Normal numbers and perfect necklaces

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Midwest Computability Seminar - 15 February 2021

Borel normal numbers

Let b be an integer greater than or equal to 2.

A real number is normal to base b if in its base-b expansion every block of digits occurs with the same limiting frequency as every other block of the same length.

Couterxamples:
$0.010010001000001\ldots$

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Couterxamples:
$0.010010001000001\ldots$
$0.0101010101010101\ldots$

In 1909 Borel gave this definition, proved that almost all real numbers are nomal to all integer bases, and he asked for an example.

Borel's question

All Martin-Löf random reals are normal to every base, in particular Ω-numbers.

Constructions

Lebesgue 1909, Sierpinski 1916, Champernowne 1933, Turing 1937, Copeland and Erdős 1946, Davenport and Erdős 1952, W. Schmidt 1960, M. B. Levin 1970, Stoneham 1973, M. B. Levin 1999, ...

Borel's question is essentially open.

Champernowne's example

Theorem (Champernowne 1933)

$0.1234567891011121314151617181920212223\ldots$ *is normal to base* 10*.*

ARITHMÉTIQUE. — *On suppose écrite la suite naturelle des nombres; quel est le* $(10^{1000})^{\text{ième}}$ *chiffre écrit?* Note de M. Ém. **Barbier**.

« 1. Pour écrire tous les nombres inférieurs à 11, il faut 11 fois 1 caractère; il faut 111 fois 2 caractères pour écrire les nombres inférieurs à 111; 1111 fois 3 caractères pour écrire tous les nombres inférieurs à 1111.

» Généralement, il faut, pour écrire tous les nombres inférieurs au nombre qui s'écrit par $(n+1)$ chiffres 1 consécutifs, un nombre de caractères égal au produit de n par le nombre de $(n+1)$ chiffres 1 consécutifs.

» La suite des nombres qui précèdent le nombre de 665 chiffres 1 emploie le nombre (*irréalisable*) de caractères

$$664 \times 1111111\ldots = 73777\ldots77704,$$

nombre de 667 chiffres dont 663 sont des 7.

ARITHMÉTIQUE. — *On suppose écrite la suite naturelle des nombres; quel est le* $(10^{10000})^{\text{ième}}$ *chiffre écrit?* Note de M. Em. **Barbier**.

« 1. Nous avons déterminé le $(10^{10})^{\text{ième}}$, le $(10^{100})^{\text{ième}}$, le $(10^{1000})^{\text{ième}}$ chiffre; il arrive que la recherche du $(10^{10000})^{\text{ième}}$ chiffre ne demande pas un long calcul.

» Les nombres de 10000 chiffres

$$111055\ldots554445 \text{ (ou } 9995 \times 11\ldots111) \quad \text{et} \quad 999588\ldots888889$$

Journal of the London Mathematical Society, S1-8(4):254-260,1933

THE CONSTRUCTION OF DECIMALS NORMAL IN THE SCALE OF TEN

D. G. CHAMPERNOWNE*.

A decimal $\cdot S$ is said to be normal in the scale of ten if, when γ_ρ is an arbitrary sequence of an arbitrary number ρ of digits, and $G(x)$ denotes the number of times that γ_ρ occurs as ρ consecutive digits in the first x digits of S,

$$G(x) = 10^{-\rho} x + o(x)$$

as $x \to \infty$. Rules have been given for the construction of such decimals, but these have always been somewhat involved.

Actually, a very simple construction is adequate; we shall, in fact, show in the course of this paper that the decimal $\cdot 123456789101112\ldots$, composed of the natural sequence of numbers counting from 1 upwards, is itself normal in the scale of ten.

First, we shall prove

THEOREM I. *If s_r denotes the sequence*

$$\cdot 00..0, 00..1, 00..2, \ldots\ldots, 99..9,$$

* Received 19 April, 1933; read 27 April, 1933.

Champernowne's proof

Instead, the concatenation of all blocks of n symbols in lexicographic order,

$$\underbrace{0123456789}_{megablock\ 1}\ \ \underbrace{00\ 01\ldots 98\ 99}_{megablock\ 2}\ \ \underbrace{000\ 001\ \ \ldots\ \ 998\ 999}_{megablock\ 3}\ldots$$

Champernowne's proof counts:

- each digit
- each block of two digits

  ...
- each block of n digits

Difficulties:

- overlapping blocks

  $000\ 001\ 002\ 003\ldots 990\ 991\ 992\ 993\ 994\ 995\ 996\ 997\ 998\ 999$

  Inside a megablock for length n Champernowne just counts inside blocks and bounds the number of occurrences in between blocks.
- count up to an arbitrary position within a megablock. $\square$

Our observation

For simplicity consider the alphabet $\{0, 1\}$.

