The first-order part of computational problems

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Joint work with Giovanni Soldà

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Motivation

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$$a \quad \not\leq \quad b$$

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$$\begin{array}{ccc} a & \not\leq & b \\ \rotatebox{90}{$\geq$} & \rotatebox{-45}{$\not\leq$} & \\ c & & \end{array}$$

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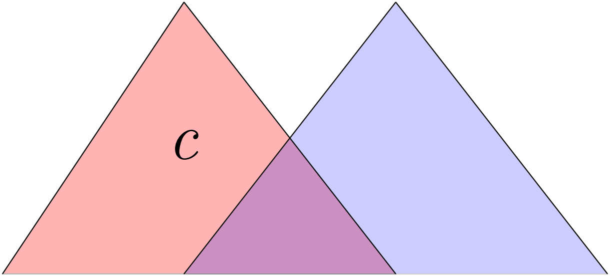

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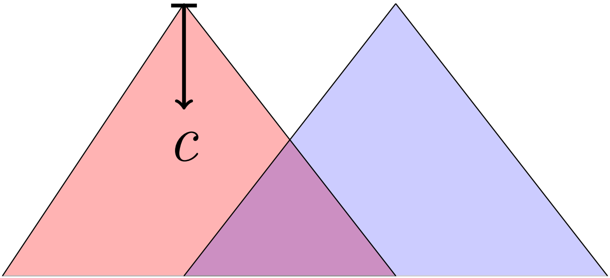

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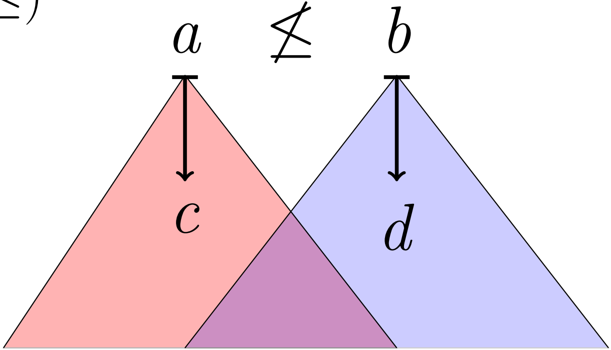

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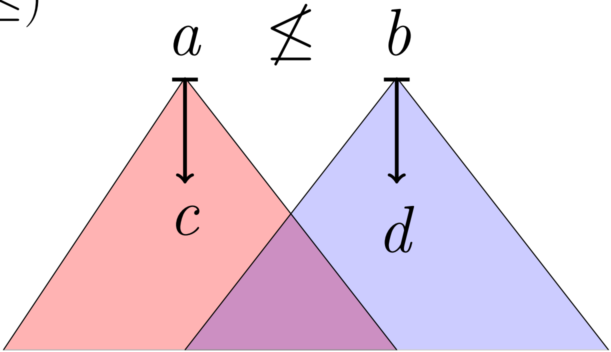

If c and d are maxima (in the resp. lower cones) satisfying some property φ then

$$c \not\leq d \Rightarrow a \not\leq b$$

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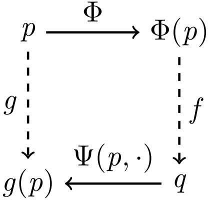

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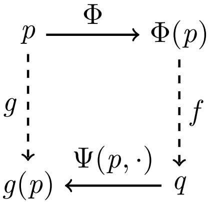

$g \le_{\mathrm{sW}} f :\iff g \le_{\mathrm{W}} f$ and Ψ does not depend on p.

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If g has codomain Y and there is a computable injection $Y \to \mathbb{N}$ with computable inverse we say that it is *first-order*.

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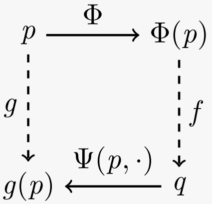

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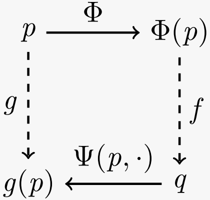

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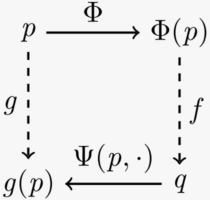
$g(p) \subset \mathbb{N}$, hence for every $q \in f(\Phi(p))$,

$$\Phi_w(q)(0) = \Psi(p, q)(0) \downarrow \in g(p)$$



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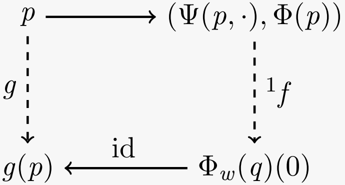
It follows that $g \le_W {}^1f \le_W f$.



