WEDNESDAY - TALK 11
COBORDISM 4

Contents

1. The SW of the trivial bundle

Example 1.1. For any X, let $\epsilon_n = \epsilon_n(X)$ be the trivial n–plane bundle on X. Then, $\epsilon_n(X) = f^*(\epsilon_n(pt))$ for $f : X \to pt$. So in particular, $w(\epsilon_n(X)) = 1$. This implies that

$$w(\xi \oplus \epsilon_n) = w(\xi)$$

for any n.

2. Normal bundles and the Whitney product formula

Now, let $M \xrightarrow{f} \mathbb{R}^{n+k}$ be the immersion of a smooth manifold M. (An immersion means that the map on tangent spaces $TM_x \to T\mathbb{R}^{n+k}_{f(x)}$ is an injection. I.e., locally, f looks like an injection.) Then we can define a k–plane bundle $\nu(M)$ which is the orthogonal complement of TM in $\mathbb{R}^{n+k}$. Note that

$$\nu(M) \oplus TM = T\mathbb{R}^{n+k} = \epsilon_{n+k}(\mathbb{R}^k).$$

Therefore, letting $w(M) = w(TM)$, we have the *Whitney product formula*

$$w(\nu(M))w(M) = 1.$$

Note that $w(\nu(M))$ does not depend on ν, so we think of $w(\nu(M))$ as the formal inverse of $w(M)$ and write

$$\overline{w}(M) = w(\nu(M)).$$

Example 2.1. The natural inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ has a trivial normal bundle, so

$$\overline{w}(S^n) = 1$$

This forces $w(S^n) = 1$, so that the SW do not see the tangent bundle of S^n. However, S^n has a non–trivial tangent bundle for $n > 1$ (you will see this for $n = 2$ in the exercises).

3. w of the tautological line bundle on $\mathbb{R}P^n$

Consider $\gamma^1_n \to \mathbb{R}P^n$ the tautological line bundle. (In fact, we have

$$\gamma^1_n \subset \mathbb{R}P^n \times \mathbb{R}^{n+1} = \epsilon_n(\mathbb{R}P^n)$$

is the set (ℓ, v) where $v \in \ell$.) Further, there's a pullback

$$\begin{array}{ccc} \gamma_1 = f^*(\gamma^1_n) & \longrightarrow & \gamma^1_n \\ \downarrow & & \downarrow \\ \mathbb{R}P^1 & \xrightarrow{f} & \mathbb{R}P^n \end{array} .$$

We have

$$f^*(w_1(\gamma_n^1)) = w_1(f^*(\gamma_n^1)) = w_1(\gamma_1) \neq 0$$

So

$$w(\gamma_n^1) = 1 + w_1.$$

4. w of the tangent bundle of $\mathbb{R}P^n$

We want to compute

$$w(\mathbb{R}P^n).$$

This takes some thought. We will describe it for $\mathbb{R}P^2$.

The tangent bundle on a manifold M at a point is given by the different directions one can move along M. Given a line in $\mathbb{R}^{n+1}$, you can tell me how to move the line by giving a linear transformation

$$\ell \to \mathbb{R}^{n+1},$$

that is, an element of $\mathrm{Hom}_{\mathbb{R}}(\ell, \mathbb{R}^{n+1})$. But,

$$\mathrm{Hom}_{\mathbb{R}}(\ell, \mathbb{R}^{n+1}) \cong \mathrm{Hom}_{\mathbb{R}}(\ell, \mathbb{R})^{\oplus n+1} \cong (\ell^*)^{\oplus n+1}.$$

So, we get a surjective map

$$(\ell^*)^{\oplus n+1} \to T\mathbb{R}P^n_\ell.$$

However, if you happened to give me a whose image was in $\ell \subseteq \mathbb{R}^{n+1}$, this will represent a zero vector in the tangent space, so we have an exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{R}}(\ell, \ell) \to (\ell^*)^{\oplus n+1} \to T\mathbb{R}P^n_\ell \to 0.$$

