UNIVERSITY OF CHICAGO SUMMER SCHOOL 2016

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1. Cobordism II: Cobordism groups are homotopy groups of Thom spaces

Last time we constructed a map from Ω_*^{fr} to π_*^s. The proof strongly used the framing on manifolds; in this talk we'll construct a similar map for the weakest form of cobordism: the one where we don't require any structure at all on the manifold.

We begin with a short discussion of transversality, as this will be very important for the constructions.

Definition 1.1. Let $f\colon Y \longrightarrow X$ is a smooth map of manifolds and $i\colon M \hookrightarrow X$ is an embedding of a submanifold, we say that f *is transverse to* i if for every $x \in M$ and every $y \in f^{-1}(x)$, the map

$$df \oplus di\colon T_pY \oplus T_xM \longrightarrow T_xX$$

is surjective.

Example: if M is a single point x, then f is transverse to i whenever x is a regular value of f.

Theorem 1.2. *If f is transverse to i then $f^{-1}(M)$ is a smooth submanifold of Y. The set of maps transverse to i is dense in the mapping space of functions $Y \longrightarrow X$.*

Since in addition the smooth maps are a dense subset of the mapping space, we see that every map can be approximated arbitrarily closely by a smooth map which is transverse to M.

The goal of this lecture is to show that $\mathfrak{N}_n$ is isomorphic to the stable homotopy group $\pi_n MO$. In order to discuss this, we first need to discuss Thom spaces. You have already implicitly seen Thom spaces in previous lectures, but right now we'll make it explicit.

Definition 1.3. Given an n-dimensional vector bundle $E \longrightarrow B$ the *Thom space of* E, written $T(E)$, is one of the following:

(1) Let $\widehat{E} \longrightarrow B$ be the S^n-bundle obtained by one-point compactifying all of the fibers. $T(E)$ is the quotient of $\widehat{E}$ by the section at infty, which glues all of the new points together into one.
(2) If B is conpact, $T(E)$ is the one-point compactification of E.
(3) If $E \longrightarrow B$ contains a sub-n-disk bundle E' then $T(E) = E'/\partial E'$.

When multiple definitions apply they agree.

EXERCISE: Given a pullback square

$$\begin{array}{ccc} E' & \xrightarrow{f} & E \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

we get a map $T(f)\colon T(E') \longrightarrow T(E)$.

Definition 1.4. Let γ_n be the universal n-plane bundle on $BO(n)$. We define $MO(n) = T(\gamma_n)$.

Earlier you saw the Thom isomorphism; we can rephrase it in terms of Thom spaces by saying that it's an isomoprhism $\Phi; H^k(B, \mathbb{Z}/2) \longrightarrow \widetilde{H}^{k+n}(T(E), \mathbb{Z}/2)$.

Now we're going to do the following. Given any n-manifold M, embed it in $\mathbb{R}^{n+k}$ for a high enough k, and let ν be the normal bundle. Let E be a tubular neighborhood for M, and consider $E/\partial E$. Since E is an embedding of ν into $\mathbb{R}^{n+k}$,

$$T(\nu) \cong E/\partial E \cong \mathbb{R}^{n+k}/(\mathbb{R}^{n+k} \backslash E) \cong S^{n+k}/(S^{n+k} \backslash E).$$

However, ν is a vector bundle and therefore comes with a classifying pullback square

$$\begin{array}{ccc} \nu & \xrightarrow{\alpha} & \gamma_k \\ \downarrow & & \downarrow \\ M & \longrightarrow & BO(k) \end{array}$$

and thus there exists a map $T(\alpha): T(\nu) \longrightarrow T(\gamma_k)$. Thus the choice of $M \hookrightarrow \mathbb{R}^{n+k}$ produces a map

$$S^{n+k} \longrightarrow S^{n+k}/(S^{n+k} \backslash E) \cong T(\nu) \xrightarrow{T(\alpha)} T(\gamma_k).$$

Write $MO(k) = T(\gamma_k)$, and $\pi_n MO = \operatorname{colim}_k \pi_{n+k} T(\gamma_k)$. Thus we have a map $\pi_{n+k} MO(k) \longrightarrow \pi_n MO$, and the above construction gives us a map $\mathfrak{N}_n \longrightarrow \pi_n MO$. Note that we have several things we need to check to show that this is well-defined:

(1) Making k larger does not change the stable homotopy type of the map.
PROOF: If we change the embedding by composing with the usual embedding $\mathbb{R}^{n+k} \longrightarrow \mathbb{R}^{n+k+1}$ then ν changes to $\nu \oplus \mathbb{R}$, which changes $T(\nu)$ to $\Sigma T(\nu)$, and the map that is produced to $T(\gamma_{k+1})$ is exactly the composition

$$\Sigma T(\nu) \xrightarrow{\Sigma T(\alpha)} \Sigma T(\gamma_k) \longrightarrow T(\gamma_{k+1}).$$

Thus the stable homotopy class of the map does not change.