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${}^1(\cdot)$ is an interior operator:

- ${}^1({}^1 f) \equiv_\mathrm{W} {}^1 f \le_\mathrm{W} f$
- $f \le_\mathrm{W} g \Rightarrow {}^1 f \le_\mathrm{W} {}^1 g$

In particular, ${}^1 f \not\equiv_\mathrm{W} {}^1 g \Rightarrow f \not\equiv_\mathrm{W} g$.

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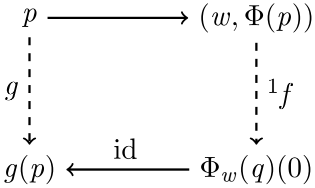

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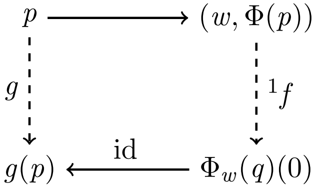
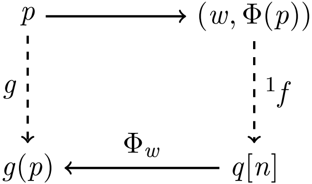


By the continuity of $\Phi_w = \Psi(p, \cdot)$, only a prefix of q is needed to solve g.

$q[n]$ is sufficiently long so that Φ_w converges on 0.



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- $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$: given a tree $T \subset \mathbb{N}^{<\mathbb{N}}$, find a path $p \in [T]$.
- $\boldsymbol{\Sigma}^1_1$-$\mathsf{C}_{\mathbb{N}}$: given a list of subtrees of $\mathbb{N}^{<\mathbb{N}}$, find the index of an ill-founded one.

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f'	:	*jump in the Weihrauch lattice*
		name of input: a sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathbb{N}^{\mathbb{N}}$ s.t. $\lim_n p_n$ is a name for an instance x of f; output : $f(x)$
$\vdots$		

Examples

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  Being $\leq_W \mathsf{C}_\mathbb{N}$ corresponds to being uniformly computable with finitely many mind changes, hence ${}^1\mathsf{lim} \leq_W \mathsf{C}_\mathbb{N}$. The other reduction follows from $\mathsf{lim} \equiv_W \widehat{\mathsf{C}_\mathbb{N}}$.

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Is there a general rule?

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$\mathbf{\Pi}^1_1$-CA can solve it, but WF^* cannot.

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$$(w, (x_n)_{n\in\mathbb{N}}) \mapsto \{(y_n)_{n<k} : (\forall n < k)(y_n \in f(x_n)) \text{ and } \Phi_w(\langle y_n \rangle_{n<k})(0)\downarrow\}$$

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$(\cdot)^{u*}$ is a closure operator:

- $f \le_{\mathrm{W}} f^{u*} \equiv_{\mathrm{W}} (f^{u*})^{u*}$
- $f \le_{\mathrm{W}} g \Rightarrow f^{u*} \le_{\mathrm{W}} g^{u*}$

Moreover $f^* \le_{\mathrm{W}} f^{u*} \le_{\mathrm{W}} \widehat{f}$

FOP and parallelization

Theorem (Soldà, V.)

For every f, ${}^1(\widehat{f}) \equiv_{\mathrm{W}} {}^1(f^{u*})$.

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For every f, ${}^1(\widehat{f}) \equiv_{\mathrm{W}} {}^1(f^{u*})$.

Proof (Sketch):

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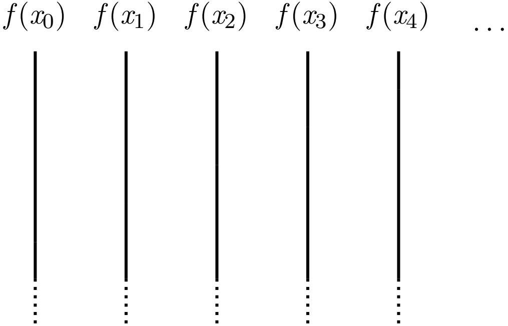

