \times_W \kappa(w)$ at the x_i. If $\kappa(w)$ is finite, then these tangent spaces together contain at most $s(q^{d+1}-1)/(q-1)$ points. Here s is the number of the x_i in this fibre and q is the number of elements of $\kappa(w)$;

(2) the images of the x_i under the projection are distinct, iff P_w does not lie on any line through two of the points. If $\kappa(w) = \mathbb{F}_q$, then these together contain at most $(q+1)s(s-1)/2$ points.

Now s is bounded by the degree of $\kappa(w)$ over its prime field, hence $s(s+1)/2 \leqslant q$. Let $\kappa(w)$ be finite. Then this shows that the number of points which P_w must avoid is less than $(q^{N+1}-1)/(q-1)$. Hence there exists a centre P_w which fulfils the requirements. □

By applying the lemma $N-d$ times, we get a morphism $X \to \mathbb{P}^d_W$ which is étale at all the x_i and maps the x_i to distinct points.

Let F be the function field of W. The field F is a global field or a function field of one variable over K. It is Hilbertian according to [FJ, 13.4.2]. Apply Lemma 23 to the induced map $Y \times_W \operatorname{Spec} F \to \mathbb{P}^d_F$. It gives an F-rational point P of $\mathbb{P}^d_F$ which specialises to the image of x_i in $\mathbb{P}^d(\kappa(w_i))$ for $i = 1, \dots, m$, and to a point distinct from the image of x_i in $\mathbb{P}^d(\kappa(w))$ for $i = m+1, \dots, l$. Furthermore P is inert in $Y \times_W \operatorname{Spec} F \to \mathbb{P}^d_F$. The closure of P in $\mathbb{P}^d_W$ gives rise to a section $W \to \mathbb{P}^d_W$, since W is regular, which passes through the images of the x_i for $i = 1, \dots, m$, and does not contain the images of the points x_i for $i = m+1, \dots, l$. The section defines a regular curve $D \subseteq \mathbb{P}^d_W$, whose generic point is inert in $Y \to \mathbb{P}^d_W$ by the choice of P. Let C be the unique irreducible curve on X which maps to D. The generic point of C is inert in $Y \to X$. Since D contains the images of the x_i for $i = 1, \dots, m$, the curve C contains the x_i for $i = 1, \dots, m$. Since $X \to \mathbb{P}^d_W$ is étale at x_i and D is regular, C is regular at these x_i. Since the images of x_i for $i = m+1, \dots, l$, are not contained in D, the curve C does not contain x_i, $i = m+1, \dots, l$. □

The following generalises a lemma of Bloch [Bl, Lemma (3.1)].

Lemma 23. *Let* F *be a Hilbertian field. Let* S *be a finite set of discrete valuations of* F *and* F_v *the completion of* F *at* $v \in S$. *Let* V *be an irreducible variety over* F *of dimension* d *and* $\pi : V \to \mathbb{P}^d_F$ *a quasi-finite, dominant and separable map. Let* $I \subseteq \mathbb{P}^d(F)$ *be the set of rational closed points* x *such that* $\pi^{-1}(x)$ *consists of a unique closed point* y *with* $[\kappa(y) : \kappa(x)] = \deg \pi$. *Then the image of* I *in* $\prod_{v \in S} \mathbb{P}^d(F_v)$ *is dense.*

Proof. After replacing V by an open subset it is finite and étale over its image $U \subseteq \mathbb{A}^d_F \subseteq \mathbb{P}^d_F$ which is open. Let $f(t_1, \dots, t_d, x) \in k(t_1, \dots, t_d)[x]$ be an irreducible polynomial describing the extension of the generic fibre of $V \to \mathbb{P}^d_F$. After further shrinking V we may assume the polynomial f describes the cover $V \to U$. As $U(F_v)$ is dense in $\mathbb{P}^d(F_v)$ we may assume there are given $(a_v) \in \prod_{v \in S} U(F_v) \subseteq \prod_{v \in S} \mathbb{A}^d(F_v)$ and $m \in \mathbb{N}$. We may assume that m is sufficiently large, such that any $b \in \mathbb{P}^d(F)$ which approximates the a_v to order m lies in $U(F)$. We have to prove that there is a $b \in I$ such that $b \equiv a_v \bmod \pi_v^m$, where π_v is a prime element of F_v.

