University of Chicago Lectures.

Lecture (3): onwards and upwards.

29 July 2016.

Algebraic K-theory

1. Rings and projective modules.

2. K_0, Swan's theorem, and examples.

3. K-theory via group-completion, $K(\mathrm{Fin}) \simeq \mathbb{S}$.

4. $K(\mathbb{F}_q)$, $K(\mathbb{C})/\ell$.

5. Recent progress and conjectures.

1. Rings and projective modules.

R an associative ring,

P a right R-module.

<u>Def.</u> P is <u>projective</u> if dotted lifts exist in the

$$\begin{tikzcd} & M \arrow[d] \\ P \arrow[r] \arrow[ur, dashed] & N \arrow[d] \\ & 0 \end{tikzcd}$$

solid-arrow diagrams, when the vertical column is exact ($M \to N$[redacted] is surjective).

<u>Lemma</u>. TFAE:

(i) P is projective;

(ii) $\mathrm{Hom}_R(P, -)$ is exact;

(iii) P is a summand of $R^{\oplus I}$ for some set I.

<u>Remark</u>. Projective modules are algebraic analogues of (possibly infinite-dimensional) vector bundles. [redacted] When R is commutative, this analogy becomes quite precise.

<u>Def.</u> A right R-module N is <u>finitely</u> <u>presented</u> if it can be written as a cokernel

$$R^m \to R^n \to N \to 0,$$

with m, n non-negative integers.

<u>Exs.</u> (a) R^n for all $n \geq 0$; especially k^n.

(b) Not $\mathbb{Z}/(17)$. Not $k[x]/(x^n)$, $n \geq 1$ over $k[x]$.

(c) The ideal $(2, 1+\sqrt{-5}) \subseteq \mathbb{Z}[\sqrt{-5}]$ is projective but not free. This reflects that the class group $\mathrm{Cl}(\mathbb{Z}[\sqrt{-5}]) \cong \mathbb{Z}/2$.

2. K_0, Swan's theorem, and examples.

Def. $K_0(R)$ is the group-completion of the monoid of isomorphism classes of finitely presented projective right R-modules.

Exs. (i) $K_0(k) \cong \mathbb{Z}$ when k is a field.

(ii) $K_0(\mathbb{Z}) \cong \mathbb{Z}$.

(iii) $K_0(\mathbb{Z}[\sqrt{-5}]) \cong \mathbb{Z} \oplus \mathbb{Z}/2$.

(iv) $K_0(k[x_1, \ldots, x_n]) \cong \mathbb{Z}$.

Def. X compact, Hausdorff.

$\mathcal{C}(X)$ = the ring of $\mathbb{C}$-valued continuous functions on X.

Construction. $\mathcal{E} \to X$ a $\mathbb{C}$-vector bundle. We have seen that there exists $\mathcal{F}$ s.t. $\mathcal{E} \oplus \mathcal{F} \cong \mathbb{1}^n$.

$\mathcal{E}(X)$ = $\mathbb{C}$-vector space of continuous sections of $\mathcal{E} \to X$.

$\mathbb{1}^n(X) \cong \mathcal{C}(X)^n$.

Hence, $\mathcal{E}(X) \oplus \mathcal{F}(X) \cong \mathcal{C}(X)^n$, so $\mathcal{E}(X)$ is projective over $\mathcal{C}(X)$.

Conversely, let P be a projective $\mathcal{C}(X)$-module of f.p. Write P as a summand of $\mathcal{C}(X)$. There is a matrix $e \in M_n(\mathcal{C}(X))$ s.t. the image of e is $P \subseteq \mathcal{C}(X)^n$. View this as a matrix of n^2 continuous functions on X, or as a morphism $\mathbb{1}^n \to \mathbb{1}^n$. The image is a v.b. on X.

Serre-Swan **Proposition.** These constructions induce an equivalence of categories

$$\mathrm{Vect}(X) \xrightarrow[\sim]{\text{sec}} \mathrm{fProj}(X).$$

Cor. $KU^0(X) \cong K_0(\mathcal{C}(X))$ when X is compact Hausdorff. **Ex.** $K_0(\mathcal{C}(S^1)) \cong \mathbb{Z}$.

3. K-theory via group-completion, $K(\mathrm{Fin}_*) \simeq \mathbb{S}$.

$$\mathbb{Z}_{\geq 0} \text{ addition monoid } \underset{\text{group completion}}{\rightsquigarrow} \mathbb{Z}.$$

This can be done for <u>topological</u> <u>monoids</u> as well.