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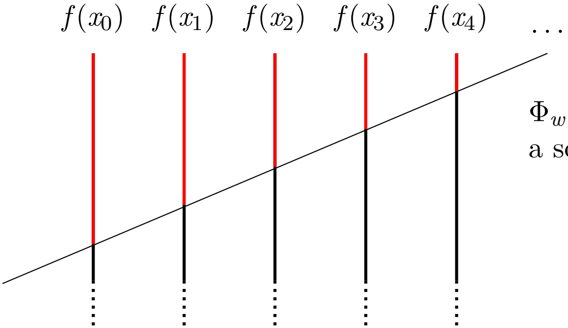

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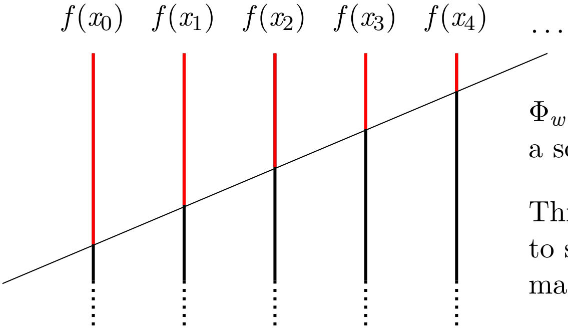

Φ_w selects a prefix of a solution.

This corresponds to selecting finitely many columns.

FOP and parallelization

Theorem (Soldà, V.)

For every f, if $f \equiv_{\mathrm{W}} \widehat{g}$ for some first-order g, then

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If $\mathrm{id} \le_{\mathrm{sW}} f$ *then this lifts to jumps: for every n*

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Is this peculiar of first-order problems?

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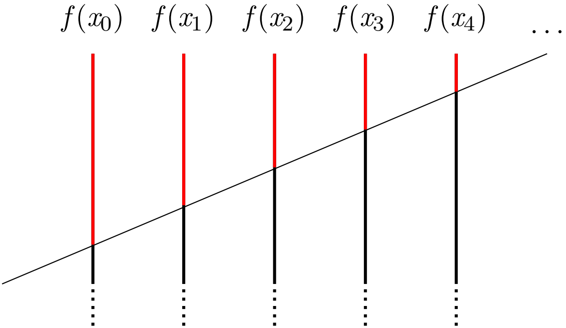

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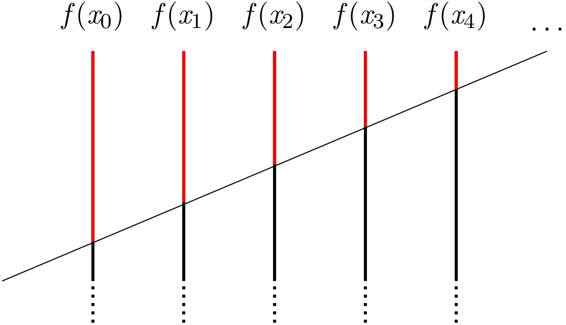

Φ_w selects a prefix of a solution.

The prefix of $f(x_i)$ may depend on the solution to x_j.

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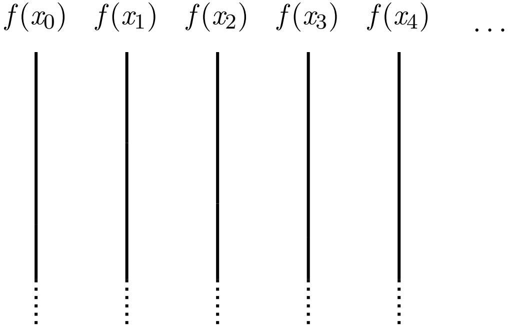
On the other hand, an input for $({}^1 f)^{u*}$ is $(w, (w_n, x_n)_n)$ s.t.



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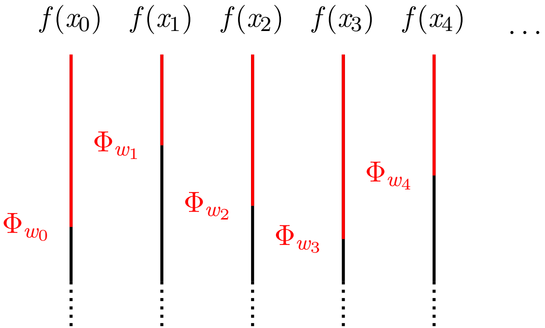