Use the weak approximation theorem for a finite set of discrete valuations to find $(a_i) \in F^d$: $a_i \equiv (a_v)_i \bmod \pi_v^m$. Put $g(t_1, \dots, t_d, x) := f(a_1 + \prod \pi_v^m \cdot t_1, \dots, a_d + \prod \pi_v^m \cdot t_d, x)$.

As F is Hilbertian [FJ, 13.4.1] shows that there are elements $c_1, \dots, c_d \in F$, which are integers in F_v such that $g(c_1, \dots, c_d, x)$ is irreducible. This implies that the point $b = (a_i + \prod \pi_v^m \cdot c_i) \in I$ fulfils the requirements. $\square$

6. Construction of weak solutions

Proposition 24. *Let P be a covering problem on a regular arithmetic scheme or a regular variety X. Let $Y \to X$ be étale. If the induced covering problem (Remark 6) on Y is trivial, then P has a weak solution.*

Proof. If $\dim(X) = 0$, then the proposition is obvious. So assume $\dim(X) \geqslant 1$. Proposition 19 shows that we may replace X by any open subscheme X' during the proof. After shrinking X we may assume that $Y \to X$ is finite and étale. After replacing Y by the Galois hull of $Y \to X$, we may assume $Y \to X$ Galois with group G. Furthermore we may assume that there is a smooth morphism $X \to W$ to a regular curve W.

Look at the decomposition groups G_y of various closed points $y \in Y$. By removing finitely many points of W we may assume that any appearing decomposition group occurs for y in infinitely many fibres of $Y \to W$. Let $G_i \leqslant G$ represent all the occurring decomposition groups G_y. Let $y_i \in Y$ be a closed point with decomposition group G_i. We may assume that the images $x_i \in X$, $i = 1, \dots, n$, have different images in W.

According to the Approximation Lemma 20 there is an irreducible curve C through the x_i which is regular in these points and whose generic point is inert in the cover $Y \to X$. By shrinking X further we may assume that C is nonsingular. Let D be the (irreducible) pre-image of C on Y. We have an exact sequence

$$1 \to \pi_1(D) \to \pi_1(C) \to G \to 1. \tag{5}$$

Since C and hence D are regular and the covering problem is trivial over Y, the induced subgroup $N_D \trianglelefteq \pi_1(D)$ is the whole group. Therefore the image of $\pi_1(D)$ is contained in N_C and N_C defines a normal subgroup N' of G. For an $x \in X$ consider the commutative diagram

$$\begin{array}{ccc} \pi_1(X) & \longrightarrow & G. \\ \uparrow & \nearrow & \\ \pi_1(x) & & \end{array} \tag{6}$$

Let N be the pre-image of N' in $\pi_1(X)$. To prove the proposition we need to show for all points $x \in X$ (which we may assume to have an image in W distinct from the images of the x_i) that N_x is the pre-image of N. Due to the commutativity of the above diagram this is equivalent to showing that N_x is the pre-image of N' via the map $\pi_1(x) \to G$.

Let $G_y \leqslant G$ be the decomposition group of a point y over x, say $G_y = G_i$. Because of the factorisation $\pi_1(x) \twoheadrightarrow G_y \hookrightarrow G$ we may replace X by the quotient scheme Y/G_y of Y by G_y and replace x by the image of y. Then x is inert in the extension $Y \to X$. According to the choice of the G_i the point $x' = x_i \in C$ is inert in $Y \to X$. The points x and x' map to distinct points in W. By construction N' induces $N_{x'} \trianglelefteq \pi_1(x')$ (x' is a regular point of C). The Approximation Lemma gives a curve $C' \subseteq X$ which contains x and x' as regular points. We may assume that C' is regular by removing some points from X different from x and x'. C' contains points which are inert in $Y \to X$, whence its generic point is inert in $Y \to X$. By the same construction as above, let $N'' \trianglelefteq G$ be the normal subgroup inducing $N_{C'} \trianglelefteq \pi_1(C')$. As N'' induces $N_{x'} \trianglelefteq \pi_1(x')$ (x' is a regular point of C'), we have $N'' = N'$ because $\pi_1(x') \to G$ is surjective. But N'' induces N_x since x is a regular point of C'. This shows the claim. □

