GEOMETRIC GROUP THEORY — LECTURE NOTES

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CONTENTS

1. INTRODUCTION

1.1. **The geometry of infinite groups.** Why study the geometry of infinite groups? One motivation comes from Riemannian geometry. One takes a special interest in homogeneous spaces, and in particular, in symmetric spaces, which are quotients of finite dimensional Lie groups by compact subgroups. This theory has many connections with algebraic geometry, representation theory and other areas of mathematics. On the other hand, the range of examples is limited, and it is natural to look for a broader collection of spaces to work with.

Infinite groups give rise to an enourmous supply of homogeneous geometric spaces to study. There are many natural functors in geometry or topology which assign a group to a space. By studying the essential geometric properties of the group so obtained one acquires insight into the geometry of the original space, with all the small–scale thorny or cranky phenomena factored out.

On the other hand, groups are almost *too* general to study. Markov's theorem on the undecidability of the word problem for general finitely presented groups discourages one from trying to develop a theory which encompasses *all* the geometry, and hence the algorithmic complexity, of such groups, and indicates that there are definite limits to what one should hope to accomplish.

This motivates us to consider coarser equivalence relations on groups, and to study proper subclasses of groups for which one can hope to have a more informative theory.

One essential requirement is to define *computable invariants*. Another is to understand the relationship of geometry with algebraic properties, like having a finite presentation.

2. Coarse geometry

2.1. **Hausdorff and Lipschitz geometry.** There are two basic ingredients involved in coarse geometry:

(1) Ignore everything on a sufficiently small scale
(2) Allow linear changes of global scale

The first idea is captured by the Hausdorff distance, the second by Lipschitz geometry.

Definition 2.1.1. Let A, B be closed subsets of a metric space X. The *Hausdorff distance* between A and B, denoted $d_{\text{Haus}}(A, B)$ is defined by

$$d_{\text{Haus}}(A, B) = \inf\{k | N_k(A) \subset B \text{ and } N_k(B) \subset A\}$$

If no such k exists, the distance is infinite.

Here $N_k(A)$ denotes the k–neighborhood of the set A. That is, the set of points in X within distance k to some point in A.

Notice that if X is *compact* and $d_{\text{Haus}}(A, B) = t$ is finite, then the infimum is actually realized. It is easy to see that a sequence $A_i \subset X$ converges to A with respect to d_{Haus} iff the following two conditions are satisfied:

(1) For all $a \in A$ there are points $a_i \in A_i$ with $a_i \to a$
(2) For all $a_i \in A_i$ with $a_i \to a$, we must have $a \in A$

For a metric space X, let $\mathscr{C}(X)$ denote the set of closed subsets of X.

Theorem 2.1.2 (Hausdorff)**.** *Let X be a metric space. Then d_{Haus} defines a metric on $\mathscr{C}(X)$, which possibly takes on infinite values. Moreover, if X is compact, then with respect to d_{Haus} the space $\mathscr{C}(X)$ is compact.*

Proof. Suppose $d_{\text{Haus}}(A, B) = 0$. Then for every point $p \in A$ there is a sequence $p_i \in B$ with $d_X(p, p_i) \to 0$. Since B is compact, $p \in B$, and $A \subset B$. By symmetry, $B \subset A$, so they are equal. The symmetry of d_{Haus} is evident, so one just needs to check the triangle inequality to verify that it is a well–defined distance function. But if $A, B, C \in \mathscr{C}(X)$ with $d_{\text{Haus}}(A, B) = r$ and $d_{\text{Haus}}(B, C) = s$ then every point in A is within distance r of some point in B, and every point in B is within distance s of some point in C, so $A \subset N_{r+s}(C)$.

The key nontrivial point is to show that if X is compact, then so is $\mathscr{C}(X)$. Let U_j be a countable basis for the topology of X. Then for each j, if $A_i \cap U_j \neq \emptyset$ for infinitely many i, we can pass to a subsequence where $A_i \cap U_j \neq \emptyset$ for all but finitely many i. We choose a diagonal subsequence i_n of i with the property that for every open set U_j, either only finitely many A_i intersect U_j, or all but finitely many A_i intersect U_j. Let A be the set of points a with the property that every U_j containing a intersects all but finitely many A_i. Then A is compact and satisfies the properties for convergence $A_{i_n} \to A$. $\square$

Note in fact that for X merely separable and locally compact we can metrize the one point compactification and define a topology on $\mathscr{C}(X)$ for which is it compact and metrizable.

Definition 2.1.3. We say that a complete metric space X is *Hausdorff equivalent* to a metric space X', denoted $X \sim_H X'$, if there is a metric space Y and isometric embeddings $i_X : X \to Y$, $i_{X'} : X' \to Y$ such that

$$d_{\text{Haus}}(i_X(X), i_{X'}(X')) < \infty$$

with respect to the Hausdorff distance on $\mathscr{C}(Y)$.

Note that we are free to choose both the space Y and the isometric embeddings $i_X, i_{X'}$.

Definition 2.1.4. Let X, X' be Hausdorff equivalent complete metric spaces. The *Gromov–Hausdorff distance* between X and X', denoted $d_{\text{Haus}}(X, X')$, is the infimum with respect to all Y and isometric embeddings $i_X, i_{X'}$ of $d_{\text{Haus}}(i_X(X), i_{X'}(X'))$.

The analogue of Hausdorff's theorem for this generalized distance function is the following theorem of Gromov:

Theorem 2.1.5 (Gromov)**.** *The set of all isometry classes of complete compact metric spaces is a locally compact metric space with respect to d_{Haus}.*

Definition 2.1.6. A map $f : X \to Y$ between metric spaces is *k–Lipschitz* for some $k \geq 0$ if for all $p, q \in X$ we have

$$d_Y(f(p), f(q)) \leq k d_X(p, q)$$

That is, f increases distances by a factor of at most k.

For example, a map is 1–Lipschitz iff it is distance non–decreasing. Notice that a Lipschitz map is continuous. A map is *Lipschitz* if it is k–Lipschitz for some k.

