University of Chicago Lectures.

Lecture (2): K-theory and the Hopf invariant one problem.

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1. The Hopf invariant one problem.

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4. The proof.

1. The Hopf invariant one problem.

$\underline{n \geq 2}$: $f: S^{2n-1} \longrightarrow S^n$

$X = X_f =$ cone of f

$$\begin{tikzcd} S^{2n-1} \arrow[r] \arrow[d] & S^n \arrow[d] \\ D^{2n} \arrow[r] & X_f \end{tikzcd}$$

$H^n(X, \mathbb{Z}) \cong H^{2n}(X, \mathbb{Z}) \cong \mathbb{Z}$,

$H^i(X, \mathbb{Z}) = 0$ for $i > 0$, $i \neq n, 2n$.

Pick $x \in H^n(X, \mathbb{Z})$ a generator, $y \in H^{2n}(X, \mathbb{Z})$ another generator. Then

$$x^2 = H(f)\, y,$$

where $H(f) \in \mathbb{Z}$ is the Hopf invariant of f.

To rigidify the problem, can choose y in the pullback

$$H^{2n}(S^{2n}, \mathbb{Z}) \xrightarrow{\cong} H^{2n}(X_f, \mathbb{Z})$$

to be the pullback of the dual to a choice of $[S^{2n}] \in H_{2n}(S^{2n}, \mathbb{Z})$

$\underline{\text{Fact}}$. The Hopf invariant induces a group homomorphism

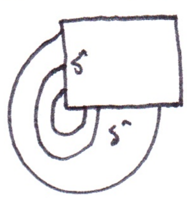
$$H: \pi_{2n-1}(S^n) \longrightarrow \mathbb{Z}.$$

In particular, it is homotopy invariant.

$\underline{\text{Exs}}$ (i) If $f \simeq *$, then $X_f \simeq S^n \vee S^{2n}$, $H(f) = 0$.

(ii) If n is odd, then $x^2 = -x^2$, so $H(f) = 0$ since $H^{2n}(X_f, \mathbb{Z})$ is torsion free.

(iii) If n is even, then $\operatorname{im}(H) \supseteq 2\mathbb{Z}$. Take $J_2(S^n) = S^n \times S^n / (x,e) \sim (e,x)$



Fact, we can view this as X_f for some f, and $H(f) = \pm 2$.

The identification identifies the two n-cells $S^n \times \{e\}$, $\{e\} \times S^n$ inside $S^n \times S^n$.

$\underline{\text{Cor}}$. $\pi_{2n-1}(S^n)$ contains a copy of $\mathbb{Z}$ when n is even.

(iv) $\underline{n=2}$. If $S^3 \xrightarrow{f} S^2$ is the Hopf map, then

$$X_f \cong \mathbb{C}P^2,$$

and ~~[illegible]~~ hence $H^*(X_f, \mathbb{Z}) \cong \mathbb{Z}[x]/(x^3)$.

Hence, $H(f) = \pm 1$.

(v) $\underline{n=4}$. $S^7 \xrightarrow{f} S^4$ glues to give $\mathbb{H}P^2$. $H(f) = \pm 1$.

(vi) $\underline{n=8}$. $S^{15} \to S^8$ glues to give $\mathbb{O}P^2$. $H(f) = \pm 1$.

$\underline{Q}$. For which $\underline{\text{even}}$ n is there an $f: S^{2n-1} \to S^n$ of Hopf invariant 1?

$\underline{\text{Thm}}$ (Adams, 1960). $n = 2, 4, 8$.

2. Consequences.

Thm. The following are true only for $n = 1, 2, 4, 8$.

(a) $\mathbb{R}^n$ is a division algebra.

(b) S^{n-1} is parallelizable.

(c) S^{n-1} is an H-space.

proof. That (a) – (c) hold when $n = 1, 2, 4, 8$ follows from the division algebra structures $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ on $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^4, \mathbb{R}^8$, respectively.

Lemma 1. If $\mathbb{R}^n$ is a division algebra or if S^{n-1} is parallelizable, then S^{n-1} is an H-space.

proof. If $\mathbb{R}^n$ is a division algebra, $(x,y) \longmapsto \frac{xy}{|xy|}$ gives S^{n-1} an H-space structure.

