Problems for the UChicago ATSS-Thursday

July 28, 2016

It will make your life easier to remember the following fact. If

$$F \longrightarrow E \longrightarrow B$$

is a fibration (whatever that is), then, for any space X, you get a 'long exact sequence':

$$\cdots \longrightarrow [X, \Omega^2 B] \longrightarrow [X, \Omega F] \longrightarrow [X, \Omega E] \longrightarrow [X, \Omega B] \longrightarrow [X, F] \longrightarrow [X, E] \longrightarrow [X, B]$$

In particular, when X is a sphere, you can use the adjunction between Σ and Ω to get a long exact sequence:

$$\cdots \longrightarrow \pi_{n+1} B \longrightarrow \pi_n F \longrightarrow \pi_n E \longrightarrow \pi_n B \longrightarrow \pi_{n-1} F \longrightarrow \cdots \longrightarrow \pi_0 E \longrightarrow \pi_0 B \longrightarrow 0$$

Where 'long exact sequence' means what you think it means when the terms are groups, and means something else when they are pointed sets.

Problem 1. Prove 'by hand' that:

(a) $\widetilde{K}(S^1) = 0$,

(b) $\widetilde{K}(S^2) = \mathbb{Z}$,

(c) $\widetilde{KO}(S^1) = \mathbb{Z}/2$.

Give explicit vector bundles that generate these groups.

Problem 2. Show that the inclusion induces an isomorphism $\pi_k O(n) \cong \pi_k O(n+1)$ for $n > k+1$.

Problem 3. Show that $\pi_2 BO = \pi_1 O = \mathbb{Z}/2$.

Problem 4. Show that $\pi_3 BO = \pi_2 SO(4)$. Now compute this in any way you like.

Problem 5. Can you catch 'em all? Can you make it to $\pi_8 BO$? (That would earn you a Pokébadge).

Problem 6. Find the group completion of the multiplicative abelian monoid $\mathbb{Z}_{>0}$.

Problem 7. Prove using an adjoint functor theorem that the group completion of an abelian monoid exists.

Problem 8. Prove that if 2^m divides $3^m - 1$ for some $m \geq 0$, then $m = 1, 2, 4$. Hint: prove by induction that if $m = 2^\ell n$ with n odd, then the highest power of 2 dividing $3^m - 1$ is 2 for $\ell = 0$ and $2^{\ell+2}$ for $\ell > 0$.

Problem 9. Show by hand (arguing with vector bundles) that if $A \subset X$ is a closed inclusion (of, say, compact Hausdorff spaces) then the following sequence is exact:

$$K(X) \to K(A) \to \widetilde{K}(X/A)$$

Definition. A map $f : X \to Y$ is a **(weak)** n**-equivalence** if $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection and if for every choice of basepoint $x \in X$ the map $\pi_n(f) : \pi_i(X, x) \to \pi_i(Y, f(x))$ is an isomorphism for $i < n$ and a surjection for $i = n$.

Problem 10. (a) Show that there is a homotopy fiber sequence

$$U(n) \to U(n+1) \to S^{2n+1}$$

for all $n \geq 0$.

(b) Show that $BU(n) \to BU(n+1)$ is a $2n+1$-equivalence for all $n \geq 0$.

Problem 11. Suppose that X is a $2n$-dimensional CW complex. Prove that the maps

$$[X, BU(n)] \to [X, BU(n+1)] \to [X, BU(n+2)] \to \cdots \to [X, BU]$$

are all bijections.

These problems show that there are strong stable range results for vector bundles, even without the help of Bott periodicity. Of course, Bott periodicity makes these results much stronger.

Problem 12. Use Bott periodicity and Postnikov towers to compute $[X, BU(2)]$ when

(a) $\dim X \leq 4$,

(b) $\dim X \leq 5$, and

(c) $\dim X \leq 6$.

Note that part (a) gives a computation of $\widetilde{K}(X)$ when X is connected and $\dim X \leq 4$ by Problem 11.

The following problems will only make sense after Friday's talk.

Problem 13. When G acts trivially on X we have: (i) for $G = C_2$, $K\mathbf{R}^0(X) = KO^0(X)$, and (ii) $K^0_G(X) = R(G) \otimes_{\mathbb{Z}} K^0(X)$.

Problem 14. If G acts freely on X then $K^0_G(X) = K^0(X/G)$.

Problem 15. The previous result is *not* true for $K\mathbf{R}$ (a counterexample is the space $S^1 \times S^1$ where C_2 acts by the antipodal action on the second factor; but it's hard to show this directly.) Nevertheless, prove that $K\mathbf{R}^0(X \times C_2) \cong K^0(X)$.

Problem 16. If Y is a space with an action of H, define $G \times_H Y$ as the quotient of $G \times Y$ by the relation $(gh, y) \sim (g, hy)$. Then $K^0_G(G \times_H Y) \cong K^0_H(Y)$.

Problem 17. Let ρ denote the regular representation of C_2 (i.e. the action on $\mathbb{R}^2$ given by permuting the basis vectors.) Show that there is an equivariant homeomorphism $S^\rho \cong \mathbb{C}P^1$ where C_2 acts by conjugation on $\mathbb{C}P^1$. (Recall that, for a vector space V, S^V denotes the one-point compactification.)