Letting ℓ vary and noting that $\mathrm{Hom}_{\mathbb{R}}(\gamma_1, \gamma_1) = \epsilon_1$ and $\gamma_1^* \cong \gamma_1$, we get

$$0 \to \epsilon_1 \to (\ell)^{\oplus n+1} \to T\mathbb{R}P^n \to 0.$$

So

$$T\mathbb{R}P^n \oplus \epsilon_1 = (\ell)^{\oplus n+1}.$$

This means that

$$w(\mathbb{R}P^n) = (1 + w_1)^{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} w_1^i.$$

Example 4.1. If $n + 1 = 2^k$, then

$$w(\mathbb{R}P^n)(1 + w_1)^{n+1} \equiv 1 + w_1^{2^k} \mod (2) = 1$$

since $w_1^{2^k} = 0$ in $H^*(\mathbb{R}P^n)$. However, if $n+1$ is not a power of 2, then one of $\binom{n+1}{i} \neq 0$ for some $0 < i < n+1$ and hence $w_i(\mathbb{R}P^n) \neq 0$. Therefore, if $\mathbb{R}P^n$ has a trivial tangent bundle, $n = 2^k - 1$. The fact that this only happens if $n = 1$, 3 and 7 is a deep fact of mathematics proved by Adams called the Hopf Invariant one problem.

Example 4.2. We have

$$w(\mathbb{R}P^2) = \sum_{i=0}^{3} w_1^i = 1 + w_1 + w_1^2$$

(since $w_1^3 = 0$).

We can use this to compute

$$\overline{w}(\mathbb{R}P^2) = 1 + w_1$$

since

$$(1 + w_1 + w_1^2)(1 + w_1) = 1.$$

Note in particular that if $\mathbb{R}P^2 \subset \mathbb{R}^{2+k}$ is an immersion, then $w_1(\nu(\mathbb{R}P^2)) \neq 0$, so $\nu(\mathbb{R}P^2)$ has dimension at least 1 and hence, $k \geq 1$. We conclude that if $\mathbb{R}P^2$ immerses into $\mathbb{R}^m$, then $m \geq 3$. (It's a theorem of Whitney that $\mathbb{R}P^2$ can be immersed in $\mathbb{R}^3$).

5. Stiefel-Whitney numbers

Let $r_1, \ldots, r_n$ be such that $r_1 + 2r_2 + \ldots + nr_n = n$. Then for ξ a vector bundle over X,

$$w_1(\xi)^{r_1} \ldots w_n(\xi)^{r_n} \in H^n(X)$$

Now, if $\xi = TM$, then we get

$$w_1(M)^{r_1} \ldots w_n(M)^{r_n} \in H^n(M)$$

so we can pair it with the fundamental class

$$w_1^{r_1} \ldots w_n^{r_n}[M] := \langle w_1(M)^{r_1} \ldots w_n(M)^{r_n}, [M] \rangle \in \mathbb{Z}/2.$$

We get a set of numbers called the Stiefel-Whitney numbers of M, and these form an important invariant of M.

Theorem 5.1 (Pontrjagin-Thom). *Let M be a smooth compact closed manifold. Then the Stiefel-Whitney numbers of M are zero if and only if M is the boundary of some smooth compact manifold.*

First, we make some observations. Let W be an $n+1$–dimensional manifold with boundary $M = \partial W$. The orientation on W induces an orientation on its boundary M. In fact, the orientation of W is a choice of fundamental class

$$[W] \in H_{n+1}(W, M),$$

and $[W]$ restricts to a fundamental class $[M]$ of M under the boundary homomorphism

$$H_{n+1}(W, M) \xrightarrow{\partial} H_n(M),$$

so

$$\partial([W]) = [M].$$

Intuitively, a triangulation of W induces a triangulation of M. If you think of the sum of the $n+1$-simplices of W as giving the fundamental class in homology for W, then you can see that $\partial([W])$ will give the sum of the n–simplices of M, and hence represent a fundamental class. To actually prove it, you need to use excision a couple times.