In the megablock n viewed circularly, each block of length n occurs exactly n times at different positions modulo n.

position	$12\ \ 34\ \ 56\ \ 78$	
	$00\ 01\ 10\ 11$	
	$00\ 01\ 10\ 11$	00 occurs twice, at positions different modulo 2

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	$00\ 01\ 10\ 11$	
	$00\ 01\ 10\ 11$	10 occurs twice, at positions different modulo 2
	$00\ 01\ 10\ 11$	
	$00\ 01\ 10\ 11$	11 occurs twice, at positions different modulo 2

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position	12 34 56 78	
	00 01 10 11	
	00 01 10 11	00 occurs twice, at positions different modulo 2
	00 01 10 11	
	00 01 10 11	01 occurs twice, at positions different modulo 2
	00 01 10 11	
	00 01 10 11	10 occurs twice, at positions different modulo 2
	00 01 10 11	
	00 01 10 11	11 occurs twice, at positions different modulo 2

Neither Barbier nor Champernowne noticed this!

In the megablock n viewed circularly, each block of length n occurs exactly n times at different positions modulo n.

000 001 010 011 100 101 110 111	000 occurs three times,
000 001 010 011 100 101 110 111	at different positions modulo 3
000 001 010 011 100 101 110 111	
000 001 010 011 100 101 110 111	001 occurs three times
000 001 010 011 100 101 110 111	at different postions modulo 3
000 001 010 011 100 101 110 111	

...

However, not every permutation of the blocks of length n has the property:

$$00\ 10\ 11\ 01$$

Perfect necklaces

Definition (Alvarez, Becher, Ferrari and Yuhjtman 2016)

A necklace over a b-symbol alphabet is (n, k)-perfect if each block of length n occurs k times, at different position modulo k for any convention of the starting point.

De Bruijn sequences are exactly the $(n, 1)$-perfect sequences.

The (n, k)-perfect necklaces have length kb^n.

Megablocks are perfect necklaces

Identify the blocks of length n over a b-symbol alphabet with the set of non-negative integers modulo b^n according to representation in base b.

Theorem (Alvarez, Becher, Ferrari and Yuhjtman 2016)

Let r coprime with b. The concatenation of blocks corresponding to the arithmetic sequence $0, r, 2r, ..., (b^n - 1)r$ yields an (n, n)-perfect necklace.

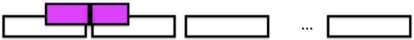
With $r = 1$ we obtain the lexicographically ordered sequence, this is the magablock for length n.

Megablocks for length n are perfect necklaces

A bijection $\sigma : \{0, ..b-1\}^n \to \{0, ..b-1\}^n$ is a cycle if $\{\sigma^j(w) : j = 0, ..., b^n - 1\}$ is the set of all blocks of length n.

Lemma

Let σ be a cycle over blocks of length n and let v be one block. The necklace $[\sigma^0(v)\sigma^1(v)\ldots\sigma^{b^n-1}(v)]$ is (n,n)-perfect if and only if for every $\ell = 0, 1, \ldots n-1$ for every block x of length ℓ and for every block y of length $n-\ell$, there is a unique block w of length n such that $w(n-\ell \ldots n-1) = x$ and $(\sigma(w))(0 \ldots n-\ell-1) = y$.

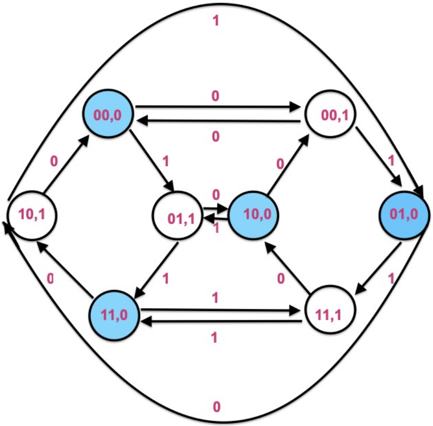


For every length-n block splitted in two parts, there is exactly one matching in the cycle (a tail of a block and the head of next block).

