COMPLEX VARIABLES, FALL 2017, FINAL

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This final exam was posted online on Thursday, November 30, and is due before 11:00 on Thursday, December 7.

Collaboration is not allowed, nor is the use of outside materials and textbooks. Marsden/Hoffman and your class notes may be used to remember definitions, but not to copy calculations or proofs.

Problem 1. For each of the following statements, say whether it is true or false, and give a justification for your answer:

(1) If f is a non-constant analytic function on a connected open domain D then for any open subset $U \subset D$, the image $f(U)$ is open.
(2) Let $\mathbb{D}$ denote the unit disk. If $g : \mathbb{D} \to \mathbb{D}$ analytic satisfies $g(0) = 0$ then $g(z)/z$ has a removable singularity at $z = 0$.
(3) Suppose f is analytic on $\mathbb{C}^*$ and has residue 1 at the point 0. Then the residue of $g(z) := f(z)^2$ at 0 is also 1.
(4) If f is entire and non-constant, then $f(z)(f(z) - 1)$ has a zero somewhere.

Problem 2. Evaluate the following residues; here $\text{Res}(f; w)$ means the residue of the function f at the point w.

(1) $\text{Res}\left(\frac{e^z}{z(1-z)^3}; 1\right)$
(2) $\text{Res}\left(\frac{e^z-1}{\sin(z)}; 0\right)$
(3) $\text{Res}(\tan(z); \pi/2)$
(4) Compute the residues of $\frac{z^2-1}{\cos(\pi z)+1}$ at *all* its singularities.

Problem 3. Compute the following integrals

(1) Evaluate
$$\int_\gamma \frac{3z^2 - 7z + 2}{(z-1)^3} dz$$
where γ is a positively oriented simple loop around 1.
(2) Compute
$$\int_0^\infty \frac{\log x}{(x^2+1)^2} dx$$
(3) Compute
$$\int_0^{2\pi} \frac{1}{5 + 3\sin(\theta)} d\theta$$
(4) Show that
$$\int_0^\infty \frac{x \sin x}{x^4+1} dx = \frac{\pi}{2} e^{-1/\sqrt{2}} \sin \frac{1}{\sqrt{2}}$$

Problem 4. Recall that a *Möbius transformation* is a map of the Riemann sphere to itself of the form
$$z \to \frac{az+b}{cz+d}$$
for some a, b, c, d complex numbers with $ad - bc = 1$.

(1) Show that a map is a Möbius transformation if and only if it is meromorphic (as a map from the Riemann sphere to itself) of degree 1 (i.e. one to one).
(2) If p_1, p_2, p_3 are three *distinct* points in the Riemann sphere $\mathbb{C} \cup \infty$, show that there is a *unique* Möbius transformation which takes p_1, p_2, p_3 to $0, 1, \infty$ (in that order).
(3) Determine which Möbius transformations take the unit disk $\mathbb{D}$ to itself (by a 1-1 and onto map), and show that for any $p \in \mathbb{D}$ there is a Möbius transformation taking $\mathbb{D}$ to itself and taking p to 0.
(4) The *Poincaré metric* on $\mathbb{D}$ is obtained from the usual Euclidean metric by (infinitesimally) scaling the length at a point $z \in \mathbb{D}$ by the factor $2/(1-|z|^2)$; that is, if $\gamma : [0,1] \to \mathbb{D}$ is a smooth curve, the "hyperbolic length" of the curve γ is

$$\text{hyperbolic length of } \gamma = \int_0^1 \frac{2|\gamma'(t)|}{1-|\gamma(t)|^2} dt$$

If $\alpha : \mathbb{D} \to \mathbb{D}$ is a Möbius transformation taking $\mathbb{D}$ to itself, show that the hyperbolic length of γ is equal to the hyperbolic length of $\alpha \circ \gamma$.

Problem 5. Prove that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$$

Problem 6. Show that the function

$$w(z) = \int_0^z \frac{(1-\zeta^5)^{2/5}}{(1+\zeta^5)^{4/5}} d\zeta$$

maps the unit disk $|z| < 1$ onto the interior of a *pentagram* — i.e. a regular five-pointed star. (Draw the pentagram!)

Problem 7. Recall that if f is analytic, the *nonlinearity* $N(f) := f''/f'$ has the property that $N(f) = N(af+b)$ for any constants a, b where $a \neq 0$.

(1) Define the *Schwarzian derivative* $S(f)$ by the formula

$$S(f) := N'(f) - N(f)^2/2$$

Show that $S(1/f) = S(f)$ and deduce that

$$S\left(\frac{af+b}{cf+d}\right) = S(f)$$

for any complex a, b, c, d with $ad - bc \neq 0$.
(2) Let T be a domain bounded by circular arcs meeting at angles $\pi\alpha_1, \cdots, \pi\alpha_n$. Let $\mathbb{H}$ denote the upper half-plane (i.e. the set of complex numbers with positive imaginary part) and let $f : \mathbb{H} \to T$ uniformize T; i.e. it is analytic, one-to-one and onto (the existence of f follows from Riemann's uniformization theorem). Show that $S(f)$ extends to a single-valued meromorphic function in $\mathbb{C}$, which is real-valued on the real axis, with poles of order 2 at the preimages of the vertices.
(3) **(Bonus question):** Suppose that the preimages p_j of the vertices of T are real and finite. Show that

$$S(f(z)) = \sum_{j=1}^{n} \left(\frac{(1-\alpha_j^2)}{2(z-p_j)^2} + \frac{\beta_j}{z-p_j} \right)$$

where the β_j satisfy the equations

$$\sum \beta_j = 0, \quad \sum 2p_j\beta_j + (1-\alpha_j^2) = 0, \quad \sum p_j^2\beta_j + p_j(1-\alpha_j^2) = 0$$

Note: in general the determination of the p_j and the constants β_j is difficult. But if $n = 3$ — i.e. if T is bounded by three circular arcs — then the β_j are completely determined by the equations above, and by a Möbius transformation we can move the p_j to $-1, 0, 1$. Thus we get a completely explicit formula for $S(f)$ in this case.
(4) **(Bonus question):** Again in the case $n = 3$, so that T is bounded by three circular arcs, suppose that $\alpha_j = 1/n_j$ for positive integers n_j (or $+\infty$ if $\alpha_j = 0$). Show that the inverse function f^{-1} can be extended by analytic continuation to a single-valued function in some domain:

- if $1/n_1 + 1/n_2 + 1/n_3 > 1$ then f^{-1} is meromorphic on the entire Riemann sphere, and is a rational function;
- if $1/n_1 + 1/n_2 + 1/n_3 = 1$ then f^{-1} is meromorphic on all of $\mathbb{C}$ and is doubly-periodic for some period lattice Λ;
- if $1/n_1 + 1/n_2 + 1/n_3 < 1$ then f^{-1} is meromorphic on all of $\mathbb{H}$ and is a modular function for the group Γ^+ of index 2 consisting of orientation-preserving transformations in the group Γ generated by inversions in the sides of the "triangle" T.

Since $S(f)$ only determines f up to composition with a Möbius transformation, we have to choose a suitable T so that the domain is $\mathbb{C}$ or $\mathbb{H}$ in cases 2 and 3 above. Note that the modular functions J and λ arise in this way (as degenerate cases where some of the angles are zero).

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