COMPLEX VARIABLES, WINTER 2017, FINAL

DANNY CALEGARI

This final exam was posted online on Friday, March 10, and is due before 12:30 on Friday, March 17.

Collaboration is not allowed, nor is the use of outside materials and textbooks. Marsden/Hoffman and your class notes may be used to remember definitions, but not to copy calculations or proofs.

Problem 1. For each of the following statements, say whether it is true or false, and give a justification for your answer:

(1) If f is a non-constant analytic function on a connected open domain D then for any open subset $U \subset D$, the image $f(U)$ is open.
(2) The function $z^2/\sin(z)$ has a removable singularity at $z = 0$.
(3) Suppose f is analytic on $\mathbb{C}^*$ and has residue 1 at the point 0. Then the residue of $g(z) := f(z)^2$ at 0 is also 1.
(4) If f is entire, and $|f(z)| \geq 1$ for all z then f is constant.

Problem 2. Evaluate the following residues; here $\text{Res}(f; w)$ means the residue of the function f at the point w.

(1) $\text{Res}(e^z/(z^3 - 1); 1)$
(2) $\text{Res}(1/(z^7 - w^7); w)$
(3) $\text{Res}(\tan(z); \pi/2)$
(4) $\text{Res}(z^2; 0)$

Problem 3. Compute the following integrals

(1) Evaluate
$$\int_\gamma \frac{5z^2 - 3z + 2}{(z-1)^3} dz$$
where γ is a positively oriented simple loop around 1.
(2) Compute
$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+2x+2)} dx$$
(3) Compute
$$\int_0^{2\pi} \frac{1}{5 + 3\sin(\theta)} d\theta$$
(4) Show that
$$\int_0^{\infty} \frac{\cos(mx)}{x^2+1} dx = \frac{\pi}{2} e^{-m}$$

Problem 4. The *Gamma function* is defined by the formula
$$\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} dx$$

(1) Show that for complex numbers z with positive real part this integral converges, and the resulting function is holomorphic in the domain $\text{Re}(z) > 0$.
(2) Using integration by parts, show that $\Gamma(z+1) = z\Gamma(z)$ in the domain of definition.
(3) Compute $\Gamma(1)$ directly, and deduce that $\Gamma(n+1) = n!$ for positive integers n.

(4) Using the identity $\Gamma(z) = \Gamma(z+1)/z$ repeatedly, show that Γ has an analytic continuation to a single-valued meromorphic function defined on all of $\mathbb{C}$, with poles at non-positive integers.
(5) Compute the residues at the poles.

Problem 5. Recall that if f is analytic, the *nonlinearity* $N(f) := f''/f'$ has the property that $N(f) = N(af+b)$ for any constants a, b where $a \neq 0$.

(1) Define the *Schwarzian derivative* $S(f)$ by the formula
$$S(f) := N'(f) - N(f)^2/2$$
Show that $S(1/f) = S(f)$ and deduce that
$$S\left(\frac{af+b}{cf+d}\right) = S(f)$$
for any complex a, b, c, d with $ad - bc \neq 0$.
(2) Let T be a domain bounded by circular arcs meeting at angles $\pi\alpha_1, \cdots, \pi\alpha_n$. Let $\mathbb{H}$ denote the upper half-plane (i.e. the set of complex numbers with positive imaginary part) and let $f : \mathbb{H} \to T$ uniformize T; i.e. it is analytic, one-to-one and onto (the existence of f follows from Riemann's uniformization theorem). Show that $S(f)$ extends to a single-valued meromorphic function in $\mathbb{C}$, which is real-valued on the real axis, with poles of order 2 at the preimages of the vertices.
(3) **(Bonus question):** Suppose that the preimages p_j of the vertices of T are real and finite. Show that
$$S(f(z)) = \sum_{j=1}^{n} \left(\frac{(1-\alpha_j^2)}{2(z-p_j)^2} + \frac{\beta_j}{z-p_j} \right)$$
where the β_j satisfy the equations
$$\sum \beta_j = 0, \quad \sum 2p_j\beta_j + (1-\alpha_j^2) = 0, \quad \sum p_j^2\beta_j + p_j(1-\alpha_j^2) = 0$$
Note: in general the determination of the p_j and the constants β_j is difficult. But if $n = 3$ — i.e. if T is bounded by three circular arcs — then the β_j are completely determined by the equations above, and by a Möbius transformation we can move the p_j to $-1, 0, 1$. Thus we get a completely explicit formula for $S(f)$ in this case.
(4) **(Bonus question):** Again in the case $n = 3$, so that T is bounded by three circular arcs, suppose that $\alpha_j = 1/n_j$ for positive integers n_j (or $+\infty$ if $\alpha_j = 0$). Show that the inverse function f^{-1} can be extended by analytic continuation to a single-valued function in some domain:
   - if $1/n_1 + 1/n_2 + 1/n_3 > 1$ then f^{-1} is meromorphic on the entire Riemann sphere, and is a rational function;
   - if $1/n_1 + 1/n_2 + 1/n_3 = 1$ then f^{-1} is meromorphic on all of $\mathbb{C}$ and is doubly-periodic for some period lattice Λ;
   - if $1/n_1 + 1/n_2 + 1/n_3 < 1$ then f^{-1} is meromorphic on all of $\mathbb{H}$ and is a modular function for the group Γ^+ of index 2 consisting of orientation-preserving transformations in the group Γ generated by inversions in the sides of the "triangle" T.

   Since $S(f)$ only determines f up to composition with a Möbius transformation, we have to choose a suitable T so that the domain is $\mathbb{C}$ or $\mathbb{H}$ in cases 2 and 3 above. Note that the modular functions J and λ arise in this way (as degenerate cases where some of the angles are zero).

Department of Mathematics, University of Chicago, Chicago, Illinois, 60637
E-mail address: dannyc@math.uchicago.edu