UNIVERSITY OF CHICAGO SUMMER SCHOOL 2016

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1. Whitehead, CW complexes, homology, cohomology

Spaces are built up out of cells: disks attached to one another. The CW approximation theorem states that for every space X there exists a CW complex Z and a map $Z \longrightarrow X$ such that $\pi_i(Z) \longrightarrow \pi_i(X)$ is an isomorphism for all $i \geq 0$. (In the case $i = 0$ by "isomorphism" we mean "bijection.") This means that in order to study spaces up to weak equivalence, it suffices to study CW complexes.

(This is a bit of a cheat: we say that homotopy groups are the things we care about. These only see maps from spheres into the space.... so we can only detect what spaces look like up to attaching spheres. So this shouldn't be too surprising.)

We may ask why we care about weak equivalences, if the question we were asking originally was to study spaces up to homotopy equivalence. It turns out that between CW complexes, weak equivalences and homotopy equivalences are the same thing:

Theorem 1.1 (Whitehead)**.** *If a map $f\colon X \longrightarrow Y$ is a weak equivalence then f is a homotopy equivalence.*

This follows from the cellular approximation theorem:

Theorem 1.2. *Any map $f\colon X \longrightarrow Y$ between CW complexes is homotopic to a cellular map. Moreover, if the map is already cellular on a subcomplex A of X then the homotopy can be chosen to be the identity on A.*

Thus studying CW complexes up to homotopy eqiuvalence is exactly the same as studying CW complexes up to weak equivalence. Since most spaces we care about are CW complexes already, it isn't that big of a deal.

So now we want to define some invariants on spaces up to homotopy equivalence. Actually, in the spirit of category theory and focusing not just on objects but on morphisms, we want an invariant of spaces and maps.

Definition 1.3. A *(reduced) homology theory* is a sequence of functors $\widetilde{h}_n\colon \mathbf{Top} \longrightarrow \mathbf{Ab}$ satisfying the following axioms:

1. If $f \simeq g$ then the induced maps $f_* = h_n f$ and g_* are equal.
2. For every CW pair (X, A) there exist boundary homomorphisms $\partial\colon \widetilde{h}_n(X/A) \longrightarrow \widetilde{h}_{n-1}(A)$ which fit into an exact sequence
$$\longrightarrow \widetilde{h}_n A \longrightarrow \widetilde{h}_n X \longrightarrow \widetilde{h}_n(X/A) \xrightarrow{\partial} \widetilde{h}_{n-1} A \longrightarrow .$$
These are natural in the sense that for every map of CW pairs $f\colon (X, A) \longrightarrow (Y, B)$ there exists a square
$$\begin{array}{ccc} \widetilde{h}_n(X/A) & \xrightarrow{\partial} & \widetilde{h}_{n-1}A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{h}_n(Y/B) & \xrightarrow{\partial} & \widetilde{h}_{n-1}B \end{array}$$

(3) For any wedge sum $X = \bigvee X_\alpha$ with inclusions $i_\alpha \colon X_\alpha \longrightarrow X$ the induced map $\bigoplus_\alpha \widetilde{h}_n(X_\alpha) \longrightarrow \widetilde{h}_n(X)$ is an isomorphism.

These axioms are a form of the Eilenberg–Steenrod axioms. (I won't go into the historical details, but these are the most modern version of these.)

We're going to construct a homology theory. Let X be a CW complex. We define a chain complex $C_n(X)$ by letting $C_n(X)$ be the free abelian group generated by the n-cells of X. We need to define a map $C_nX \longrightarrow C_{n-1}X$. This map is defined as $d_n[e^n_\alpha] = \sum_\beta c_{\alpha\beta}[e^{n-1}_\beta]$, so we just need to figure out what the coefficients $c_{\alpha\beta}$ are. For an n-cell e^n_α and an $n-1$-cell e^{n-1}_β we define $c_{\alpha\beta}$ to be the degree of the map

$$S^{n-1} \xrightarrow{\chi_\alpha} X_{n-1} \longrightarrow X_{n-1}/(X_{n-1} \backslash e^\beta_{n-1}) \longrightarrow S^{n-1},$$

where the first map is the attaching map of e^n_α, the second map collapses all of X_{n-1} except e^{n-1}_β to a point, and the last map is given by the characteristic map of e^{n-1}_β. We then define $\widetilde{h}_n$ to be the n-th homology of this complex. For $n = -1$ we define $C_{-1}(X) = \mathbb{Z}$ and the map d_0 takes the sum of the coefficients. All other negative n's are 0.

We're going to skip verifying axiom (2). For (3), note that for $n \neq 0$ the generators satisfy the conditions given, and for a cell in X_α the characteristic map only touches cells in X_α. Thus (3) works. To see (1), by cellular approximation we can assume that f and g are homotopic by a cellular homotopy $X \times I \longrightarrow Y$. The chain complex for $X \times I$ is given in the following way: truncate the chain complexes $C_n(X)$ and $C_n(I)$ so that they are 0 at negative dimensions. Now take their tensor product: this has as the k-th group $\oplus_{i+j=k} C_k(X) \otimes C_k(I)$, with the boundary given by $d(x \otimes y) = d(x) \otimes y + (-1)^{\deg x} x \otimes d(y)$. Then augment the chain complex by adding a $\mathbb{Z}$ in degree -1. (EXERCISE: check this). We have a homotopy between cellular maps, so by cellular approximation it homotopic to a cellular map, so we get a cellular map $X \times I \longrightarrow Y$, which gives a map between these chain complexes. It turns out (EXERCISE: check this) that when restricted to the subcomplex of the form $C_*(X) \otimes C_*(0)$ this is f and when restricted to $C_0(X) \otimes C_*(1)$ this is g, and the map gives a chain homotopy (EXERCISE: how?). Two chain-homotopic maps give the same map on homology, so we're done.