Now, choose a parallelization of S^{n-1} such that

$$x, v_1(x), \dots, v_{n-1}(x)$$

is orthonormal for all $x \in S^{n-1}$ and such that

$$v_1(e), \dots, v_{n-1}(e)$$

is the standard basis at $e = (1, 0, \dots, 0)$.

Then, $x, v_1(x), \dots, v_{n-1}(x)$ is an orthonormal basis for all x, and hence differs from the standard basis by a unique $\alpha_x \in SO_n$.

Setting $(x,y) \longmapsto \alpha_x(y)$ gives an H-space structure with identity e.

Lemma 2. S^{2n} is not an H-space for $n>0$.

Lemma 3. If S^{n-1} is an H-space with $n>0$ even, then there exists $f: S^{2n-1} \to S^n$ with $H(f) = \pm 1$.

End of proof. Lemmas 1 and 2 together show that (a)–(c) fail for odd n. Now, Lemma 1 implies it is enough to check that S^{n-1} is not an H-space when $n-1$ is odd and $n \neq 2,4,8$. Lemma 3 shows that if S^{n-1} is an H-space, then Hopf invariant 1 is true for S^n, which only happens for $n=2,4,8$ by Adams.

3. λ-rings and ψ-operations.

$E \to X$ a vector bundle, $\Lambda^i E$ is another vector bundle.

$$\Lambda^i(E \oplus F) \cong \bigoplus_{m+n=i} \Lambda^m E \otimes \Lambda^n F.$$

This structure gives $\mathrm{Vect}_{\mathbb{C}}(X)$ a λ-semiring structure.

Def. A λ-(semi)ring is a commutative (semi)ring R together with operations $\lambda^i : R \to R$, $i \geq 0$, s.t.

$$\lambda^0(x) = 1,$$
$$\lambda^1(x) = x,$$
$$\lambda^i(x+y) = \sum_{m+n=i} \lambda^m(x)\lambda^n(y).$$

Last condition is equivalent to $\lambda_t(x) = \sum \lambda^i(x) t^i$ defining a monoid homomorphism

$$R \longrightarrow 1 + tR[[t]].$$

Lemma. If R is a λ-semiring, then $R^{-1}R$ is a λ-ring.

proof.

$$\begin{tikzcd} R \arrow[r] \arrow[d, dash] & 1 + tR[[t]] \arrow[d, dash] \\ R^{-1}R \arrow[r, dashed] & 1 + tR^{-1}R[[t]] \end{tikzcd}.$$

Cor. $KU(X)$ is a λ-ring.

$KU(X) \xrightarrow{i^*} \mathbb{Z} \cong KU(*)$ is an augmentation of λ-rings.

If $n \in \mathbb{Z}$, $\lambda^i(n) = \binom{n}{i}$, i^* commutes with the λ^i.

<u>Adams operations</u>.
$$\psi^0(x) = i^*(x).$$
$$\psi^1(x) = x.$$
$$\psi^2(x) = x^2 - 2\lambda^2(x).$$
$$\psi^k(x) = (-1)^{k-1} k \lambda^k(x) + \sum_{i=1}^{k-1} (-1)^{i-1} \lambda^i(x) \psi^{k-i}(x).$$

~~Def. $\ell \in KU(X)$ is a line element if $i^*(\ell) = 1$.~~
~~This makes the most sense when X is connected.~~

<u>Facts</u>. (1) Each ψ^k is a ring endomorphism.

(2) $\psi^j \psi^k = \psi^{jk} = \psi^{kj}$.

(3) If L is a line bundle, then $\psi^k(L) = L^{\otimes k}$.
(Easy to check from the formulas.)

Prove (1) and (2) using splitting principle.

<u>Lemma</u>. $\psi^p(x) \equiv x^p \mod p$ when p is prime.