Definition 2.1.7. We say that a complete metric space X is *Lipschitz equivalent* to a metric space X', denoted $X \sim_L X'$, if there is a Lipschitz homeomorphism $f : X \to X'$ with a Lipschitz inverse.

Definition 2.1.8. Two metric spaces X, Y are *quasi–isometric* if there are spaces X', Y' such that

$$X \sim_H X' \sim_L Y' \sim_H Y$$

We denote this $X \sim_Q Y$.

By writing out the definitions, on sees that two spaces X, Y are quasi–isometric if there exists a $k \geq 1, \epsilon \geq 0$ and $f : X \to Y$ satisfying the following two properties:

(1) f is *coarsely onto*. That is, there is a $\delta > 0$ such that for all $p \in Y$, there is $q \in X$ with $d_Y(p, f(q)) \leq \delta$.
(2) The following inequality holds:

$$\frac{1}{k} d_X(p, q) - \epsilon \leq d_Y(f(p), f(q)) \leq k d_X(p, q) + \epsilon$$

It is easy to see from this second, equivalent definition that $\sim_Q$ defines an *equivalence relation* on metric spaces.

A map $f : X \to Y$ satisfying the second condition but not necessarily the first is called a quasi–isometry. Equivalence classes of quasi–isometric spaces, and quasi–isometries between them form a category which is much easier to work with than the category of metric spaces and isometries. Throughout the sequel, it will frequently be useful to work in this category.

Definition 2.1.9. Let X be a metric space. A *δ–net* is a subset $S \subset X$ such that $N_\delta(S) = X$.

In particular, a map $f : X \to Y$ is coarsely onto iff the image is a δ–net for some δ.

2.2. **Groups and generating sets.**

Definition 2.2.1. A *presentation* of a group G is a collection of generators $g_i, i \in I$ and a collection of finite words in the generators and their inverses $w_j, j \in J$ so that if F denotes the free group generated by letters g_i, and N denotes the normal subgroup generated by the words w_j, there is an exact sequence

$$0 \to N \to F \to G \to 0$$

If I is finite, we say G is *finitely generated*, and if both I and J are finite, we say G is *finitely presented*.

We denote a finitely presented group in the following way:

$$G = \langle g_1, g_2, \ldots, g_n | w_1, w_2, \ldots, w_m \rangle$$

The set $S = \{g_1, g_2, \ldots, g_n\}$ is called a *generating set* for G.

Definition 2.2.2. The *Cayley graph of G with respect to a generating set S*, denoted $C(G)_S$, is the graph with one vertex for each element of G, and an edge between $a, b \in G$ if $b = g_i a$ for some $g_i \in S$.

It is also possible to think of $C(G)_S$ as an oriented graph. Paths in the 1–skeleton of G beginning at the identity correspond to words in the generators and their inverses, and two such words represent the same element of G if they end at the same vertex.

We think of $C(G)_S$ as a *path metric space*, by declaring that the length of every edge is 1. With respect to this metric, the distance between two vertices $a, b \in G$ is equal to the length of the shortest word in the generators which represent the element ba^{-1}.

Now, for a given group G, there might be many different choices of a finite generating set. The following theorem shows that such differences are immaterial in the quasi–isometry category:

Theorem 2.2.3. *Let G be a finitely generated group, and S_1, S_2 two generating sets. Then*

$$C(G)_{S_1} \sim_Q C(G)_{S_2}$$

Proof. Let f be a map from $C(G)_{S_1}$ to $C(G)_{S_2}$ which is the identity on the zero skeleton. Each generator g_i^1 of S_1 can be expressed as a word w_i^2 in the generators g_j^2 of S_2 and their inverses. So if a, b are elements of G satisfying $ba^{-1} = g_i^1$, we map the corresponding edge in $C(G)_{S_1}$ to the path in $C(G)_{S_2}$ between a and b corresponding to the word w_i.

This map is obviously coarsely onto, since it maps onto the 0–skeleton of $C(G)_{S_2}$, which is a $1/2$–net for $C(G)_{S_2}$. To see that it is a quasi–isometry, rewrite each generator g_i^1 in S_1 as a word w_i^2 and each generator g_i^2 as a word w_i^1. Let

$$k = \max\{|w_1^1|, |w_2^1|, \ldots, |w_1^2|, |w_2^2|, \ldots\}$$

where for a word w, $|w|$ denotes the number of letters in w. Then every path w in $C(G)_{S_1}$ can be approximated by a path of length kw in $C(G)_{S_2}$, by replacing each edge in one Cayley graph by the corresponding word in the other. In particular, f is a k bilipshitz map between 0–skeleta, and therefore extends to a quasi–isometry, as claimed. $\square$

It follows that we can define unambiguously the quasi–isometry type of a finitely generated group. If G, H are finitely generated groups, we write $G \sim_Q H$ and say G, H are quasi–isometric if they have quasi–isometric Cayley graphs with respect to some choice of generating sets.

Theorem 2.2.4. *Suppose G is finitely generated by S, and acts properly, isometrically, and cocompactly on some complete Riemannian manifold M. Then $C(G)_S \sim_Q M$.*

Proof. Let $p \in M$ be some point. Then we map $f : G \to M$ by $g \to g(p)$. Since G sits in $C(G)_S$ as a $1/2$–net, it suffices to prove that $G \sim_Q M$.

The map f is coarsely onto, since G is co–compact and therefore the diameter of M/G is finite. Moreover, if $g_i \in S$, then $\max_i d_M(p, g_i(p)) = t$ is finite, so f is t–Lipschitz. It remains to show that there are constants k, ϵ such that if $d_M(p, g(p)) = d$, then g can be expressed as a word of length $\leq k|d| + \epsilon$ in the generators of G and their inverses.

