Pinned Distance Sets Using Effective Dimension

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Kolmogorov complexity in Euclidean space

Fix a universal TM U. Let $n, r \in \mathbb{N}$, and $x \in \mathbb{R}^n$. The *Kolmogorov complexity of x at precision r* is

$$\begin{aligned} K_r(x) &= \text{length of the shortest input } \pi \text{ such that } U(\pi) = d_x \\ &= \text{the minimum number of bits to specify } x \text{ to precision } 2^{-r}. \end{aligned}$$

where $d_x = (\frac{m_1}{2^r}, \dots, \frac{m_n}{2^r})$ is the closest dyadic rational at precision r to x.

The *Kolmogorov complexity of x at precision r given y at precision t* is

$$\begin{aligned} K_r(x) &= \text{length of the shortest input } \pi \text{ such that } U(\pi, d_y) = d_x \\ &= \text{the minimum number of bits to specify } x \text{ to precision } 2^{-r} \text{ if you know} \\ &\qquad y \text{ to precision } 2^{-t}. \end{aligned}$$

where $d_y = (\frac{m_1}{2^t}, \dots, \frac{m_n}{2^t})$ is the closest dyadic rational at precision t to y.

Kolmogorov complexity in Euclidean space

- For every $x \in \mathbb{R}^n$ and $r \in \mathbb{N}$, $0 \leq K_r(x) \leq nr + O(\log r)$.
- Symmetry of information: For every $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $r, t \in \mathbb{N}$,
$$K_{r,t}(x, y) = K_t(y) + K_{r,t}(x \mid y) + O(\log r + t).$$
- We can *relativize* the definitions in the natural way to get $K_r^A(x), K_{r,t}^A(x \mid y)$, etc.

Effective Dimensions of Points

Definition (Lutz '03, Mayordomo '03)

Let $n \in \mathbb{N}$, and $x \in \mathbb{R}^n$. The *(effective Hausdorff) dimension of* x is

$$\dim(x) = \liminf_{r \to \infty} \frac{K_r(x)}{r}.$$

Definition (Athreya et al. '07, Lutz and Mayordomo '08)

Let $n \in \mathbb{N}$, and $x \in \mathbb{R}^n$. The *(effective) strong dimension of* x is

$$\mathrm{Dim}(x) = \limsup_{r \to \infty} \frac{K_r(x)}{r}.$$

The effective dimensions of a point x measure the density of algorithmic information in x.

The Point-to-Set Principle

Theorem (J. Lutz and N. Lutz, '16)

For every set $E \subseteq \mathbb{R}^n$,

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x), \textit{ and}$$

$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \mathrm{Dim}^A(x).$$

- The Hausdorff and packing dimension of a *set* is characterized by the corresponding dimension of the *points* in the set.
- Allows us to use algorithmic techniques to answer questions in geometric measure theory.

Distance sets

Let $E \subseteq \mathbb{R}^n$. The distance set of E is

$$\Delta E = \{\|x - y\| \mid x, y \in E\}.$$

More generally, if $x \in \mathbb{R}^n$, the pinned distance of E w.r.t. x is

$$\Delta_x E = \{\|x - y\| \mid y \in E\}.$$

Question: How do the sizes of ΔE and $\Delta_x E$ relate to the size of E?

Distance sets

When E is a finite set, Erdös conjectured that $|\Delta E|$ is nearly linear in terms of $|E|$.

- In a breakthrough paper, Guth and Katz proved this in the plane.
- Still an important open problem for $\mathbb{R}^n$ with $n \geq 3$.

Falconer posed an analogous question for the case that E is infinite, known as Falconer's *distance set problem*.

- If $E \subseteq \mathbb{R}^n$ has $\dim_H(E) > n/2$, then ΔE has positive measure.
- Still open in all dimensions.
- Guth, Iosevich, Ou and Wang, proved that if $E \subseteq \mathbb{R}^2$ and $\dim_H(E) > 5/4$, then $\mu(\Delta E) > 0$.

Distance sets

Substantial progress has been made in a slightly different direction, on the Hausdorff dimension of *pinned distance sets* in the plane.

- Orponen proved that if E is Ahlfors regular and $\dim_H(E) > 1$, then for "most" points $x \in \mathbb{R}^2$, $\dim_H(\Delta_x E) = 1$.
- Shmerkin weakened the regularity assumption of Orponen's result to simple regularity, i.e., $\dim_H(E) = \dim_P(E)$.
- Liu showed that, if $\dim_H(E) = s \in (1, 5/4)$, then for most x, $\dim_H(\Delta_x E) \geq \frac{4}{3}s - \frac{2}{3}$.
- Shmerkin improved this bound when $\dim_H(E) = s \in (1, 1.04)$, by proving that
$$\dim_H(\Delta_x E) \geq 2/3 + 1/42 \approx 0.6904$$

Pinned distance sets using effective dimension

Using effective dimension, we are able to improve these bounds, when the dimension of E is close to 1.