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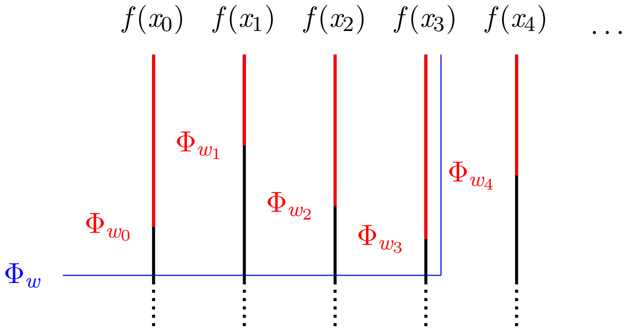

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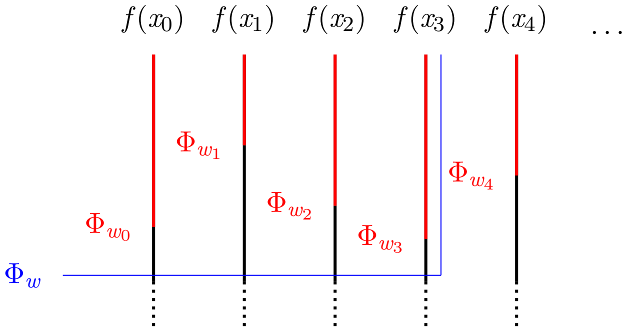

The prefix of $f(x_i)$ is independent of the solution of x_j.

FOP and unbounded-*

In some cases, we have a work around. E.g. if $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ is finitely valued (for every $p \in \mathrm{dom}(f)$, $|f(p)| < \infty$) then

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Lemma

There are two sequences $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}$ s.t.

- *for every n, $\emptyset' \not\leq_T A_n$, $\emptyset' \not\leq_T B_n$, but $\emptyset' \leq_T A_n \oplus B_n$;*
- *for every n and every computable functional Ψ s.t. $\emptyset' = \Psi(\langle A_i \rangle, B_n)$, the map sending x to the prefix of B_n used in the computation of $\emptyset'(x)$ is not B_n-computable.*

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- $\mathbf{\Pi}^1_1\text{-}\mathsf{CA} \equiv_W \widehat{\mathsf{WF}}$, hence $^1\mathbf{\Pi}^1_1\text{-}\mathsf{CA} \equiv_W \mathsf{WF}^{u*}$.

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$g \le_{\mathrm{W}} f^\diamond$ iff Player 2 has a computable winning strategy for $G(f \to g)$

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Besides, if $\operatorname{ran}(f) = k$ *then* $f^{*} \equiv_{\mathrm{W}} f^{\diamond}$.

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Idea: we guess the possible answers to the oracle calls and use the effective closedness of $\operatorname{Graph}(f)$ to discard wrong guesses.

Examples: C_k for every $k \in \mathbb{N}$.

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(Brattka, Gherardi) The *completion* of a represented space X is

$$\overline{X} := X \cup \{\bot\}$$

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Question: can we do better?

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$$\mathsf{WF}^* <_{\mathrm{W}} \mathsf{WF}^{u*} \equiv_{\mathrm{W}} \mathsf{WF}^{\diamond} \equiv_{\mathrm{W}} {}^1\mathbf{\Pi}^1_1\text{-}\mathsf{CA}$$

Ramsey's theorem

Theorem (Brattka, Rakotoniaina)

For every $n > 1$ *and* $k \geq 2$, $\mathsf{C}_k^{(n)} \leq_{\mathrm{W}} \widehat{\mathsf{SRT}_k^n} \leq_{\mathrm{W}} \widehat{\mathsf{RT}_k^n} \equiv_{\mathrm{W}} \mathsf{WKL}^{(n)}$

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Are the last two reductions strict?

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Open question (Brattka, Rakotoniaina): $\mathsf{C}_{\mathbb{N}}' \le_{\mathrm{W}} \mathsf{RT}^2_2$?

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For every n and $k > 1$, $\mathsf{C}_{\mathbb{N}}^{(n)} \not\le_{\mathrm{W}} \mathsf{RT}^{n+1}_k$.

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In particular, for $n = 2$, ${}^1\mathsf{SRT}^2_2 <_{\mathrm{W}} {}^1\mathsf{RT}^2_2$ (Brattka, Rakotoniaina), hence

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Can we fully characterize ${}^1\mathsf{RT}^n_k$?

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