Let γ be a geodesic joining p to $g(p)$, and let $p_1, p_2, \ldots, p_{\lfloor d \rfloor}$ be points spaced 1 apart on γ, which start at p and end within distance 1 of $g(p)$. For each p_i, let σ_i be a path of length $\leq$ diam(M/G) from p_i to some $h_i(p)$ where $h_i \in G$. Then we can estimate

$$|\sigma_{i-1} \cup \gamma[i-1, i] \cup \sigma_i| \leq 2\text{diam}(M/G) + 1$$

Since G is cocompact, there is some k such that there are only k translates of p within distance $2\text{diam}(M/G) + 1$. It follows that $h_i h_{i-1}^{-1}$ can be expressed as a word of length at most k in the generators of G and their inverses. In particular, g can be expressed as a word of length $\leq k(\lfloor d \rfloor + 1)$ and the theorem is proved. □

Example 2.2.5. If N is a closed Riemannian manifold, then $\pi_1(N)$ is quasi–isometric to the universal cover $\widetilde{N}$.

Lemma 2.2.6. *Let G be finitely generated, and let $H < G$ be a subgroup with finite index $[G : H] = k < \infty$. Then H is finitely generated, and $H \sim_Q G$.*

Proof. Let $\{g_1, \ldots, g_n\}$ be a generating set for G. Since H has finite index in G we can choose a finite set of coset representatives $s_1, \ldots, s_k$, expressed as words in the g_i and their inverses of length $\leq n$ for some n. Let $h \in H$ be expressed as a word $w = a_1 a_2 \ldots a_m$ in the g_i and their inverses. Let w_i denote the subword of w of the form $a_1 a_2 \ldots a_i$. Then $w_i = h_i s_{j_i}$ for some $h_i \in H$ and some coset s_{j_i}. So we rewrite w as the following word

$$w_h = a_1 s_{j_1}^{-1} s_{j_1} a_2 s_{j_2}^{-1} s_{j_2} \ldots a_m$$

Since $w \in H$, we actually have $s_{j_m} = \text{id}$. Now notice that each subword $s_{j_l} a_{l+1} s_{j_{l+1}}^{-1}$ is a word of length $\leq 2n+1$ in the g_i and their inverses which is in H. So every element of H can be expressed as a product of elements of H of length a most $2n+1$ in the generators of G and their inverses. In particular, H is finitely generated, and moreover with respect to this generating set S_H we estimate that for all $h_1, h_2 \in H$,

$$\frac{1}{2n+1} d_{C(G)_S}(h_1, h_2) \leq d_{C(H)_{S_H}}(h_1, h_2) \leq d_{C(G)_S}(h_1, h_2)$$

so the inclusion of H into G is a quasi–isometry.

Moreover, the inclusion $H \to G$ is coarsely onto, so we are done. □

Definition 2.2.7. Two groups G_1, G_2 are *commensurable* if they contain isomorphic subgroups of finite index.

Example 2.2.8. If G_1, G_2 are finitely generated and commensurable, then $G_1 \sim_Q G_2$.

Definition 2.2.9. Let G be a semisimple Lie group. A *lattice* $\Gamma < G$ is a discrete subgroup with finite covolume. Γ is *uniform* if it is cocompact.

In particular, all uniform lattices in a given semisimple Lie group are quasi–isometric to each other. On the other hand, there may be many different quasi–isometry classes of *nonuniform lattices* in a given semisimple Lie group. This subtlety is very important.

Example 2.2.10. Let $\mathbb{H}^2$ be described as the upper half–plane, which is a homogeneous space for the group $\text{SL}(2, \mathbb{R})$. Thinking of $\mathbb{H}^2$ as the set of complex numbers with imaginary part > 0, the action of $\text{SL}(2, \mathbb{R})$ is the standard projective action:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

The kernel of the action is the subgroup $\pm\mathrm{Id}$ and the stabilizer of a point is the subgroup $\mathrm{SO}(2,\mathbb{R}) < \mathrm{SL}(2,\mathbb{R})$. So we can identify $\mathbb{H}^2$ with the space of left cosets

$$\mathbb{H}^2 = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$$

The subgroup $\mathrm{SL}(2,\mathbb{Z}) < \mathrm{SL}(2,\mathbb{R})$ is discrete, and acts on $\mathrm{SL}(2,\mathbb{R})$ on the left in a way which commutes with the action of $\mathrm{SO}(2,\mathbb{R})$ on the right. So the action descends to an action on $\mathbb{H}^2$ where the action is still discrete, since $\mathrm{SO}(2,\mathbb{R})$ is compact. A fundamental domain D for the action is the intersection of the vertical strip of complex numbers with real part between $-\frac{1}{2}$ and $\frac{1}{2}$, and the complex numbers of positive imaginary part, and absolute value at least 1. The boundary of this fundamental domain is a (partially ideal) hyperbolic polygon with 3 sides, two of them sharing the same ideal vertex. Subdivide the bounded edge by adding a new vertex at i. This makes D into a (degenerate) hyperbolic quadrilateral with one ideal vertex. Then the action of $\mathrm{SL}(2,\mathbb{Z})$ identifies the four sides of ∂D in pairs. The quotient orbifold is the $2,3,\infty$ triangle orbifold. This is noncompact, so $\mathrm{SL}(2,\mathbb{Z})$ is a nonuniform lattice.

The edge–pairing maps are the elements

$$\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Notice that though β has order 4 in $\mathrm{SL}(2,\mathbb{Z})$, it has order 2 in $\mathrm{PSL}(2,\mathbb{Z})$. The elements α,β generate $\mathrm{PSL}(2,\mathbb{Z})$ and give it the presentation

$$\mathrm{PSL}(2,\mathbb{Z}) = \langle \alpha,\beta | \beta^2 = \mathrm{id}, (\beta\alpha)^3 = \mathrm{id}\rangle$$

There is an obvious surjective homomorphism from $\mathrm{PSL}(2,\mathbb{Z})$ to $\mathrm{PSL}(2,\mathbb{Z}/2\mathbb{Z})$ given by considering matrices mod 2; the kernel of this homomorphism is isomorphic to the free group on 2 generators.

We will see in the sequel that the free group and a closed surface group are *not* quasi–isometric.

