COMPLEX VARIABLES, FALL 2017, MIDTERM

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This midterm exam was posted online on Thursday, October 26, and is due before 11:00 on Thursday, November 2.

Collaboration is not allowed, nor is the use of outside materials and textbooks. Marsden/Hoffman and your class notes may be used to remember definitions, but not to copy calculations or proofs.

Problem 1. For each of the following statements, say whether it is true or false, and give a justification for your answer:

(1) $f(z) = \sin(z)$ is a bounded function.
(2) Let $\mathbb{Z} \subset \mathbb{C}$ denote the set of integers. If f is complex differentiable on $\mathbb{C} - \mathbb{Z}$, and continuous at every point of $\mathbb{Z}$, then f is analytic on all of $\mathbb{C}$.
(3) $f(z) = e^z$ is one-to-one.
(4) a map $f : X \to Y$ is continuous if and only if for all closed sets $K \subset Y$, the preimage $f^{-1}(K) \subset X$ is closed.

Problem 2. The *open unit disk* D is the set of complex numbers z with $|z| < 1$.

Find an analytic map $f : D \to \mathbb{C}$ which is one-to-one and onto or prove that such a map does not exist.

Problem 3. Let a be a nonzero complex number.

(1) what is the definition of a^i?
(2) show that if one value of a^i is real, then all other values are real.
(3) if a^i is real for some (all) values, what are the possible values of a?

Problem 4. Let $f(z)$ be analytic for $|z| < R$ and continuous for $|z| \le R$. If γ is the circle $|z| = R$ traversed anticlockwise show that $\int_\gamma f = 0$.

Problem 5. Let γ be a simple closed curve bounding an open region A and going around it in the anticlockwise direction, and let $P(z)$ be a polynomial which is never zero on γ. Show that

$$\frac{1}{2\pi i} \int_\gamma \frac{P'(z)}{P(z)} dz$$

is equal to the number of roots of P inside the region A.

Problem 6. Let γ denote the unit circle in $\mathbb{C}$, traversed anticlockwise; i.e. $\gamma : [0, 2\pi] \to \mathbb{C}$ is defined by $\gamma(\theta) = e^{i\theta}$.

Let f be the function defined on the image of γ by the formula

$$f(\zeta) = \begin{cases} 1 \text{ if } \arg(\zeta) \in [0, \pi) \text{ and} \\ 0 \text{ if } \arg(\zeta) \in [\pi, 2\pi) \end{cases}$$

(1) Define

$$F(z) := \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)} d\zeta$$

Show that F is analytic in the open unit disk — i.e. where $|z| < 1$ (Warning: f is not continuous on γ. Is this a problem?)

(2) Use the Cauchy integral formula for derivatives to give a power series expansion for $F(z)$ around 0, and show that it converges absolutely and uniformly on compact subsets of the open unit disk.
(3) Use your power series formula for F to write down a closed-form expression for F, valid in the unit disk. (Hint: compute the derivative of F, and then try to identify the resulting power series as a familiar function).
(4) How does F map the open unit disk into $\mathbb{C}$? Draw a picture! What happens as z gets close to 1 or -1?

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