UNIVERSITY OF CHICAGO SUMMER SCHOOL 2016

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1. Cobordism I: Introduction, framed cobordism

Reference: *Algebraic and Geometric Surgery*, by Andrew Ranicki. See chapters 2 and 6, especially.

Here is a new relation between manifolds. Two manifolds M and N are *cobordant* if there exists a manifold W whose boundary is $M \sqcup N$. (If the manifolds are oriented and we care about orientation, wee want the boundary of W to be $M \sqcup -N$.) We will write (W, M, N) for a cobordism W from M to M'. What are some examples of cobordism?

(1) A manifold is cobordant to itself because the boundary of $M \times I$ is $M \sqcup M$.
(2) For any manifold with boundary W, ∂W is cobordant to the empty set.
(3) n circles are cobordant to m circles, by "dressing" an alien with n legs, $m-1$ arms and one head.
(4) (Composition) Given a cobordism from M to M' and another cobordism from M' to M'' we can produce a cobordism from M to M'' by gluing them together along the M'. (Note that here we are assuming that we have a collar of the boundary of W. This is a standard assumption to make, and we will not be mentioning it again.)
(5) (Surgery) Suppose that we have an n-manifold M, and an embedding $\iota\colon S^p \times D^q \longrightarrow M$, where $p + q = n$. Note that

$$\partial(S^p \times D^q) = S^p \times S^{q-1} = \partial(D^{p+1} \times S^{q-1}.$$

Thus we can cut out the interior of $S^p \times D^q$ and glue in $D^{p+1} \times S^{q-1}$ along the same boundary (in any way that we like). This is called a *surgery* on M.[a] (For some nice pictures, see Wikipedia.) Then we can write down the manifold

$$W = (M \times I) \cup_{S^p \times D^q \times \{1\}} (D^{p+1} \times D^q).^{\text{b}}$$

W is called the *trace* of the surgery.

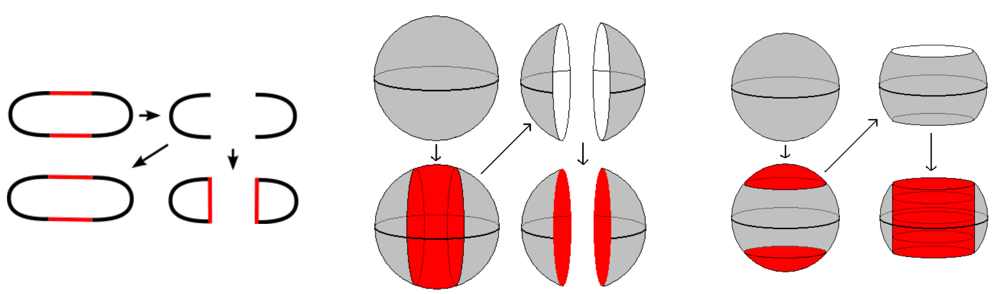


Figure 1. Can remove a $S^0 \times D^1$ from a circle, attach in one of two ways. From a sphere, can remove either $S^0 \times D^2$ or $S^1 \times D^1$. Note that the cylinder does NOT need to be added in such a nice manner; if we attach the bottom circle with the opposite orientation we get the Klein bottle. Images are from Wikipedia.

[a]If you have seen it before, compare this to handle attaching. They are related but not the same.

[b]Note that here we are actually attaching a handle.

In fact, it turns out that all cobordisms can be obtained by doing compositions of surgeries. This requires some Morse theory, but is not very difficult. (See, for example, Chapter 2 of Ranicki.)

But there are lots and lots of ways of doing surgery, and that doesn't give a great classification of cobordism. (We would, ideally, like it to be computable, for example.) Thus, instead of classifying cobordisms, let's try to classify cobordism classes of manifolds. We define Ω_n to be the set of n-dimensional equivalence classes of manifolds. Note that we can put an addition on this by taking disjoint union of manifolds. Because we can think of different types of structures of manifolds we can have different structures on this, including non-oriented, oriented, complex, framed, spin, etc. They all have slightly different theories. Usually these are distinguished by decoration on the Ω, although unoriented cobordism is often written $\mathfrak{N}_n$. Note that with this operation Ω_n becomes a group, since $[M] + [-M] = [\emptyset]$, and $[\emptyset]$ is the identity.

We can now go further and define $\Omega_* = \oplus_{n\geq 0}\Omega_n$. This is actually a ring, with $[M][N] = [M \times N]$. (Note that if M is cobordant to M' via W then $W \times N$ is a cobordism from $M \times N$ to $M' \times N$, so this is well-defined.) Again, we can also do this with decorations. Tomorrow we're going to talk more about unoriented cobordism, but today we're going to talk instead about *framed* cobordism.

Definition 1.1. A *framing* of an n-manifold M is an embedding $M \hookrightarrow \mathbb{R}^{n+k}$ together with a choice b of trivialization of the normal bundle. (NB: *not* the tangent bundle.) Note that if we have a trivialization of the normal bundle $M \hookrightarrow S^{n+k}$ then we can construct one for $M \hookrightarrow S^{n+k+\ell}$ for all $\ell \geq 0$. [c]

If we let ν be the normal bundle to M inside $\mathbb{R}^{n+k}$, we can think of the framing as a pullback square

$$\begin{array}{ccc} \nu & \xrightarrow{b} & \epsilon^k \\ \downarrow & & \downarrow \\ M & \longrightarrow & * \end{array}$$

One way to get a framed manifold is to find a differentiable function $f\colon \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k$ for which 0 is a regular value. Then the preimage of 0 is a framed manifold.