Example 2.2.11 (Bieberbach theorem). Let G be a discrete group of isometries of Euclidean space $\mathbb{E}^n$. Then G contains a free abelian subgroup of finite index. Moreover, if G is cocompact, then G contains a subgroup isomorphic to $\mathbb{Z}^n$ of finite index. In particular, if G_1, G_2 are cocompact groups of isometries of $\mathbb{E}^n$, then G_1 and G_2 are abstractly commensurable, and therefore quasi–isometric to each other and to $\mathbb{E}^n$.

Example 2.2.12. Let G_1, G_2 be two cocompact groups of isometries of $\mathbb{H}^3$. Then G_1 and G_2 are not always commensurable. To see this we will distinguish them by means of the geometry of the corresponding quotient orbifolds $M_i = \mathbb{H}^3/G_i$. Mostow rigidity implies that if G_1, G_2 are abstractly commensurable, they contain subgroups which are conjugate *in Isom*$(\mathbb{H}^3)$.

If G is a cocompact group of isometries of $\mathbb{H}^3$, then a fundamental dichotomy due to Margulis implies that either G is *arithmetic*, or else the commensurator is *discrete*. That is, there is a *maximal discrete* subgroup $G' < \mathrm{Isom}^+(\mathbb{H}^3)$ for each commensurability class such that every element G of the commensurability class is conjugate to a subgroup of G'. In particular, for each non–arithmetic commensurability class, there is a lower bound on the length of the shortest closed geodesic.

Moreover, a theorem of Borel implies that there are only finitely many arithmetic groups with covolume less than any given bound, so for any given $V > 0$, there is a lower bound on the length of the shortest geodesic in an arithmetic orbifold with volume $\leq V$.

For a fixed G_1, to find an incommensurable G_2, one just needs to find infinitely many G_i such that the quotients $M_i = \mathbb{H}^3/G_i$ are orbifolds with shorter and shorter closed geodesics, while at the same time vol$(M_i) \leq V$ for some V. If N is a hyperbolic manifold with a cusp, Thurston's hyperbolic Dehn surgery theorem implies that for all but finitely many Dehn fillings $N(p/q)$ of N, the resulting manifold is hyperbolic with vol$(N(p,q)) <$ vol(N). Moreover, the length of the core geodesic of the filling torus $\to 0$ as $p, q \to \infty$ in Dehn surgery space. In particular, the resulting manifolds $N(p/q)$ have fundamental groups which fall into infinitely many commensurability classes.

In fact, even stronger than this, one obtains the following observation of Craig Hodgson:

Theorem 2.2.13 (Hodgson)**.** *Let M_i be the manifolds obtained by hyperbolic Dehn surgery on some hyperbolic knot. Then there are only finitely many M_i in every commensurability class.*

On the other hand, all the groups $\pi_1(M_i)$ are quasi–isometric to $\mathbb{H}^3$ and therefore to each other.

2.3. **Large scale connectedness.** We begin to develop some invariants of quasi–isometry type. Notice that since a quasi–isometry is typically not even continuous, classical algebraic topological invariants like homotopy groups or homology groups are not well–defined at the level of quasi–isometry type. On the other hand, one can introduce coarse versions of these invariants, which detect obstructions to connectedness at large scales.

Definition 2.3.1. A *thickening* of a metric space X is an isometric embedding $i : X \to Y$ which is coarsely onto.

Definition 2.3.2. A metric space X is called *large scale k–connected* if for every thickening Y of X there is a thickening Z of Y such that every map $\sigma : S^l \to Y$ can be extended to a map $\sigma : B^{l+1} \to Z$ for $l \leq k$.

Theorem 2.3.3. *A subgroup G of a finitely presented group H is finitely generated iff it is large scale connected with respect to the metric induced as a subspace of H. A finitely generated group G is finitely presented iff it is large scale simply–connected.*

Proof. We prove the second statement first.

Suppose G is finitely generated by S. Let X be a thickening of $C(G)_S$, so that there is a t with $N_t(C(G)_S) = X$. Let $\gamma : S^1 \to X$ be a loop. Then every point in γ can be joined to a vertex of $C(G)_S$ by a path of length $\leq t$, so we subdivide γ in $\lfloor |\gamma| \rfloor$ segments of length ≤ 1 and join the vertices to vertices of $C(G)_S$. Adjacent vertices of the subdivision are joined to pairs of points in $C(G)_S$ which are distance at most $2t + 1$ apart, and therefore they are joined by paths in $C(G)_S$ of length at most $2t+1$. It follows that by adding cones to every loop in X of length $\leq 4t + 2$ we can build a new thickening Y in which every such γ can be replaced by a simplicial $\gamma' : S^1 \to C(G)_S$.

Since G is finitely presented, every simplicial loop in the 1–skeleton is a word in S and its inverses which can be reduced to a product of commutators of finitely many relations:

$$w = g_1 r_{i_1}^{\pm 1} g_1^{-1} g_2 r_{i_2}^{\pm 1} g_2^{-1} \ldots g_k r_{i_k}^{\pm 1} g_k^{-1}$$

Where each r_{i_l} is one of $r_1, r_2, \ldots, r_m$. Since there is a uniform bound T on the length of each r_m, by attaching cones to each loop of length $\leq T$ we get a new thickening Z such that every conjugate of r_m bounds a disk. It follows that γ' is contractible in Z, and we are done.

Conversely, if there is a thickening X of $C(G)_S$ such that every loop in $C(G)_S$ can be contracted in X, then there is a t such that every loop $\gamma : S^1 \to C(G)_S$ bounds a disk $f : D \to N_t(C(G)_S)$. Let τ be a triangulation of D such that each edge gets mapped to an edge of length at most 1. Then we can join each vertex of the triangulation to a vertex of G by a path of length $\leq t$. As before, vertices of τ which share an edge, are joined to vertices of G which are distance at most $2t+1$ apart, and therefore can be joined by a path in $C(G)_S$ of length at most $2t+1$. It follows that there is a map $f' : \tau^1 \to C(G)_S$ from the 1–*skeleton* of the triangulation of D, whose boundary agrees with γ, and with the property that every edge gets mapped to a path of length at most $2t+1$. But this triangulation lets us express the word corresponding to w as a product of conjugates of words which correspond to loops of length $\leq 6t+3$. In particular, G is finitely presented by the words in the generators and their inverses which evaluate to id and which are of length $\leq 6t+3$. In particular, G is finitely presented.