<u>Exs.</u> (i) $\coprod_{n\geq 0} BS_n$, $\quad S_m \times S_n \longrightarrow S_{m+n}$,

(ii) $\coprod_{n\geq 0} BGl_n(R)$, $\quad Gl_m(R)\times Gl_n(R) \longrightarrow Gl_{m+n}(R)$,

(iii) $\coprod_{n\geq 0} BO_n$, $\quad O_m \times O_n \longrightarrow O_{m+n}$,

(iv) $\coprod_{n\geq 0} BU_n$. $\quad U_m \times U_n \longrightarrow U_{m+n}$.

<u>Ex.</u> $\coprod_{\substack{P \text{ f.p. right} \\ \text{projective } R\text{-mod.}}} BAut_R(P)$ $\quad Aut_R(P) \times Aut_R(Q) \longrightarrow Aut_R(P\oplus Q)$.

One can group complete each of these <u>homotopy commutative H-spaces</u> obtaining an <u>infinite loopspace</u>.

<u>Def.</u> $K(R) := \left(\coprod BAut_R(P)\right)^{gp}$.

<u>Exs.</u> These are all hard. Few complete examples are known.

(i) $(\coprod BS_n)^{gp} \simeq \Omega^\infty \mathbb{S}$. So, $\pi_i(\coprod BS_n)^{gp} \cong \pi_i^s$. Barratt-Priddy-Quillen.

(iii) $(\coprod BO_n)^{gp} \simeq \mathbb{Z}\times BO$.

(iv) $(\coprod BU_n)^{gp} \simeq \mathbb{Z}\times BU$.

4. $K(\mathbb{F}_q)$, $K(\mathbb{C})/_\ell$.

<u>Thm</u> (Quillen). $K(\mathbb{F}_q) \simeq F\psi^q$, where $F\psi^q$ is the homotopy fiber of $1-\psi^q : BU \to BU$.

<u>Cor.</u>

$$K_0(\mathbb{F}_q) \cong \mathbb{Z},$$
$$K_{2n-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^n-1),$$
$$K_{2n}(\mathbb{F}_q) = 0, \quad n \geq 1.$$

Now, $K(\bar{\mathbb{F}}_p) \simeq \operatorname{colim}_{n\to\infty} K(\mathbb{F}_{p^n})$. Choose $(\ell, p) = 1$. Then,

$$\pi_i K(\bar{\mathbb{F}}_p)/_\ell \cong \begin{cases} 0 & i \text{ odd}, \\ \mathbb{Z}/\ell & i \text{ even}. \end{cases}$$

<u>Thm</u> (Suslin) k a field. ℓ prime to char k.

Then, $K(\bar{k})/_\ell \simeq ku/\ell$.

<u>Ex.</u> $K(\mathbb{C})/_\ell \simeq ku/\ell$.

[redacted]

5. Recent progress and conjectures.

<u>Conjecture</u> (Bass). Let R be a commutative ring f.g. over $\mathbb{Z}$.
Then, $K_i(R)$ is f.g.

<u>Ex</u>. (i) (Serre). π_i^s is finite for $i>0$.

(ii) $K_i(\mathbb{F}_q)$ by Quillen.

(iii) $K_i(\mathcal{O}_K)$ K a number field by Quillen.

(iv) $K_i(\mathbb{F}_q(C))$ C a curve, Quillen/Grayson.

Otherwise, this is completely open.

<u>Thm</u> (Rognes/Blumberg–Mandell). p an odd prime.

additive complete cokernel-of-j spectrum

$K(\mathbb{Z})^\wedge_p \simeq j \vee \tilde{K}(\mathbb{Z})$

$$\mathrm{tor}_p\left(K_*(\mathbb{S})\right) \cong \mathrm{tor}_p\left(\pi_* \mathbb{S} \oplus \pi_{*-1} c \oplus \pi_{*-1} \overline{\mathbb{CP}^\infty_{-1}} \oplus \pi_* \tilde{K}(\mathbb{Z})\right).$$

Some Thom spectrum.

This reduces $K(\mathbb{S})$ to understanding
(i) stable homotopy, hard,
(ii) $\pi_{*-1}\overline{\mathbb{CP}^\infty_{-1}}$, should be easy, and
(iii) $K_*(\mathbb{Z})$, hard.

<u>Vandiver's conjecture</u>. p does not divide the class number of the maximal real subfield of $\mathbb{Q}(\zeta_p)$.

<u>Thm</u>. VC is equivalent to $K_{4n}(\mathbb{Z})=0$ $\forall n\geq 1$!