Theorem (S. '22)

Let $E \subseteq \mathbb{R}^2$ *be an analytic set with* $\dim_H(E) > 1$*. Then, for all* $x \in \mathbb{R}^2$ *outside a set of Hausdorff dimension at most* 1*,*

$$\dim_H(\Delta_x E) \geq \frac{s}{4} + \frac{1}{2},$$

where $s = \dim_H(E)$*.*

In particular, for most points $x \in E$, $\dim_H(\Delta_x E) \geq \frac{s}{4} + \frac{1}{2}$.

Pinned distance sets using effective dimension

Theorem (S. '22)

Suppose that $x, y \in \mathbb{R}^2$, $e_1 = \frac{y-x}{\|y-x\|}$ *satisfy the following.*

(C1) $\dim(x), \dim(y) > 1$

(C2) $K^x_r(e_1) = r - O(\log r)$ *for all* r.

(C3) $K^x_r(y) \geq K_r(y) - O(\log r)$ *for all sufficiently large* r.

(C4) $K_r(e_1 \mid y) = r - o(r)$ *for all* r.

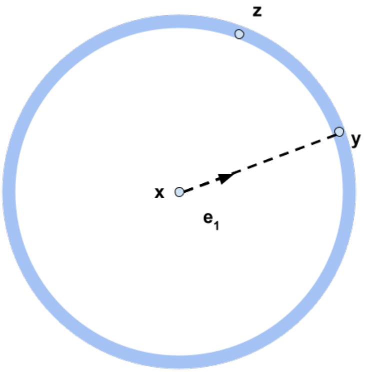
Then

$$\dim^x(\|x - y\|) \geq \frac{3}{4}.$$

We reduce our main theorem on the Hausdorff dimension of pinned distance sets to this pointwise analog.

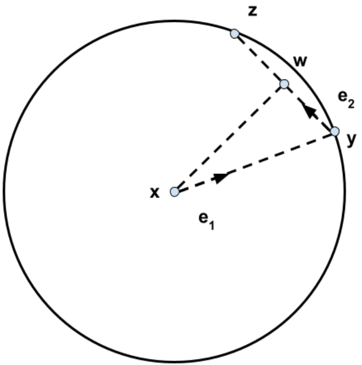
1. Orponen's theorem on radial projections.
2. Point-to-set principle.

Fix points $x, y \in \mathbb{R}^2$ satisfying conditions (C1)-(C4). We will prove that $\dim^x(\|x-y\|) \geq 3/4$.



- Fix a precision $r \in \mathbb{N}$. Suffices to show that $K_r^x(\|x-y\|) \gtrsim \frac{3}{4}r$.
- By symmetry of information, $K_r^x(y \mid \|x-y\|) \approx K_r^x(y) - K_r^x(\|x-y\|)$.
  - $K_r^x(y \mid \|x-y\|) \approx$ the amount of information needed to compute y if you know x and $\|x-y\|$.
- A *lower* bound $K_r^x(\|x-y\|)$ is equivalent to an *upper* bound on $K_r^x(y \mid \|x-y\|)$.
- Suffices to bound the set of $z \in \mathbb{R}^2$ such that $\|x-z\| = \|x-y\|$ and $K_r(z) \lesssim K_r(y)$.

Goal: Bound the size of the set of points z such that $\|x - z\| = \|x - y\|$ and $K_r(z) \leq K_r(y)$.



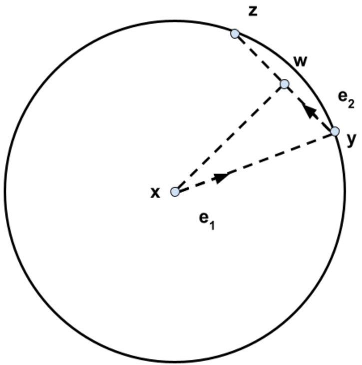
For any $e \in \mathcal{S}^1$ and $x \in \mathbb{R}^2$, $p_e x = e \cdot x$.

1. $|p_{e_1} y - p_{e_1} z| \lesssim \|y - z\|^2$
2. $p_{e_2} x = p_{e_2} w$

- Reduce this to *projections*.
- Divide and conquer.
  - Use (1) to bound the set of points z which are "far" away from y.
  - Use (2) to bound the set of points z which are "close" to y.
- Using (1), we get a weak lower bound on the complexity of $\|x - y\|$:
$$K_t^x(\|x - y\|) \gtrsim \frac{K_t(y)}{2}$$
- Choosing t such that $K_t(y) \approx \frac{r}{2}$ gives $K_t^x(\|x - y\|) \gtrsim \frac{r}{4}$.
- Use (2) to show handle the points "close" to y.

Reducing to projection theorem

Fixed appropriate t. Proved $K_t^x(\|x-y\|) \gtrsim \frac{r}{4}$. Need to show that $K_{r,t}^x(\|x-y\|) \gtrsim \frac{r}{2}$.



