Twisted cohomology of configuration spaces of the plane and spaces of maximal tori via point-counting

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Abstract

We consider two families of algebraic varieties $Y_n$ indexed by natural numbers $n$: the configuration space of unordered $n$-tuples of distinct points on $\mathbb{C}$, and the space of unordered $n$-tuples of linearly independent lines in $\mathbb{C}^n$. Let $W_n$ be any sequence of virtual $S_n$-representations given by a character polynomial, we compute $H^i(Y_n; W_n)$ for all $i$ and all $n$ in terms of double generating functions. One consequence of the computation is a new recurrence phenomenon: the stable twisted Betti numbers $\lim_{n \to \infty} \dim H^i(Y_n; W_n)$ are linearly recurrent in $i$. Our method is to compute twisted point-counts on the $\mathbb{F}_q$-points of certain algebraic varieties, and then pass through the Grothendieck-Lefschetz fixed point formula to prove results in topology. We also generalize a result of Church-Ellenberg-Farb about the configuration spaces of the affine line to those of a general smooth variety.

1 Introduction

We consider two families of spaces indexed by natural numbers $n$. The first family is the configuration space of ordered $n$-tuples of distinct points in a manifold $M$:

$$\text{PConf}_n M := \{(x_1, \cdots, x_n) \in M^n : x_i \neq x_j, \ \forall i \neq j \}.$$ 

The symmetric group $S_n$ acts freely on $\text{PConf}_n M$ by permuting the ordered points. The quotient $\text{Conf}_n M := \text{PConf}_n M/S_n$ is the configuration space of unordered $n$-tuples of distinct points. The second family is the space of $n$ linearly independent lines in $\mathbb{C}^n$:

$$\tilde{T}_n(\mathbb{C}) := \{(L_1, \cdots, L_n) : L_i \text{ a line in } \mathbb{C}^n, L_1, \cdots, L_n \text{ linearly independent}\}.$$ 

$S_n$ acts freely on $\tilde{T}_n(\mathbb{C})$ by permuting the ordered lines. The quotient $\mathcal{T}_n(\mathbb{C}) := \tilde{T}_n(\mathbb{C})/S_n$ can be identified with the space of maximal tori in $\text{GL}_n(\mathbb{C})$. See Section 3 for more details.

Every normal $S_n$-cover $X \to Y$ gives a natural bijection between representations of $S_n$ and local systems on $Y$ that become trivial when restricted to $X$. Thus, every $S_n$-representations give rise to a local system on $\text{Conf}_n M$ and on $\mathcal{T}_n(\mathbb{C})$.

Question 1 (Twisted Betti numbers). What are the twisted Betti numbers $\dim H^i(\text{Conf}_n M; W_n)$ and $\dim H^i(\mathcal{T}_n(\mathbb{C}); W_n)$ for each $i$ and $n$, and for each representation $W_n$ of $S_n$?
These twisted Betti numbers have geometric, arithmetic, and combinatorial meaning (see e.g. Sections 2 and 5 in \[10\]). The program of computing these numbers dates back to the work of Arnol’d in the 1960s. For example, in the simplest nontrivial case when \( M = \mathbb{C} \), and when \( W_n \) is the trivial, the sign, or the standard representations of \( S_n \), the numbers \( \dim H^i(\text{Conf}_n(\mathbb{C});W_n) \) have been calculated by Arnol’d \[1\], Cohen \[8\], and Vassiliev \[18\]. However, there is no known formula of \( \dim H^i(\text{Conf}_n(\mathbb{C});W_n) \), for all \( i \) and \( n \) and \( W_n \). In his 2014 ICM talk, Farb proposed a list of problems, one of which (Problem 2.1 in \[10\]) is equivalent to Question 1. See Remark 1 below for more details. The present paper contains two collections of results: one topological and one arithmetic. We will use the arithmetic results to obtain results in topology.

**Topological results:**

- Theorem 1 computes the generating functions of \( \dim H^i(\text{Conf}_n(\mathbb{C});W_n) \) and of \( \dim H^i(T_n(\mathbb{C});W_n) \), giving systematic answers to Question 1 for \( M = \mathbb{C} \).

- In Corollary 2 we discover a new recurrence phenomenon: the stable twisted Betti numbers \( \lim_{n \to \infty} \dim H^i(\text{Conf}_n(\mathbb{C});W_n) \) and \( \lim_{n \to \infty} \dim H^i(T_n(\mathbb{C});W_n) \) eventually satisfy linear recurrence relations in \( i \).

**Arithmetic results:**

- Theorem 3 computes weighted point-counts on the \( \mathbb{F}_q \)-points of \( \text{Conf}_n V \) where \( V \) is any smooth, connected variety over \( \mathbb{F}_q \) of positive dimension.

- Corollary 4 states that when \( n \to \infty \), the weighted point-counts on the \( \mathbb{F}_q \)-points of \( \text{Conf}_n V \) converges in some appropriate sense. This gives a new proof of a recent theorem of Farb-Wolfson and generalizes a theorem of Church-Ellenberg-Farb.

### 1.1 Computing twisted Betti numbers.

We will consider Question 1 in a more general setting where \( W_n \) is allowed to be a virtual \( S_n \)-representation, i.e. a formal \( \mathbb{Q} \)-linear combination of \( S_n \)-representations. Virtual representations are in natural bijection with the set of class functions of \( S_n \). In this case, \( \dim H^i(\text{Conf}_n(\mathbb{C});W_n) \) and \( \dim H^i(T_n(\mathbb{C});W_n) \) are now well-defined rational numbers since the cohomology functor is additive when taking direct sums in coefficients.