We can choose a smooth tubular neighborhood V of M in W, and because $V \cong M \times \mathbb{R}_{\geq 0}$, this gives an inward pointing trivial normal vector field on M, which splits a trivial line bundle off of the tangent bundle of W restricted to M:

$$i^*TW = TW|_M = TM \oplus \epsilon.$$

So we have

$$w_k(M) = w_k(TM) = w_k(TM \oplus \epsilon) = w_k(i^*TW) = i^*w_k(TW) = i^*w_k(W).$$

Now we evaluate the SW numbers. Let $r_1 + 2r_2 + \ldots + nr_n = n$

$$\begin{aligned} w_1^{r_1} \ldots w_n^{r_n}[M] &= \langle w_1^{r_1}(M) \ldots w_n^{r_n}(M), [M] \rangle \\ &= \langle i^*w_1^{r_1}(W) \ldots i^*w_n^{r_n}(W), \partial[W] \rangle \\ &= \langle i^*(w_1^{r_1}(W) \ldots w_n^{r_n}(W)), \partial[W] \rangle . \end{aligned}$$

You need to verify that, given $f : X \to Y$ the following diagram commutes

$$\begin{array}{ccc} H^n(Y) \otimes H_n(X) & \xrightarrow{f^* \otimes 1} & H^n(X) \otimes H_n(X) \\ \downarrow{\scriptstyle 1 \otimes f_*} & & \downarrow{\scriptstyle \langle -,- \rangle} \\ H^n(Y) \otimes H_n(Y) & \xrightarrow{\langle -,- \rangle} & H^n(X) \otimes H_n(X). \end{array}$$

That is,

$$\langle f^*\alpha, a \rangle = \langle \alpha, f_*a \rangle .$$

But if you do, you'll see that

$$\begin{aligned} w_1^{r_1} \dots w_n^{r_n}[M] &= \langle i^*(w_1^{r_1}(W) \dots w_n^{r_n}(W)), \partial[W] \rangle \\ &= \langle w_1^{r_1}(W) \dots w_n^{r_n}(W), i_* \partial[W] \rangle \\ &= 0 \end{aligned}$$

since $i_* \partial = 0$.

Corollary 5.2. *If M and M' are cobordant, then their SW numbers are equal.*

This will help us show the other implication. We want to show that if the all the Stiefel–Whitney numbers of M are zero, then M is a boundary. That is, the image of M in $\mathcal{N}_n$ is zero.

Let $BO = Gr(\mathbb{R}^\infty) = \bigcup_{k \geq 0} Gr_k(\mathbb{R}^\infty)$. Then recall that

$$H^*(BO) = \mathbb{Z}/2[w_1, w_2, \dots].$$

Since two cobordant manifolds have equal SW numbers, there is a well defined pairing

$$H^n(BO) \otimes \mathcal{N}_n \xrightarrow{\#} \mathbb{Z}/2,$$

which sends the cobordism class of a manifold M to $w_1^{r_1} \dots w_n^{r_n}[M]$.

If all of the Stiefel–Whitney numbers M were zero but M was not a boundary (i.e. $[M] \neq 0$ in $\mathcal{N}_n$), then we would conclude that this pairing is singular. Indeed, for this M, we would have that for all $w_1^{r_1} \dots w_n^{r_n} \in H^n(BO)$, $\#(w_1^{r_1} \dots w_n^{r_n} \otimes [M]) = 0$ even though $[M] \neq 0$. So, it's enough to show that this can't happen.

We appeal to things Zhouli told us about. We have maps:

(a) $\alpha : \mathcal{N}_n \xrightarrow{\cong} \pi_n(MO)$ the Pontrjagin-Thom collapse.
(b) $h : \pi_n(MO) \hookrightarrow H_n(MO)$ the Hurewicz homomorphism
(c) $\Phi : H_n(MO) \xrightarrow{\cong} H_n(BO)$ the stable Thom isomorphism.

We know that the map

$$\mathcal{H} : \mathcal{N}_n \xrightarrow{\alpha} \pi_n(MO) \xrightarrow{h} H_n(MO) \xrightarrow{\Phi} H_n(BO)$$

is injective, and one can show that the diagram

$$\begin{array}{ccc} H^n(BO) \otimes \mathcal{N}_n & \xrightarrow{1 \otimes \mathcal{H}} & H^n(BO) \otimes H_n(BO) \\ & \searrow^{\#} & \downarrow \langle -,- \rangle \\ & & \mathbb{Z}/2 \end{array}$$

commutes. So # cannot be singular!