Astute graphs

Fix b-symbol alphabet. The astute graph $G_{b,n,k}$ is directed, with kb^n vertices.
The set if vertices is $\{0,..b-1\}^n \times \{0,..,.k-1\}$.
An edge $(w,m) \to (w',m')$ if $w(2,,n) = w'(1..n-1)$ and $(m+1) \mod k = m'$

This is $G_{2,2,2}$



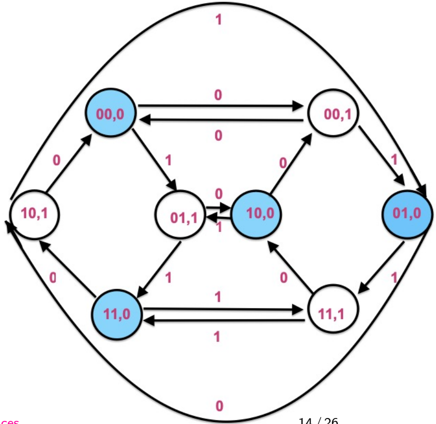
Astute graphs

Observation

$G_{b,n,1}$ is the de Bruijn graph of blocks of length n over b-symbols.

Observation

$G_{b,n,k}$ is Eulerian because it is strongly regular and strongly connected.



Eulerian cycles in astute graphs

Each Eulerian cycle in $G_{b,n-1,k}$ gives one (n,k)-perfect necklace.

Each (n,k)-perfect necklace can come from many Eulerian cycles in $G_{b,n-1,k}$

Theorem (Alvarez, Becher, Ferrari and Yuhjtman 2016)

The number of (n,k)-perfect necklaces over a b-symbol alphabet is

$$\frac{1}{k}\sum_{d_{b,k}|j|k} e(j)\varphi(k/j)$$

where

- $d_{b,k} = \prod p_i^{\alpha_i}$*, such that* $\{p_i\}$ *is the set of primes that divide both* b *and* k*, and* α_i *is the exponent of* p_i *in the factorization of* k*,*
- $e(j) = (b!)^{jb^{n-1}} b^{-n}$ *is the number of Eulerian cycles in* $G_{b,n-1,j}$
- φ *is Euler's totient function*

Normal sequences as sequences of Eulerian cycles

Theorem (proved first by Ugalde 2000 for de Bruijn)

The concatenation of (n,k)-perfect necklaces over a b-symbol alphabet, for increasing (n,k) –at most arithmetically– is normal to the b-symbol alphabet.

Proof of Theorem

A number is normal to base b if in its base-b expansion every block of digits occurs with the same limiting frequency as every other block of the same length.

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A number is normal to base b if in its base-b expansion every block of digits occurs with the same limiting frequency as every other block of the same length.

Instead, an equivalent formulation of normality but simpler to test:

Lemma (Piatetski-Shapiro 1951)

A sequence $a_1 a_2 a_3 \ldots$ is normal to a b-symbol alphabet if and only if there is positive constant C such that for every bock w,

$$\limsup_{n \to \infty} \frac{\textit{number of occurrences of } w \textit{ in } a_1 \ldots a_n}{n} < C\ b^{-|u|}.$$

To prove that the sequence of megablocks is normal the count at an anbritrary position is bounded by considering the count at the end of the megablock.

$\square$

The concatenation of (n, n)-perfect necklaces, n linearly increasing, is normal. Applying the same modification that Champernowne did we also obtain his result.

Corollary

Champernowne's sequence $0.12345678910112\ldots$ *is normal to base* 10.

End of the first part of the talk

Speed of convergence to normality

- A real x is normal to base b if the fractional parts of $x, bx, b^2x, \ldots$, that is $(b^n x \mod 1)_{n \geq 0}$, is uniformly distributed in the unit interval, Wall 1949.

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- A sequence $(x_n)_{n\geq 1}$ is uniformly distributed in the unit interval if $D_N((x_n)_{n\geq 1}) = \sup_{[\alpha,\beta)} \left| \frac{\#\{n \leq N : x_n \in [\alpha,\beta)\}}{N} - \gamma \right|$ goes to 0 as N to ∞.

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- Schmidt 1972 proved that there is constant C such that for every $(x_n)_{n \geq 1}$ there are infinite Ns, $D_N((x_n)_{n \geq 1}) > C \frac{\log N}{N}$.
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- It is still unknown whether the optimal order of discrepancy can be achieved by $(b^n x \mod 1)_{n\geq 0}$ for some real x, Korobov 1956.