If $G < H$ where H is finitely presented, and G is large scale connected, then for any two $g_1, g_2 \in G$ there is a bound t and elements

$$g_1 = g_{11}, g_{12}, \ldots, g_{1m} = g_2$$

such that $d(g_{1i}, g_{1j}) \leq t$. But then G is generated by the paths in H of length $\leq t$ which start at the identity and end at a vertex of G. In particular, G is finitely generated. Conversely, if G is finitely generated, every word $g \in G$ can be expressed as a product in a finite set of generators g_i and their inverses. Each g_i can be realized as a path in H of bounded length, so G is large scale connected. □

Let $G = \langle S|R \rangle$ be a finitely presented group. Let K^1 be a wedge of circles, with one circle for each generator $g_i \in G$. Let $K = K^2$ be obtained from K^1 by attaching one 2–cell for each relation $r_i \in R$. Then $\pi_1(K) = G$, and the universal cover $\widetilde{K}$ is a 2–complex whose 1–skeleton is exactly $C(G)_S$, and on which G acts properly discontinuously, cellularly, and co–compactly. A description of a reduced word w representing a loop as a product of conjugates of relations

$$w = g_1 r_{i_1}^{\pm 1} g_1^{-1} g_2 r_{i_2}^{\pm 1} g_2^{-1} \ldots g_k r_{i_k}^{\pm 1} g_k^{-1}$$

corresponds to a combinatorial map of a disk to $\widetilde{K}$. Since $\widetilde{K}$ is simply connected, the simplicial approximation theorem says that such disks exist for all loops in $C(G)_S$. For a reduced loop w, such a combinatorial disk is called a *van–Kampen diagram for w*.

Definition 2.3.4. A group G is of type $\mathbf{F}_n$ if it admits a $K(G,1)$ complex $\mathbf{X}$ with finite n–skeleton. A group G is of type $\mathbf{FP}_n$ if and only if $\mathbb{Z}$ has a projective resolution as a trivial $\mathbb{Z}G$ module that is finite through dimension n.

It is clear that $\mathbf{F}_n$ implies $\mathbf{FP}_n$, and $\mathbf{FP}_1$ implies $\mathbf{F}_1$. This last property is precisely the property that G is finitely generated. $\mathbf{F}_2$ is the property that G is finitely presented. It is not too hard to generalize theorem 2.3.3 to show that G satisfies $\mathbf{F}_{n+1}$ iff it is large scale n–connected. Bestvina and Brady showed that $\mathbf{FP}_2$ does not imply $\mathbf{F}_2$, by considering right–angled Artin groups. Here a *right–angled Artin group* is a group with a presentation $G = \langle g_1, \ldots, g_n | r_1, \ldots, r_m \rangle$ where every r_i is a commutator of two generators.

3. Groups acting on trees

3.1. **Graphs of groups.**

Definition 3.1.1. A *graph of groups* is a connected graph Γ together with the following structure:

(1) For each vertex v, an associated group G_v
(2) For each edge e, an associated group G_e and a pair of injective homomorphisms

$$i_{v_1} : G_e \to G_{v_1}, i_{v_2} : G_e \to G_{v_2}$$

for the two vertices v_1, v_2 of e

Example 3.1.2. If Γ is a single edge e with vertices v_1, v_2 then we can define the *amalgamated free product* G of the vertex groups over the edge groups, denoted

$$G = G_{v_1} *_{G_e} G_{v_2}$$

to be the group defined by the presentation

$$G = \langle G_{v_1}, G_{v_2} | i_1(G_e) = i_2(G_e) \rangle$$

Theorem 3.1.3. *Let $G = A *_C B$ be an amalgamated free product. Then the natural inclusion homomorphisms from A and B to G are injective.*

Proof. Let $A_C = \{\alpha_1, \dots\}$ be a set of nonzero right coset representatives of $i_A(C)$ in A, and $B_C = \{\beta_1, \dots\}$ a set of nonzero right coset representatives of $i_B(C)$ in B. A *reduced word* is a word of the form

$$w = c\alpha_{i_1}\beta_{i_1}\alpha_{i_2}\beta_{i_2}\dots\alpha_{i_n}$$

where $c \in C$, and where possibly $\alpha_{i_1}, \alpha_{i_n} = \text{id}$. It is clear that every element in G has such an expression. Let X denote the set of such expressions. It turns out that such an expression is unique.

To see this it suffices to define a transitive action of A and of B on X. Without loss of generality, let α_{i_1} be nontrivial. Let $a \in A$. There are two cases. If $ai_A(c)\alpha_{i_1} = i_A(c')\alpha_j$ where α_j is a nonzero right coset representative. Then

$$\begin{aligned} a \cdot w &= ac\alpha_{i_1}\beta_{i_1}\dots\alpha_{i_n} \\ &= c'\alpha_j\beta_{i_1}\dots\alpha_{i_n} \end{aligned}$$

On the other hand, if $ai_A(c)\alpha_{i_1} = i_A(c') \in i_A(C)$ then

$$a \cdot w = c'\beta_{i-1}\dots\alpha_{i_n}$$

If $b \in B$ then if $b \in B \backslash i_B(C)$, then $bi_B(c) = i_B(c')\beta_j$ for some β_j. Then

$$b \cdot w = c'\beta_j\alpha_{i_1}\dots\alpha_{i_n}$$

If $b \in B \cap i_B(C)$, then $bi_B(c) \in i_B(C)$, so

$$b \cdot w = bc\alpha_{i_1}\dots\alpha_{i_n}$$

If $a = i_A(c)$ and $b = i_B(c)$ for some c, then one can check that they act in the same way on X. So the action of $A * B$ on X factors through $A *_C B$. It is clear that the action is transitive, and therefore $A *_C B$ is in bijective correspondence with X. □