7. The main results

Theorem 25. *Assume X is a regular arithmetic scheme. Then every bounded covering problem and every abelian covering problem on X has a unique solution.*

Theorem 26. *Assume X is a regular variety over K.*

(a) *Assume K has enough inert points, e.g. K is finite, Hilbertian or PAC such that all Sylow subgroups of G_K are infinite. Then every tame, bounded covering problem on X has a unique solution.*
(b) *Assume K is finite. Then every tame, abelian covering problem on X has a unique solution.*

Proof for both theorems. If the given covering problem P is abelian then under the assumptions Proposition 28 shows that P is bounded. According to Proposition 17 there is an étale morphism $Y \to X$, such that the induced covering problem becomes weakly trivial over Y. Lemma 14 shows that the induced covering problem even becomes trivial. Now Proposition 24 shows that P has a weak solution. Another application of Lemma 14 shows that this weak solution is unique and a solution. □

Remark 27. In Theorem 26 the assumption on tameness can be dropped, if X is proper over a curve. This follows by the same proof in view of Proposition 17.

8. Abelian covering problems

Proposition 28. *Let X be a regular arithmetic scheme or a regular variety over the finite field K. Let P be an abelian covering problem on X. If X is a variety assume P is tame or X is proper over a curve. Then P is bounded.*

Proof.

Claim 1. *It is enough to show that there exists $Y \to X$ étale such that the exponents of $\pi_1(\tilde{C})/N_C$ are bounded for the induced covering problem on Y.*

This is true because the boundedness assumption only entered the proof of the main result via Proposition 17 where the weaker assumption suffices.

Claim 2. *It is enough to show that the N_x have bounded indices for all $x \in X$: If x is a regular point on C, then $\pi_1(x)/N_x \leqslant \pi_1(\tilde{C})/N_C$. The density theorem of Čebotarev shows that*

$\pi_1(\tilde{C})/N_C$ can be covered by $\pi_1(x)/N_x$ for some regular $x \in C$. If B is a bound for the indices $[\pi_1(x) : N_x]$, then B is a bound for the exponents of the $\pi_1(\tilde{C})/N_C$.

We prove the proposition by induction on the dimension of X. If the dimension is 0 or 1, then the claim is obvious. So let $\dim(X) \geqslant 2$.

By Lemma 15 we may assume there is an elementary fibration $X \subseteq \bar{X} \to W$ as in that lemma.

By induction hypothesis we may assume that the covering problem restricted to $s : W \to X$ has bounded indices and by Proposition 17 and Lemma 14 is even trivial on W.

We first prove the proposition in the case all quotients $\pi_1(\tilde{C})/N_C$ are l-groups for a fixed prime l. Let $x_0 \in X$ be a closed point in the image of the section s and $w_0 \in W$ its image, hence $\kappa(x_0) = \kappa(w_0)$.

Claim 3. *There is a finite étale Galois cover $X' \to X$ such that the indices of N_x are bounded for those points $x \in X$, which have the same decomposition group in* $\mathrm{Gal}(X'|X)$ *as x_0.*

Proof. In case $l = \operatorname{char} \kappa(W)$ we may assume the fibers C_w are projective. It is enough to show there is an étale cover $W' \to W$ such that for $w \in W$ with the same decomposition group as w_0, the indices of N_{C_w} for the fibre C_w over w are bounded. Then take $X' = X \times_W W'$.

For this it is enough to bound

$$\mathrm{H}^1\big(C_w \times_{\kappa(w)} \kappa(w)_{\mathrm{alg}}, \mathbb{Q}_l/\mathbb{Z}_l\big)^{G_{\kappa(w)}}$$

for suitable $w \in W$. As the point $w \in C_w$ is completely split in the cover defined by N_{C_w}, I may look at the geometric curve. But

$$\mathrm{H}^1\big(C_w \times_{\kappa(w)} \kappa(w)_{\mathrm{alg}}, \mathbb{Q}_l/\mathbb{Z}_l\big)$$

is nothing else than the (ind-)constructible sheaf $R^1 f_\star \mathbb{Q}_l/\mathbb{Z}_l$. For $l = \operatorname{char} \kappa(W)$ this follows from the proper base change theorem [Mi1, VI, Theorem 2.1]; for $l \neq \operatorname{char} \kappa(W)$ we additionally need purity [Mi1, VI, §5]. As the rank is bounded, $R^1 f_\star \mathbb{Q}_l/\mathbb{Z}_l$ is (ind-)locally constant after W is replaced by an open subscheme.

