Effectivizing the theory of Borel equivalence relations

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Joint work with Uri Andrews

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Today’s menu

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- Opening
    - Borel and computable reductions

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- Developing a computable analog of the Borel theory
    - Dichotomies
    - Orbit equivalence relations
    - Isomorphism relations

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Since then, Borel reductions have been widely explored, showing deep connections with topology, group theory, combinatorics, model theory, and ergodic theory – to name a few.

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The history of $\leq_c$ is intricate. Despite being introduced in the 1970s (hence, even *before* $\leq_B$) and being examined, among others, by **Ershov** in the East and **Lachlan** in the West, it was forgotten and rediscovered multiple times, often reappearing under a different name.

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Computable reductions found remarkable applications in various fields, including the theory of numberings, proof theory, computable structure theory, combinatorial algebra, and theoretical computer science. But a systematic study of $\leq_c$ has really begun to take off only recently.

Developing a computable analog of the Borel theory

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- *There is an embedding from $\langle \mathcal{P}(\omega), \subseteq^* \rangle$ into the Borel hierarchy.*
  Louveau, Velickovic (1994)

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Theorem (Kechris, Louveau)

E_1 is minimal above E_0. In fact, let $E \leq_B E_1$. Then, exactly one of the following holds:

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Following Coskey, Hamkins, and R. Miller (2012), we adapt benchmark relations from the Borel theory by restricting them to the c.e. sets. This naturally give rise to equivalence relations on the natural numbers. Indeed, if E is on the c.e. sets, then we let, for all $e, i \in \omega$,

$$e \, E^{ce} \, i \Leftrightarrow W_e \, E \, W_i.$$

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So, $\mathsf{Id}(2^\omega)$ translates to the equality of c.e. sets, given by

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$$e \, E_1^{ce} \, i \Leftrightarrow (\forall^\infty n)(W_e^{[n]} = W_i^{[n]}).$$

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- $\mathrm{Id}(\omega)$ is Δ^0_1,
- $=^{ce}$ is Π^0_2,
- E_0^{ce} is Σ^0_3.

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Theorem (Ianovski, R. Miller, Ng, Nies)

E_0^{ce} is Σ_3^0 universal.

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Orbit equivalence relations

Countable Borel equivalence relations

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Let G be a group acting on a standard Borel space. Then the orbit equivalence relation E_G is given by

$$x \, E_G \, y \Leftrightarrow (\exists \gamma \in G)(\gamma \cdot x = y).$$

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- For each countable group G, the shift action of G on the space 2^G is given by

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for $g, h \in G$ and $p \in 2^G$. (If $G = \mathbb{Z}$, this corresponds to left shift of doubly-infinite binary sequences).

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If E is a cber on a standard Borel space X, then there is a countable group G and a Borel action of G on X such that $E = E_G$.

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Theorem (Dougherty, Jackson, Kechris)

E_∞ is a universal cber *(that is, $E \leq_B E_\infty$ for all cbers E).*

Orbit equivalence relations under computable lenses

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One way to see this is by using the following lemma. Say that a given E_G^{ce} is permutation induced if there is a computable subgroup H of S_∞ so that

$$x \, E_G^{ce} \, y \Leftrightarrow (\exists \pi \in H)(W_y = \{\pi(n) : n \in W_x\}).$$

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So, when dealing with E_G^{ce}, we shall assume that G is a subgroup of S_∞ whose action on the c.e. sets is given, for all $\pi \in G$, by

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- $|W_i \cap [0, n]| = |W_j \cap [0, n]|$,
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Hence, E_∞^{ce} has many natural realizations.

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Anyway, the analog of **Feldman-Moore** theorem fails also working up to $\leq_c$, e.g., E_{min} and E_{max} are enumerable indices but, being stricly below $=^{ce}$, they cannot be equivalent to any E^{ce}_G.

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Thus, there is no computable analog of **Glimm-Effros** dichotomy.

Isomorphism relations

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- For a countable language L, let $\mathsf{Mod}(L)$ denote the collection of all countable L-models with universe ω. Each element of $\mathsf{Mod}(L)$ can be viewed as an element of the product space
$$X_L := \prod_{i \in I} 2^{\omega^{n_i}},$$
which is homeomorphic to the Cantor space.

Isomorphism relations, II

- The logic action of S_∞ on X_L is given as follows:

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The most complex classes of countable structures, I

Say that a class $\mathbb{K}$ of countable structures is on top for $\leq_B$ if, for all countable languages L, $\cong_L$ Borel reduces to $\cong_{\mathbb{K}}$. (This is the same of asking that every S_∞-relation reduces to $\cong_{\mathbb{K}}$).

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- *Torsion-free abelian groups.*
  Paolini, **Shelah** (preprint)

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1. Classes on top are not Borel (but analytic);
2. There are Borel equivalence relations which don't admit a classification by countable structures, e.g., Kechris and Louveau showed that E_1 is not Borel reducible to the isomorphism of countable graphs.

Isomorphism relations under computable lenses, I

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Then, to compare isomorphism relations on computable structures, one considers partial computable reductions with domain containing the relevant set $I(\mathbb{K})$.

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2. Torsion abelian groups are *not* on top for $\leq_B$, but computable torsion abelian groups are on top for $\leq_c$.

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This jump is proper on Borel equivalence relations:

Theorem (H. Friedman, Stanley)

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- *If E is universal Σ^1_1, then $E \sim_c E^+$.*

The computable FS-jump

Clemens, Coskey, and Krakoff (2022) introduced a natural computable analog of the FS-jump:

- For E on ω, E^+ is given by

$$xE^+y \Leftrightarrow [W_x]_E = [W_y]_E.$$

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Theorem (Clemens, Coskey, Krakoff)

- *If E is universal Σ^1_1, then $E \sim_c E^+$.*
- *If E is hyperarithmetic, then $E <_c E^+$.*

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- If $a = 3 \cdot 5^e$, then $E^{+a} = \bigoplus_i E^{+\varphi_e(i)}$.

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Let E on ω be hyperarithmetic. Then, there exists a notation $a \in \mathcal{O}$ such that

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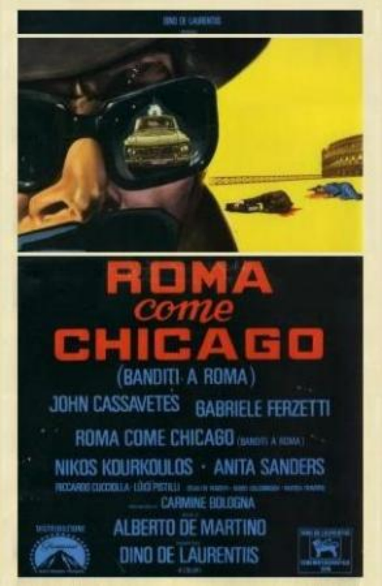
Theorem (Andrews, S.)

- *Let $a, b \in \mathcal{O}$ be notations for $\alpha < \omega^2$. Then, $E^{+a} \sim_c E^{+b}$ for all E.*
- *There are two notations $a, b \in \mathcal{O}$ for ω^2 so that $\mathrm{Id}(\omega)^{+a}$ and $\mathrm{Id}(\omega)^{+b}$ are incomparable.*

Thank you!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CHICAGO
(BANDITI A ROMA)
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