Example 3.1.4. If G_i is a family of groups with $i \in \{1, \dots, n\}$, and for some group A we have injective homomorphisms $i_{G_i} : A \to G_i$, then one can produce the amalgamated free product

$$G = *_A G_i$$

By induction on the number of G_i, it is clear that all the G_i inject into G. This example can be built up by induction from the previous example, since

$$(G_1 *_A G_2) *_A G_3 = (G_{s(1)} *_A G_{s(2)}) *_A G_{s(3)}$$

for any permutation s. G is associated to a graph of groups which is combinatorially just the interval $[1, n]$, with a vertex at every integer i and corresponding vertex group G_i, and every edge group isomorphic to A, where the inclusion maps are the injections i_{G_i}.

Example 3.1.5. Suppose Γ is a graph with one vertex v and one edge e with both ends at v. Then the we can define the *HNN extension G of the vertex group over the edge group* as follows. The edge defines two inclusions $i_1, i_2 : G_e \to G_v$. Then

$$G = \langle G_v, s | s i_1(G_e) s^{-1} = i_2(G_e) \rangle$$

This is related to infinite amalgamated free products in the following way. Take $\mathbb{Z}$ copies of G_v, labelled as G_v^i. Take $\mathbb{Z}$ copies of G_e, labelled as G_e^i. Let $\widetilde{\Gamma}$ be the universal cover of Γ. Then we can label the vertices of $\widetilde{\Gamma}$ by G_v^i, the edges with G_e^i, and inclusions

$$i_1 : G_e^i \to G_v^i, i_2 : G_e^i \to G_v^{i+1}$$

This defines an iterated amalgamated free product

$$\widetilde{G} = \langle G_v^i | i_1(G_e^i) = i_2(G_e^{i-1}) \rangle$$

There is an automorphism $\phi : \widetilde{G} \to \widetilde{G}$ which sends $G_e^i \to G_e^{i+1}$ in the obvious way. Then G is the semidirect product

$$G = \widetilde{G} \rtimes \mathbb{Z}$$

where the generator of $\mathbb{Z}$ acts on $\widetilde{G}$ by ϕ.

Notice that G_v injects into $\widetilde{G}$ and therefore also into G.

3.2. **The fundamental group of a graph of groups.** Let T be a simplicial tree, and let G act on T without inversions. That is, for each edge $e \in T$, the stabilizer of e stabilizes both the vertices. Then we can form the quotient graph $T/G = \Gamma$. Let S be a maximal tree for the quotient Γ, and let $\widetilde{S} \subset T$ be a lift. Let l_i be the edges of $\Gamma \backslash S$, and $\widetilde{l_i}$ some choice of lifts to T so that the union $\widetilde{S} \cup_i \widetilde{l_i}$ is a connected tree Z. For each $\widetilde{l_i}$, let $\widetilde{n_i}$ be the free vertex (i.e. the vertex not in $\widetilde{S}$) and $\widetilde{m_i}$ the attached vertex (i.e. the vertex in $\widetilde{S}$). Then the projection $Z \to S$ is the quotient map which identifies each $\widetilde{n_i}$ with $\widetilde{m_i}$.

For each vertex v of Γ, let G_v be the stabilizer of a lift $\widetilde{v}$ to Z, and similarly for each edge. The inclusion homomorphisms are the obvious ones for all edges e of S. For each edge l of $\Gamma \backslash S$ with vertex $v \in \Gamma$, the injective homomorphisms to G_v are defined as follows. Let $g \in G$ map $\widetilde{n_i}$ to $\widetilde{m_i}$. There are two natural injections of G_{l_i} into G_v defined by identifying G_v as the stabilzer of $\widetilde{n_i}$ and $\widetilde{m_i}$ respectively. These stabilizers are conjugate in G:

$$\text{stab}(\widetilde{m_i}) = g \text{stab}(\widetilde{n_i}) g^{-1}$$

This conjugation isomorphism defines two injective homomorphisms of G_{l_i} to G_v, unique up to inner conjugation. In particular, the isomorphism class of the graph of groups Γ so determined is unique.

Notice that if G acts simplicially on a tree T then it acts without inversions on the tree obtained from T by barycentric subdivision.

Definition 3.2.1. Let Γ be a (finite or infinite) graph of groups. We define the *fundamental group of* Γ as follows. Let T be a maximal tree in Γ. Then there is an iterated amalgamated free product defined by

$$G_T = \langle G_{v_i} | i^1_{e_j}(G_{e_j}) = i^2_{e_j}(G_{e_j}) \rangle$$

where each e_j is an edge in T, and $i^k_{e_j}$ are the two inclusions into the relevant vertex groups, for $k = 1, 2$. Let Y be the wedge of circles obtained by collapsing T to a vertex. Notice for each edge $e \in Y$ that the edge group includes into G_T, since every vertex group of T includes into G_T by theorem 3.1.3. We define a symbol s_{e_i} for each edge e_i of Y. Then we define

$$G = \langle G_T, s_{e_i} | s_{e_i} i^1_{e_i}(G_{e_i}) s_{e_i}^{-1} = i^2_{e_i}(G_{e_i}) \rangle$$

which is an iterated HNN extension of G_T.

Theorem 3.2.2. *Descriptions of G as the fundamental group of a graph of groups are in bijective correspondence with actions of G on simplicial trees without inversions, up to isomorphism.*

Proof. We have given the construction in one direction. It suffices to show that a graph of groups Γ arises from an action of its fundamental group G on a tree T. This tree T is easy to define. The vertices are exactly the conjugacy classes of the vertex groups G_v in G, and the edges are exactly the conjugacy classes of the edge groups G_e in G. It is clear that T is connected. To see that it is a tree one uses the uniqueness of the reduced word expressions for each amalgamated free product. We leave the details to the reader. □

Definition 3.2.3. A graph of groups Γ is *minimal* if for every edge group G_e, both of the inclusion homomorphisms to the associated vertex groups are proper. A graph of groups is *trivial* if the underlying graph is a single vertex.