<u>proof</u>. We can assume that $x = [E]$. By the splitting principle, we can assume $E = L_1 + \cdots + L_n$. Then,
$$\begin{aligned}\psi^p(E) &= \psi^p(L_1) + \cdots + \psi^p(L_n)\\ &= L_1^{\otimes p} + \cdots + L_n^{\otimes p}\\ &\equiv (L_1 + \cdots + L_n)^{\otimes p} \mod p.\end{aligned}$$

Lemma. Let H be the tautological line bundle on $S^2 \simeq \mathbb{C}P^1$.

Then, $\widetilde{KU}(S^2) \cong \mathbb{Z} \cdot ([H]-1)$.

$$\psi^k([H]-1)$$
$$= k([H]-1).$$

More generally,

$$\psi^k\Big(\big([H]-1\big)^{\overset{\text{external product}}{*n}}\Big) = k^n\big([H]-1\big)^{*n}.$$

proof.
$$\begin{aligned}
\psi^k([H]-1) &= H^k - 1 \\
&= \big(1+(H-1)\big)^k - 1 \qquad (\text{since } (H-1)^2 = 0). \\
&= 1 + k(H-1) - 1 \\
&= k(H-1)
\end{aligned}$$

Inductively,
$$\begin{aligned}
\psi^k\Big(\underbrace{(H-1) * \cdots * (H-1)}_{n \text{ times}}\Big) &= k^{n-1}(H-1)^{*n-1} * k(H-1) \\
&= k^n (H-1)^{*n}.
\end{aligned}$$

4. The proof.

<u>Thm</u>. Let $n = 2m$. Then, there is a map $S^{2n-1} \to S^n$ of Hopf invariant 1 if and only if $n = 2, 4, 8$.

<u>proof</u>. We've discussed existence. So, suppose $S^{2n-1} \to S^n$ is a map with $n = 2m$. Then,

$$\widetilde{KU}(X_f) \cong \mathbb{Z} \oplus \mathbb{Z},$$

fitting into

$$0 \to \widetilde{KU}(S^{2n}) \to \widetilde{KU}(X_f) \to \widetilde{KU}(S^n) \to 0.$$

Let $\alpha \in \widetilde{KU}(S^{2n})$ denote $(H-1)^{*n}$. Let $\beta \in \widetilde{KU}(X_f)$ map to $(H-1)^{*m}$. Since $((H-1)^{*m})^2 = 0$ in $\widetilde{KU}(S^n)$,

$$\beta^2 = h(f)\,\alpha.$$

<u>Lemma</u>. $h(f) = \pm H(f)$.

<u>proof</u>. Black box: because the cohomology of X_f is concentrated in even degrees, there is a filtration on $KU(X_f)$ whose associated graded is $H^*(X_f, \mathbb{Z})$.

$$\psi^k(\alpha) = k^n \alpha = k^{2m}\alpha.$$

$$\psi^k(\beta) = k^m \beta + \mu_k \alpha.$$

$$\psi^2(\psi^3(\beta)) = \psi^2(3^m \beta + \mu_3 \alpha)$$

$$= 3^m \psi^2(\beta) + 2^{2m}\mu_3 \alpha$$

$$= 3^m(2^m \beta + \mu_2 \alpha) + 2^{2m}\mu_3\alpha.$$

$$\psi^3(\psi^2(\beta)) = \psi^3(2^m \beta + \mu_2 \alpha)$$

$$= 2^m(3^m \beta + \mu_3 \alpha) + 3^{2m}\mu_2 \alpha.$$

Hence, $3^m \mu_2 + 2^{2m}\mu_3 = 2^m \mu_3 + 3^{2m}\mu_2$, or

$$(2^{2m} - 2^m)\mu_3 = (3^{2m} - 3^m)\mu_2.$$

Now, $\psi^2(\beta) \equiv \beta^2 \mod 2$

$$= h(f)\alpha.$$

Hence,

$$h(f)\alpha \equiv 2^m \beta + \mu_2 \alpha \mod 2$$

$$\equiv \mu_2 \alpha \mod 2.$$

So, μ_2 is odd. Now, 2^m must divide $3^{2m} - 3^m = 3^m(3^m - 1)$, so 2^m divides $3^m - 1$.

<u>Lemma/Exercise</u>. If $2^m \mid 3^m - 1$, then $m = 1, 2, 4$.