We would like to identify the framed cobordism ring Ω_*^{fr}.

Theorem 1.2 (Ponrjagin)**.** *As graded rings,*

$$\Omega_*^{fr} \cong \pi_*^s.$$

Recall that $\pi_m^s = \operatorname{colim}_k[S^{m+k}, S^k]$. The map $[S^{m+k}, S^k] \longrightarrow [S^{m+k+1}, S^{k+1}]$ takes a map $f\colon S^{m+k} \longrightarrow S^k$ to $f \wedge 1\colon S^{m+k} \wedge S^1 \longrightarrow S^k \wedge S^1$.

Proof. For today we'll ignore the ring structure and focus on proving that $\Omega_m^{fr} \cong \pi_m^s S^0$. Note that it makes sense that we get *stable* homotopy groups of spheres: a manifold is framed if there is a framing in SOME dimension. Let $B_m(\mathbb{R}^{m+k})$ be the set of classes of pairs (M, b), where M is an m-manifold with an embedding $M \hookrightarrow \mathbb{R}^{m+k}$ and b is a trivialization of its normal bundle. We say that $(M, b) \sim (M', b')$ if there exists a cobordism (W, M, M') with an embedding $f\colon W \longrightarrow \mathbb{R}^{m+k+1}$ with a trivialization of the normal bundle which restricts to the trivializations of M and M' on M and M', respectively. The key observation here is that

$$\Omega_m^{fr} \cong \operatorname*{colim}_k B_m(\mathbb{R}^{m+k}),$$

where the colimit includes $B_m(\mathbb{R}^{m+k})$ into $B_m(\mathbb{R}^{m+k+1})$ by taking (M, b) to $(M, b \times \epsilon)$, where the trivialization points up along the last coordinate.

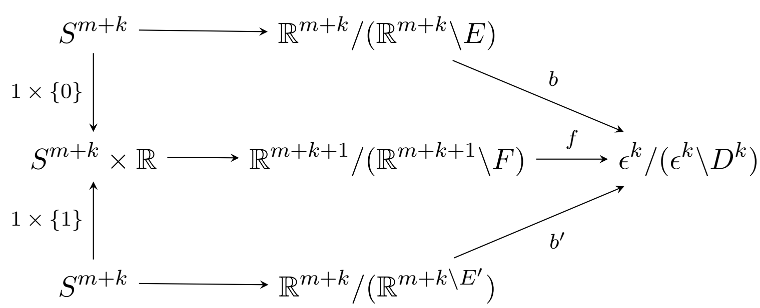
[c] A manifold can be framed if and only if its tangent bundle is stably trivial.

The key observation here is that $B_m(\mathbb{R}^{m+k}) \cong [S^{m+k}, S^k]$. We begin by constructing a map $[S^{m+k}, S^k] \longrightarrow B_m(\mathbb{R}^{m+k})$. Let $g\colon S^{m+k} \longrightarrow S^k$ be a map. If we think of S^k as the one-point compactification of $\mathbb{R}^k$ we have a 0 point in it, and we can take $g^{-1}(0)$. If 0 is a regular value of g then as we mentioned before, $g^{-1}(0)$ is an m-submanifold of S^{m+k}, and this gives a framing of $g^{-1}(0)$. It turns out that any map is homotopic to a smooth map with 0 a regular value, so if we chose the representative g correctly then this will work. (This is the Thom transversality theorem; Ranicki calls it the Sard–Thom transversality thoerem.) In addition, a homotopy between two representatives can also be chosen with this property, and the preimage of 0 will be exactly a cobordism between the two preimages. (EXERCISE: check this) Thus this direction is well-defined.

In the other direction we need the Pontrjagin–Thom construction. Suppose we are given a framed manifold (M, b) in $B_m(\mathbb{R}^{m+k})$. Let E be a tubular neighborhood of M inside $\mathbb{R}^{m+k}$ and let E' be the disk bundle sitting inside ν, which we can just think of as being the preimage of D^k inside ϵ^k. Then we take the following composite:

$$S^{m+k} \longrightarrow \mathbb{R}^{m+k}/(\mathbb{R}^{m+k}\backslash E) \cong \nu/(\nu\backslash E') \xrightarrow{b} \epsilon^k/(\epsilon^k\backslash D^k) \cong S^k.$$

This gives a map $S^{m+k} \longrightarrow S^k$. If $[(M, b)] = [(M', b')]$, let (W, M, M, f) be the cobordism and framing relating these. Let F be a tubular neighborhood of $W \times \mathbb{R}$ inside $\mathbb{R}^{m+k+1}$. Then we have a diagram



Thus the middle map is a homotopy between the map you get for (M, b) and the map you get for (M', b'); thus this is a well-defined map to homotopy classes of maps.

(EXERCISE: check that these are inverses.)

What does the colimit of the $B_m(\mathbb{R}^{m+k})$'s do? It keeps adding dimensions to the $\mathbb{R}^{m+k}$ trivially, which if you trace through the definition of the second map you see that it adds a trivial dimension, so the colimit produces the stable homotopy groups of spheres. □