$G_{\kappa(w)}$ acts on $R^1 f_\star \mathbb{Q}_l/\mathbb{Z}_l$ via its map to $\pi_1(W, w)$.

By [KL] or [Sch] the group

$$\mathrm{H}^1\big(C_w \times_{\kappa(w)} \kappa(w)_{\mathrm{alg}}, \mathbb{Q}_l/\mathbb{Z}_l\big)^{G_{\kappa(w)}}$$

is finite. Let l^n be the exponent for $w = w_0$.

Let $W' \to W$ be the étale cover trivialising $R^1 f_\star \mathbb{Z}/l^{n+1}$. If w and w_0 have the same decomposition group in $W' \to W$, then

$$\begin{aligned}\mathrm{H}^1\big(C_{w_0} \times_{\kappa(w_0)} \kappa(w_0)_{\mathrm{alg}}, \mathbb{Z}/l^n\big)^{G_{\kappa(w_0)}} &= \mathrm{H}^1\big(C_w \times_{\kappa(w)} \kappa(w)_{\mathrm{alg}}, \mathbb{Z}/l^n\big)^{G_{\kappa(w)}} \\ &= \mathrm{H}^1\big(C_w \times_{\kappa(w)} \kappa(w)_{\mathrm{alg}}, \mathbb{Z}/l^{n+1}\big)^{G_{\kappa(w)}} \\ &= \mathrm{H}^1\big(C_w \times_{\kappa(w)} \kappa(w)_{\mathrm{alg}}, \mathbb{Q}_l/\mathbb{Z}_l\big)^{G_{\kappa(w)}}\end{aligned}$$

this shows Claim 3.

Now let B be the bound in Claim 3 and d the degree of X' over X. □

Claim 4. *The index of N_x is $\leqslant Bd$ for all $x \in X$ with $f(x) \neq w_0$.*

Proof. Let $x \in X$ and $f(x) \neq w_0$. The Approximation Lemma gives an irreducible curve C on X which contains x_0 and x as regular points and whose generic point is inert in $X' \to X$. Let $C' \to C$ be the induced étale cover of curves. Let A be the set of those regular points $y \in C$, whose decomposition group equals the decomposition group of x_0. Claim 3 says that for $y \in A$ we have $[\pi_1(y) : N_y] \leqslant B$. According to the density theorem of Čebotarev for $C' \to C$, these points have a Dirichlet density $\delta \geqslant 1/d$. If the index of N_x were greater than Bd, then the exponent of $\pi_1(\tilde{C})/N_C$ would be greater than Bd. The subgroup of elements of order $\leqslant B$ of the abelian l-group $\pi_1(\tilde{C})/N_C$ would have index bigger than d in this group. The density theorem of Čebotarev for the cover of $\tilde{C}$ given by N_C shows that the density of $y \in C$, such that the index of N_y is $\leqslant B$, is smaller than $1/d$, in contradiction to the above density of A. This shows Claim 4 and the proposition in case all occurring groups are l-groups. □

Now prove the general case of the proposition. Let P be a covering problem as in the assumptions of the proposition. The decomposition of each quotient $\pi_1(\tilde{C})/N_C$ into its l-Sylow subgroups defines covering problems P_l for each l. According to what was shown, all the P_l have bounded indices. To prove the proposition, it is enough, to show that almost all P_l are trivial. This means they have trivial solutions.

Since the covering problem restricted to W is trivial, the solution of P_l defines a subgroup of $\mathrm{H}^1(X \times_W \kappa(W)_{\mathrm{alg}}, \mathbb{Q}_l/\mathbb{Z}_l)^{G_{\kappa(W)}}$. But this group is trivial (view [KL]) for almost all l. □

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