A group action of G on a tree T is *nontrivial* if it has no global fixed point.

Theorem 3.2.4. *If G is finitely generated, G admits a nontrivial action on a tree iff G admits a presentation as a minimal, nontrivial graph of groups.*

Proof. If Γ is minimal and nontrivial, every vertex group is of infinite index in G. Consequently G acts nontrivially on the tree constructed in theorem 3.2.2.

Conversely, suppose G acts nontrivially on some T. By subdivision if necessary, we may assume G acts without inversions. Without loss of generality, we can assume T is a minimal invariant subtree, and we can take the quotient graph Γ. Since G is finitely generated, Γ is finite. If Γ is not minimal, there is some edge group G_e which is taken isomorphically to one of its associated vertex groups G_v by its inclusion homomorphism. Let w be the other vertex. Then $G_v *_{G_e} G_w \cong G_w$. If v is an extremal vertex, we can collapse the edge e to the vertex w, replacing the group G_v with G_w. If v is an internal vertex, we just remove v, amalgamating the two adjacent edges into a single edge. This gives a new graph of groups with the same fundamental group, and by construction we see that Γ either fixes some proper subtree of T, or acts on some tree obtained from T by erasing the lifts of v. In the first case we obtain a contradiction. In the second case, we have simplified the graph Γ, and we can continue in this way until Γ is minimal, as required. □

In particular, we have the following corollary.

Theorem 3.2.5. *Let G be a finitely generated group. Then Γ acts nontrivially on a simplicial tree iff G admits a nontrivial description as an amalgamated free product or an HNN extension.*

3.3. **Groups acting on trees.** A tree has a natural structure as a geodesic metric space, where we declare that every edge is isometric to the unit interval. Then the group of automorphisms of the tree acts by isometries on this metric space. Typically this is the entire group of isometries, but not always. For instance, the tree T which is homeomorphic to $\mathbb{R}$ with vertices at the integers has automorphism group $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ but isometry group $\mathbb{R} \rtimes \mathbb{Z}/2\mathbb{Z}$.

The most significant feature of a tree as a geodesic metric space is the following

Lemma 3.3.1. *Let T be a connected tree. Thinking of T as a geodesic metric space, a path $\gamma : I \to T$ is a geodesic iff it is the unique embedded path between the two endpoints $\gamma(\partial I)$.*

Proof. Since T is a tree, there is an embedded path $\gamma : I \to T$, unique up to reparameterization, joining any two points $p, q \in T$. Any other path $\tau : I \to T$ contains a subpath $\tau : J \to T$ with $J \subset I$ such that if x, y are the two endpoints of J, then $\tau(x) = \tau(y)$. Then we can *trim* I along J, to produce a shorter path. Technically, we let K be the interval obtained from I by quotienting out the subinterval J. Since $\tau|_{\partial J}$ is constant, we can define a new map τ' on I by $\tau'|_J = \tau|_{\partial J}$. Then τ' factors through to a map $\tau' : K \to T$ which is shorter than τ. Since the image of τ' lies inside the image of τ, which is compact, this sequence of trimmings contains a convergent subsequence. By construction, any limiting path is embedded, and therefore it is the unique embedded path, as claimed. □

It follows that we can identify geodesic paths in T topologically by the property that they are *embedded*. Moreover, a bi–infinite line l in T is a geodesic iff it is embedded.

Definition 3.3.2. Let T be a tree, and γ an automorphism of T. An *axis* for γ is an embedded bi–infinite geodesic $l_\gamma \subset T$ on which γ acts by translation.

We establish some basic geometric facts about group actions on trees.

Lemma 3.3.3. *Let G act on T without inversions, and suppose that for some p the orbit of p is bounded. That is, there is a constant K such that $d(p, g(p)) \le K$ for all $g \in G$. Then G fixes some vertex in T.*

Proof. Let O be the orbit of p, and let T^O be the convex hull of O. Then T^O is a finite subtree of T which is fixed by G. Let $q \in T$ be the L_∞ *center of gravity of* T^O. That is, q is the unique point in T^O which minimizes

$$\inf_q \sup_p d(q, p)$$

To see that q is well–defined, observe that it can be found inductively by removing from T^O all the edges which contain a 1–valent vertex; eventually what is left is either a single vertex or a segment. Then q is either this vertex or the center of this segment.

Then since G acts by isometries, G fixes q. Either q is a vertex, or G fixes the entire edge containing q, since G acts without inversions. In particular, G fixes the vertices which are the endpoints of this interval. □

Note that we must take an L_∞ center of gravity in general, since G might be infinite and no "averaging procedure" over G might make sense.

Lemma 3.3.4. *Let γ act simplicially on T. Then either the fixed point set of γ is some proper (connected) subtree T^γ, or else γ fixes a* unique *geodesic l_γ.*

Proof. Let p be a vertex of T, and let l_p be the unique geodesic in T from p to $\gamma(p)$. Then $l_p \cap \gamma^{-1}(l_p)$ is an initial segment I of the geodesic l_p, emanating from p. Set $\partial I = p \cup q$. Then $l_q \cap \gamma^{-1}(l_q) = q$, so we assume without loss of generality that we had chosen q instead of p.

There are two cases to consider: either $\gamma(q) = q$, or else $\gamma(q) \neq q$. In the first case the fixed point set T^γ of γ is nonempty. We claim it is connected. Let $p, q \in T^\gamma$ be two points. Let I be the unique embedded path joining p to q. Then $\gamma(I) = I$, and therefore $I \subset T^\gamma$, so T^γ is connected, and therefore a tree.