1. $|p_{e_1}y - p_{e_1}z| \lesssim \|y-z\|^2$
2. $p_{e_2}x = p_{e_2}w$

- Equivalently, $K_{r,r,t}^x(y \mid \|x-y\|, y) \approx K_r(y) - r$.
  - $K_{r,r,t}^x(y \mid \|x-y\|, y) \approx$ number of bits required to compute y if you know x, $\|x-y\|$ and the first t bits of y.
- Using (2), we can compute x if we know y, z and the position of x along the line with direction $e_2^\perp$ containing x.

$$K_{r-t,r}(x \mid y) \lesssim K_r(z \mid y) + K_{r-t}(x \mid p_{e_2}x, e_2)$$

- Want to bound $K_r(x \mid p_{e_2}x, e_2)$ - the complexity of computing (an approximation of) x given (approximations of) $p_{e_2}x$ and e_2.

Projection theorem

- We want to bound $K_r(x \mid p_e x, e)$ - the complexity of computing (an approximation of) x given (approximations of) $p_e x$ and e.
- When the direction e is random relative to x, i.e., $K^x_r(e) \approx r$, we know that $K_r(x \mid p_e x, e) \approx K_r(x) - r$.
    - This is the pointwise analog of Marstrand's projection theorem.
- Unfortunately, for our application to distances, we don't have enough control over the direction to directly apply this result.
- However, we **do** have enough control to ensure that e is random *up to some initial precision*:

$$K^x_s(e) \approx s,$$

  where $s = -\log \|z - y\|$.

Projection theorem

Theorem (S. '22)

Let $x \in \mathbb{R}^2$ *such that* $\dim(x) \geq 1$. *Let* $e \in \mathcal{S}^1$ *and* $r, t \in \mathbb{N}$. *Suppose that* $K_{t,r}(e \mid x) \approx t$. *Then*

$$K_r(x \mid p_e x, e) \lesssim K_r(x) - \frac{r+t}{2}$$

- When $t = r$, i.e., e is random w.r.t. x, then this recovers the pointwise analog of Marstrand's theorem (N. Lutz and S. '18). In particular, in this case, the bound is tight.
- It is likely that this bound is **not** tight in general. Improvements on this should immediately lead to improvements on the distance set problem.

Projection theorem

Let $x \in \mathbb{R}^2$, $b \in \mathbb{N}$ and $a < b$.

1. $[a, b]$ is **teal** if $K_{b,s}(x \mid x) \lesssim b - s$ for all $a \leq s \leq b$.
2. $[a, b]$ is **yellow** if $K_{s,a}(x \mid x) \gtrsim s - a$ for all $a \leq s \leq b$.

Generalize known projection theorems to show the following. Assume $K^x_s(e) \approx s$ for all $s \leq b - a$. Then

1. If $[a, b]$ is teal, $K_{b,b,b,a}(x \mid p_e x, e, x) \approx 0$.
2. If $[a, b]$ is yellow, $K_{b,b,b,a}(x \mid p_e x, e, x) \approx K_{b,a}(x \mid x) - (b - a)$.

Read $K_{b,b,b,a}(x \mid p_e x, e, x)$ as the complexity of computing x given

- b-approximations of $p_e x$ and e
- a-approximation of x.

Projection theorem

Goal: Bound the complexity of $K_r(x \mid p_e x, e)$ when $K_t^x(e) \approx t$.
Main idea: Partition $[0, r]$ into yellow and teal intervals of length at most t, and sum the bounds.

- Naive partition does not necessarily work. The contribution of yellow intervals might be too high.
- However, we can optimize to get a partition to lower the contribution of yellow intervals enough to prove that

$$K_r(x \mid p_e x, e) \lesssim K_r(x) - \frac{r+t}{2},$$

Back to distances

Let $z \in \mathbb{R}^2$ such that $\|x - z\| = \|x - y\|$. We know that

1. $p_e x = p_e w$, where w is the midpoint of y and z and $e = \frac{z-y}{\|y-z\|}$.
2. $K_s^x(e) \approx s$, where $s = -\log \|z - y\|$
   - That is, e is random relative to x up to precision t.

Let $t \leq r$ such that $K_t(y) \approx \frac{r}{2}$. Suppose that z is as above and $s \geq t$. Using the projection theorem, we have

$$K_{r-s}(x \mid p_e x, e) \lesssim K_{r-s}(x) - \frac{r}{2}.$$

We can conclude that

$$\begin{aligned} K^x_{r,t}(\|x - y\| \mid \|x - y\|) &\gtrsim K^x_{r,t}(\|x - y\| \mid y) \\ &\gtrsim K_{r,t}(y \mid y) - (K_r(y) - r) \\ &\approx r - K_t(y) \\ &\approx \frac{r}{2} \end{aligned}$$

The End

Thank you!