- The lowest discrepancy known for $(b^n x \mod 1)_{n\geq 0}$ is $O((\log N)^2/N)$ for a real x constructed by M. Levin 1999 using the Pascal triangle modulo 2.

Nested perfect necklaces

Definition (Becher and Carton 2019)

A sequence over a b-symbol alphabet is a nested (n, k)-perfect necklace if it is (n, k)-perfect and, in case $n > 1$, it is the concatenation of b nested $(n - 1, k)$-perfect necklaces.

For example, for alphabet $\{0, 1\}$, the following is a nested $(2, 2)$-perfect necklace

$$0011\ 0110$$

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Nested (n, k)-perfect necklaces are pointed, which means an initial position

Nested perfect necklaces

These are nested $(2,4)$-perfect necklaces:

$$00001111\ 01011010$$
$$00111100\ 01101001$$
$$00011110\ 01001011$$
$$00101101\ 01111000$$

The concatenation of the first two is a nested $(3,4)$-perfect necklace.
The concatenation of the last two is a nested $(3,4)$-perfect necklace.
The concatenation of all of them is a nested $(4,4)$-perfect necklace.

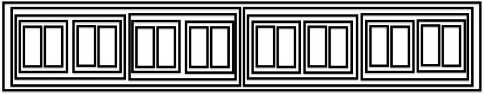
Nested perfect necklaces

Observation

Assume a b-symbol alphabet. For x a nested (n, n)-perfect necklace,

• *Since x is (n, n)-perfect, each block of length n occurs n times in x, at different positions modulo n.*

• *Since x is nested, for every $i = 1, \ldots n$, x is the concatenation of b^{n-i} nested (i, n)-perfect necklaces. So, in the prefix of x of length cnb^i each block of length i occurs $cn \pm \epsilon$ times with $\epsilon \leq 1$ ($c \pm \epsilon$ times at positions with the same congruence modulo n).*



Levin's constant and nested perfect necklaces

Theorem (Becher and Carton 2019)

The binary expansion of the number x defined by Levin 1999 using the Pascal triangle matrix modulo 2 is the concatenation of nested $(2^d, 2^d)$-perfect necklaces for $d = 0, 1, 2, \ldots$.

Theorem (Becher and Carton 2019)

For $d = 0, 1, 2, \ldots$ there are $2^{2^d - 1}$ binary nested $(2^d, 2^d)$-perfect necklaces obtained by column rotations of the Pascal triangle matrix modulo 2.

Nested perfect necklaces

Lemma

Consider the concatenation of a nested (n, n)-perfect necklace and a nested $(2n, 2n)$-perfect necklace. In any segment of length nb^n each block of length n occurs $n \pm \epsilon$ times at different positions modulo n, with $\epsilon \leq 2$.

Nested perfect necklaces

Lemma

Consider the concatenation of a nested (n, n)-perfect necklace and a nested $(2n, 2n)$-perfect necklace. In any segment of length nb^n each block of length n occurs $n \pm \epsilon$ times at different positions modulo n, with $\epsilon \leq 2$.

Leading idea with $b = 2$:

A nested $(2n, 2n)$-perfect necklace is equal to
2 nested $(2n-1, 2n)$-perfect necklaces are equal to
2^2 nested $(2n-2, 2n)$-perfect necklaces are equal to
...
2^n nested $(n, 2n)$-perfect necklaces are equal to
2^{n+1} nested $(n-1, 2n)$-perfect necklaces.

Each nested $(n-1, 2n)$-perfect necklace has length $2n2^{n-1} = n2^n$, and every block of length $(n-1)$ occurs $2n$ times, necessarily half followed by 0, the other half followed by 1. Thus, in a nested $(n-1, 2n)$-perfect necklace every block of length n occurs n times. $\square$

Nested perfect necklaces and low discrepancy

Theorem (Becher and Carton 2019)

Let b be a prime number. Every number x whose base-b expansion is the concatenation of nested $(2^d, 2^d)$-perfect necklaces for $d = 0, 1, 2 \ldots$ satifies $D_N((b^n x \mod 1)_{n\geq 0})$ is $O((\log N)^2/N)$.

We could not prove that it holds for arbitrary integer bases.

Open problems

- Give a graph interpretation to nested perfect necklaces
- Study perfect necklaces in higher dimensions
- Is there a Martin-Löf random real x such that for every N, $D_N((2^n x \mod 1)_{n\geq 1})$ is $O((\log N)^2/N)$?

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