In the second case, define

$$l_\gamma = \bigcup_{i \in \mathbb{Z}} \gamma^i(l_q)$$

We claim that l_γ is embedded. By hypothesis, $l_q \cap \gamma^{-1}(l_q) = q$, and therefore

$$\gamma^n(l_q) \cap \gamma^{n-1}(l_q) = \gamma^n(q)$$

for any $n \in \mathbb{Z}$. It suffices to show that $\gamma^n(l_q) \cap (l_q) = \emptyset$ for $n > 1$. Suppose this is not true, and let n be the smallest positive integer for which it fails. Then there is a last point $p \in l_q$ which intersects $\gamma^n(l_q)$, and we get a subinterval of l_γ from p to itself which contains $\gamma(l_q)$ as an embedded subinterval. But in a tree, any path from a point to itself contains *no* embedded subinterval, or else by collapsing one could obtain an embedded loop. Hence l_γ is embedded, and therefore a geodesic by lemma 3.3.1.

To see that l_γ is *unique*, observe that for any point p not on l_γ, that there is a unique embedded geodesic arc τ_p from p to l_γ. Then

$$l_p \cap \gamma^{-1}(l_p) = \tau_p$$

and therefore there is no axis for γ containing p. □

Theorem 3.3.5. *Let N be a finitely generated nilpotent group acting on T. Then either N fixes some subtree T^N pointwise, or else there is a unique axis l_N for N such that the action of N on l_N factors through a nontrivial homomorphism $N \to \mathbb{Z}$.*

Proof. We argue by induction. Since N is nilpotent, it has a composition sequence

$$1 = N_0 < N_1 < \cdots < N_n = N$$

where each succesive quotient N_i/N_{i-1} is cyclic. We argue by induction. If N_{i-1} fixes some maximal subtree $T^{N_{i-1}}$ pointwise, then since N_{i-1} is normal in N_i, N_i must stabilize this subtree, so we get an induced action of N_i/N_{i-1} on $T^{N_{i-1}}$. Since this quotient group is cyclic, by lemma 3.3.4 the group N_i either fixes some maximal subtree T^N of $T^{N_{i-1}}$ pointwise, or else it has a *unique* axis l_i on which N_i/N_{i-1} acts by translation.

If N_{i-1} acts on some axis l_{i-1} by translation, then since N_{i-1} is normal, by uniqueness of this axis, N_i must also fix this axis, and therefore N_i acts on l_{i-1} simplicially. The only possibilities are that this action factors through $\mathbb{Z}$ or through $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. But the latter group is not nilpotent, so the former must hold. This completes the induction step, and proves the theorem. □

In [8], Serre makes the following definition:

Definition 3.3.6. A group G has *property FA* if every simplicial action without inversions on a tree T has a global fixed point.

Example 3.3.7. A G countable, infinitely generated group has property FA.

For, we can write G as a countable increasing union

$$G_1 \subset G_2 \subset \cdots \subset G$$

where each G_i is properly contained in G_{i+1}. Let X be a graph with vertices the cosets of the G_i, and with an edge from aG_i to bG_{i+1} if $aG_i \subset bG_{i+1}$. Then X is a tree, and G acts on X by multiplication on the left. If some vertex aG_i is fixed by X, then $G_i = G$. But this implies that G is finitely generated, contrary to hypothesis.

Example 3.3.8. $\mathrm{SL(n,\mathbb{Z})}$ has property FA for $n \geq 3$.

We show this for $\mathrm{SL}(3,\mathbb{Z})$; the case for higher n is similar. $\mathrm{SL}(3,\mathbb{Z})$ is generated by the elementary matrices e_i, $1 \leq i \leq 6$ which differ from the identity matrix in exactly one off–diagonal entry which is equal to 1. For appropriate ordering of the indices, each cyclically ordered triple e_{i-1}, e_i, e_{i+1} generates a copy of the 3–dimensional Heisenberg group N_i. That is, it has a presentation of the form

$$N_i = \langle e_{i-1}, e_i, e_{i+1} | [e_{i-1}, e_i] = \mathrm{id}, [e_i, e_{i+1}] = \mathrm{id}, [e_{i-1}, e_{i+1}] = e_i^{\pm 1} \rangle$$

with sign depending on the parity of i. Each N_i is two–step nilpotent of the form

$$0 \to \mathbb{Z} \to N_i \to \mathbb{Z}^2 \to 0$$

with center generated by e_i.

By theorem 3.3.5, each N_i either has a global fixed point, or acts on some unique axis l_i by translation. But if N_i acts on l_i by translation, then e_i fixes l_i pointwise, since it is in the commutator subgroup of N_i. Since this is true for each i, we see that each e_i has a fixed point.

Now we use the fact that $\mathrm{SL}(3,\mathbb{Z})$ is *boundedly generated by the* e_i. That is, every element $g \in \mathrm{SL}(3,\mathbb{Z})$ can be expressed as a product

$$g = e_1^{n_1} e_2^{n_2} \dots e_6^{n_6}$$

for some integers $n_i, \dots, n_6$. Let p_i be some fixed point of e_i, and let $K = \sum_{i=1}^{5} d(p_i, p_{i+1})$. Then we claim that $d(g(p_6), p_6) \leq 2K$ for all g. For, $e_6^{n_6}(p_6) = p_6$. Then since e_5 fixes p_5, we have

$$d(e_5^{n_5} e_6^{n_6}(p_6), p_5) = d(e_5^{n_5}(p_6), p_5)) = d(p_6, p_5)$$

and therefore

$$d(e_4^{n_4} e_5^{n_5} e_6^{n_6}(p_6), p_4) = d(e_5^{n_5} e_6^{n_6}(p_6), p_4) \leq d(p_6, p_5) + d(p_5, p_4)$$

Continuing in this way, the inequality is established.

In particular, $\mathrm{SL}(3,\mathbb{Z})$ has a bounded orbit. It follows by lemma 3.3.3 that $\mathrm{SL}(3,\mathbb{Z})$ has a global fixed point, as claimed.

In a later section, we will see that this result follows from a much more general fact that *groups with property T have property FA*.

4. Hyperbolic groups

5. Amenability and property T

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