(2) Changing the embedding does not change the stable homotopy type of the map.
PROOF: For any two choices of embeddings $\iota: M \longrightarrow \mathbb{R}^{n+k}$ and $\iota': M \longrightarrow \mathbb{R}^{n+k'}$ there exists a k'' and a commutative square of embeddings

$$\begin{array}{ccc} M & \xrightarrow{\iota} & \mathbb{R}^{n+k} \\ {\scriptstyle \iota'}\downarrow & & \downarrow \\ \mathbb{R}^{n+k'} & \longrightarrow & \mathbb{R}^{n+k''} \end{array}$$

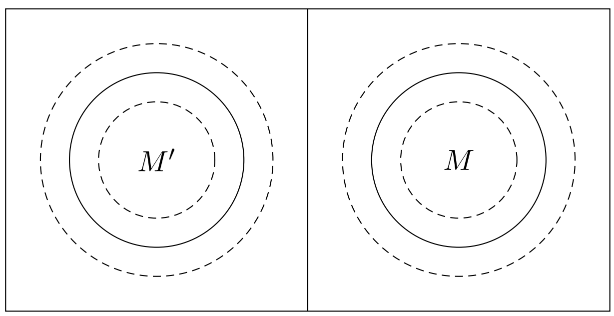
Since this diagram commutes, by the previous part the elements they produce in $\pi_{n+k''}(T(\gamma_{k''})$ are the same.

(3) If M and M' are cobordant they produce homotopic maps.
PROOF: Let (W, M, M') be a cobordism. Then pick an embedding of W into $\mathbb{R}^{m+k} \times [0,1]$ so that we have a diagram

$$\begin{array}{ccc} M & & \\ \downarrow & \searrow & \\ W & \longrightarrow & \mathbb{R}^{m+k} \times [0,1] \\ \uparrow & \nearrow & \\ M' & & \end{array}$$

Now do the Pontrjagin–Thom construction along W fiberwise along $[0,1]$, so that we get a morphism $S^{n+k} \times [0,1] \longrightarrow T(\gamma_k)$; at $\{0\}$ and $\{1\}$ it restricts to the morphisms for M, so this map gives a homotopy between the two maps, and is therefore well-defined.

Thus we see that the map we get is well-defined. Now we want to show that it's a group homomorphism (which we didn't show last time). Let M, M' be two manifolds, and pick embeddings $M \longrightarrow \mathbb{R}^{n+k}$ and $M' \longrightarrow \mathbb{R}^{n+k}$. Since these are both compact, we can assume that the $n+k$-th coordinate of the image of M is always negative, and the one for M' is always positive. Thus we have the following picture:



The dashed regions are the tubular neighborhoods of M and M' that we have chosen. We now have two ways to do the Pontrjagin–Thom construction. We can think of all of $\mathbb{R}^{n+k}$ as being S^{n+k} and then do the collapse from there; this gives us a map $S^{n+k} \longrightarrow T(\gamma_k)$. Alternately, we can think of the left half-space as one copy of S^{n+k} and the right half-space as the other and do the Pontrjagin-Thom construction in each; this gives us a map $S^{n+k} \vee S^{n+k} \longrightarrow T(\gamma_k)$. If we precompose with the pinch map $\nabla\colon S^{n+k} \longrightarrow S^{n+k} \vee S^{n+k}$ we get the same map we got before. Thus

$$PT(M \sqcup M') = (PT(M) \vee PT(M')) \circ \nabla.$$

This says exactly that we get a group homomorphism.

We would now like to show that this homomorphism is an isomorphism. Let us construct an inverse map. Given an element $\alpha \in \pi_n MO$, pick a representative for it in $\pi_{n+k} MO(k)$; this is a map $f\colon S^{n+k} \longrightarrow MO(k)$. Now since γ_k is the colimit of the universal bundles $\gamma_{m,k}$ over $Gr_k(\mathbb{R}^m)$ and S^{n+k} is compact, we know that the image of f lies in $T(\gamma_{m,k})$. Thus we have the following diagram:

$$\begin{array}{ccc} S^{n+k} & \xrightarrow{f} & T(\gamma_{m,k}) \\ & & \uparrow{\scriptstyle i} \\ & & Gr_k(\mathbb{R}^m) \end{array}$$

Here, the inclusion of $Gr_k(\mathbb{R}^m)$ is as the 0-section in $T(\gamma_{m,k})$. In analogy to the previous construction, we'd like to take the pullback of this to be the manifold. However, it doesn't necessarily need to be; this is where transversality comes in. An application of Thom's transversality theorem says that f can be homotoped to be transverse to $Gr_k(\mathbb{R}^m)$, so that this preimage will, indeed, be a manifold. This is the inverse map.

A remark about doing other kinds of cobordism: In the Pontrjagin–Thom construction, we used several facts:

- we have an embedding into $\mathbb{R}^{n+k}$ that preserves the important data,
- the normal bundle has the correct data, and
- the normal bundle is classified by a map into a classifying space.

For any other kind of cobordism (oriented, spin, framed, etc.) the embeddings and map into a classifying space use this data. However, the Pontrjagin–Thom construction is not changed. Thus to do other kinds of cobordism we just need to be able to construct the above data, and the map will work.

However, if we want to do other kinds of cobordism, for example with PL-manifolds then we need more: we need a generalization of transversality so that we can construct the inverse map. This turns out to be the hardest part, but once transversality is generalized a similar proof works to classify cobordism rings.