Let's do some calculations. Note that if X has only one 0-cell then $\widetilde{h}_0(X) = 0$, since the map from $C_0(X)$ to $C_{-1}(X)$ will always be an isomorphism. As this will be the case for almost all of our examples we're going to say this always. In addition, all negative groups are always 0, since the map $C_0(X) \longrightarrow C_{-1}(X)$ is surjective.

Let $X = S^n$. S^n has a CW structure with one 0-cell and one n-cell. If $n > 1$ then $C_m(X)$ is $\mathbb{Z}$ at n, 0 and -1 and 0 elsewhere, and therefore we have $\widetilde{h}_n(S^n) = \mathbb{Z}$ and $\widetilde{h}_m(S^n) = 0$ otherwise. When $n = 0$ we have $C_m(X)$ is $\mathbb{Z}^2$ at 0 and $\mathbb{Z}$ at -1, so $\widetilde{h}_0(X) = \mathbb{Z}$ and $\widetilde{h}_m(X) = 0$ otherwise. When $n = 1$ we have $\mathbb{Z}$ at 1, 0 and -1. Since the map from the 0-th group to the 1-st group is an isomorphism, the map from the 1-st group to the 0-th is 0. (Also, we can explicitly compute that the map is 0 explicitly, since the two endpoints of the 1-cell are attached to the same place. CHECK THIS) Thus in general we have that $\widetilde{h}_m S^n$ is $\mathbb{Z}$ at $m = n$ and 0 otherwise.

We can always prove this in a slightly different way to prove this. We will show that for all X, $\widetilde{h}_{n+1}(\Sigma X) = \widetilde{h}_n X$. We have a CW pair (CX, X) (with the CW structure coming from the CW structure on $X \times I$). Now CX is contractible, so by (1) $\widetilde{h}_n(CX) \cong \widetilde{h}_n * = 0$ for all n. Thus we have a long exact sequence

$$\widetilde{h}_n CX \longrightarrow \widetilde{h}_n(CX/X) \longrightarrow \widetilde{h}_{n-1} X \longrightarrow \widetilde{h}_{n-1} CX.$$

The two endpoints are 0, so the middle map is an isomorphism. Since $CX/X \cong \Sigma X$, this follows.

Consider $\mathbb{R}P^n$. $\mathbb{R}P^n$ can be written as $S^n / \pm 1$. If we take a CW structure on S^n with two cells in each dimension, with the -1-action swapping the cells. Thus $\mathbb{R}P^n$ has a CW structure with one cell in each dimension, and thus $C_*(X)$ has one cell in each degree $-1, \dots, n$. We now need to figure out what the boundary maps look like. Since this has a single 0-cell we don't need to worry about what's going on for $C_m(X)$ at $m = 0$, so we'll only worry about higher groups. The degree of the map is $1 + (-1)^n$, since for any point inside S^{n-1} there are two points in the preimage, one which is locally mapped to by the identity, and the other of which is mapped by $(-1)^m$. Thus when m is even the map is 2, and when m is odd it is 0. (This agrees with the other discussion that it is 0 at $m = 1$.) Thus the chain complex has maps alternating 0 and 2, and we see that $\widetilde{h}_m \mathbb{R}P^n$ for m odd less than n, and 0 otherwise. The only exception to this is when n is odd, in which case we have $\widetilde{h}_n = \mathbb{Z}$.

Consider $\mathbb{C}P^n$. This has a CW structure with a single cell in each even dimension 0 through $2n$. Thus the chain complex we construct has no consecutive groups other than the ones in dimensions -1 and 0, and thus we must have $h_m \mathbb{C}P^n$ is $\mathbb{Z}$ when $m = 2, 4, \dots, 2n$ and 0 otherwise.

The axioms we've been discussing have been for reduced homology. To get the same thing for unreduced homology, we define $H_n(X) = \widetilde{h}_n(X_+)$, where X_+ is X with a disjoint basepoint added. When working with unreduced homology there is an extra axiom needed, called excision. This is explained both in Hatcher and in Concise, (I suggest reading both together).

Now we need to talk a bit about cohomology. A cohomology theory is almost the dual of a homology theory; it's not quite the dual, because instead of taking the dual of the homology groups, we take the dual of the chain complexes that form them. This actually makes a rather large difference for computation. We can write down axioms for cohomology in the same way as the axioms for homology. To define a cohomology theory we take $C_n(X)$ and dualize it: we define $\widetilde{h}^n(X)$ to be the cohomology of the chain complex $C^n(X) = \mathrm{Hom}(C_n(X), \mathbb{Z})$. The boundary map is defined by precomposition with the boundary map on $C_n(X)$.

Just from the definitions we see that $\widetilde{h}^m(S^n)$ is $\mathbb{Z}$ if $m = n$ and 0 otherwise, and that $\widetilde{h}^m(\mathbb{C}P^n)$ is $\mathbb{Z}$ if $m = 2, 4, \dots, 2n$ and 0 otherwise. What is the cohomology of $\mathbb{R}P^n$? As before, we have a $\mathbb{Z}$ in each dimension -1 through n. The maps, however, are different: the multiplications by 2 shift, and we get $\mathbb{Z}/2$'s in the even dimensions and 0's in the odd dimensions.