COMPLEX VARIABLES, WINTER 2017, MIDTERM

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This midterm exam was posted online on Friday, February 3, and is due before 12:30 on Friday, February 10.

Collaboration is not allowed, nor is the use of outside materials and textbooks. Marsden/Hoffman and your class notes may be used to remember definitions, but not to copy calculations or proofs.

Problem 1. For each of the following statements, say whether it is true or false, and give a justification for your answer:

(1) $f(z) = \cos(z)$ is a bounded function.
(2) if $U \subset \mathbb{C}$ is open, and $f : U \to \mathbb{C}$ is complex differentiable at some point $z_0 \in U$ then f is analytic at z_0.
(3) $f(z) = e^z$ is one-to-one.
(4) a map $f : X \to Y$ is continuous if and only if for all closed sets $K \subset Y$, the preimage $f^{-1}(K) \subset X$ is closed.

Problem 2. The *open unit disk* D is the set of complex numbers z with $|z| < 1$. The *upper half plane* H is the set of complex numbers z with $\mathrm{Im}(z) > 0$.

Find an analytic map $f : D \to H$ which is one-to-one and onto.

Problem 3. Let a be a nonzero complex number.

(1) what is the definition of a^i?
(2) show that if one value of a^i is real, then all other values are real.
(3) if a^i is real for some (all) values, what are the possible values of a?

Problem 4. Consider the function $f(z) := (1 - \cos(z))/z^2$, defined for $z \neq 0$.

(1) Show that f has a well-defined limit as $z \to 0$.
(2) Define a new function F which is equal to f for $z \neq 0$, and has $F(0) = \lim_{z\to 0} f(z)$. Show that F is continuous at 0.
(3) Show that F is analytic.

We say that f has a *removable singularity* at 0.

Problem 5. Let γ denote the unit circle in $\mathbb{C}$, traversed anticlockwise; i.e. $\gamma : [0, 2\pi] \to \mathbb{C}$ is defined by $\gamma(\theta) = e^{i\theta}$.

Let f be the function defined on the image of γ by the formula

$$f(\zeta) = \begin{cases} 1 \text{ if } \arg(\zeta) \in [0, \pi) \text{ and} \\ 0 \text{ if } \arg(\zeta) \in [\pi, 2\pi) \end{cases}$$

(1) Define

$$F(z) := \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)} d\zeta$$

Show that F is analytic in the open unit disk — i.e. where $|z| < 1$ (Warning: f is not continuous on γ. Is this a problem?)

(2) Use the Cauchy integral formula for derivatives to give a power series expansion for $F(z)$ around 0, and show that it converges absolutely and uniformly on compact subsets of the open unit disk.
(3) Use your power series formula for F to write down a closed-form expression for F, valid in the unit disk. (Hint: compute the derivative of F, and then try to identify the resulting power series as a familiar function).
(4) How does F map the open unit disk into $\mathbb{C}$? Draw a picture! What happens as z gets close to 1 or -1?

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