For each positive integer \( k \), define \( X_k : \coprod_{n=1}^\infty S_n \to \mathbb{Z} \) to be the class function with \( X_k(\sigma) \) the number of \( k \)-cycles in the unique cycle decomposition of \( \sigma \in S_n \). A character polynomial is a polynomial \( P \in \mathbb{Q}[X_1,X_2,\cdots] \). It defines a class function on \( S_n \) for all \( n \). Define the degree of a character polynomial by letting each variable \( X_k \) to have degree \( k \).

Every partition \( \lambda = 1^{\lambda_1} 2^{\lambda_2} \cdots t^{\lambda_t} \) defines a character polynomial by

\[
\binom{X}{\lambda} := \binom{X_1}{\lambda_1} \binom{X_2}{\lambda_2} \cdots \binom{X_t}{\lambda_t}.
\]

\( \binom{X}{\lambda} \) has degree \( |\lambda| := \sum_{k=1}^t k \lambda_k \). For each fixed \( n \), every class function on \( S_n \) is a \( \mathbb{Q} \)-linear combination of character polynomials of the form \( \binom{X}{\lambda} \). For example, the indicator function on the conjugacy class of \( \sigma \in S_n \) is \( \binom{X}{\lambda} \) where \( \lambda = 1^{X_1(\sigma)} \cdots n^{X_n(\sigma)} \). Therefore, to answer Question 1 it suffices to restrict our attention to when \( W_n := \binom{X}{\lambda} \).
Theorem 1 (Generating function for twisted Betti numbers). Let $\mu$ be the classical Möbius function, and let $M_k(z^{-1}) := \frac{1}{k} \sum_{j \mid k} \mu\left(\frac{k}{j}\right)z^{-j}$ be the $k$-th necklace polynomial in $z^{-1}$. For any integer partition $\lambda = 1^{l_1}2^{l_2}\cdots l_{l_1}$, we have the following two equations of formal power series in two variables $z$ and $t$.

\begin{align*}
(I) \quad & \quad \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \dim H^i(\text{Conf}_n(\mathbb{C}); \left(\begin{array}{c} X \\ \frac{\lambda}{\mu} \end{array}\right))(-z)^it^n = \frac{1 - zt^2}{1 - t} \prod_{k=1}^{i} \left(\frac{M_k(z^{-1})}{\lambda_k}\right) \left(\frac{(tz)^k}{1 + (tz)^k}\right)^{\lambda_k}, \\
(II) \quad & \quad \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \dim H^{2i}(\text{Conf}_n(\mathbb{C}); \left(\begin{array}{c} X \\ \frac{\lambda}{\mu} \end{array}\right))\left(\prod_{k=1}^{i} \frac{1}{\lambda_k!}\left(\frac{t^k}{k(1 - z^k)}\right)^{\lambda_k} \cdot \prod_{j=0}^{\infty} \frac{1}{1 - tz^j}\right) \quad \text{in the decomposition of} \quad H^i(\text{PConf}_n(\mathbb{M}); \mathbb{Q}).
\end{align*}

In (I), all negative power of $z$ in $M_k(z^{-1})$ will cancel with other positive powers of $z$ so that the right-hand-side of the equality is indeed a series in $z$ and $t$. In (II), we only consider $H^{2i}(\text{Conf}_n(\mathbb{C}); \left(\begin{array}{c} X \\ \frac{\lambda}{\mu} \end{array}\right))$ because $H^{2i+1}(\text{Conf}_n(\mathbb{C}); \left(\begin{array}{c} X \\ \frac{\lambda}{\mu} \end{array}\right)) = 0$ which is a result of Borel [5].

Remark 1 (Representation stability). Farb proposed the following problem (Problem 2.1 in [10]): for a manifold $M$, compute the decomposition of $H^i(\text{PConf}_n(M); \mathbb{Q})$ into a sum of irreducible representations of $S_n$. The remarkable result of representation stability states that such a decomposition does not depend on $n$ when $n$ is sufficiently large, which was first proved by Church-Farb [4] for $M = \mathbb{C}$, and later by Church [3] for $M$ any connected orientable manifold of finite type (see also [12] for a different proof). Farb proposed a second problem (Problem 3.5 in [10]) of computing the stable decomposition of $H^i(\text{PConf}_n(M); \mathbb{Q})$ when $n$ is sufficiently large. Note that for any $S_n$-representation $W_n$, the transfer isomorphism associated to the $S_n$-cover $\text{PConf}_n(M) \to \text{Conf}_n(M)$ gives:

$$\dim H^i(\text{Conf}_n(M); W_n) = \langle H^i(\text{PConf}_n(M); \mathbb{Q}), W_n \rangle_{S_n},$$

where $\langle U, V \rangle_{S_n}$ stands for the usual inner product of two $S_n$-representations $U$ and $V$. Hence, computing the multiplicities of $W_n$ in the decomposition of $H^i(\text{PConf}_n(\mathbb{C}); \mathbb{Q})$ are equivalent to computing twisted Betti numbers of $\text{Conf}_n(M)$ in $W_n$.

The simplest nontrivial case for Farb’s two questions is when $M = \mathbb{C}$. Theorem 1 (I) reduces the questions in this case to computing Taylor expansions of rational functions. See Section 2.3 for more discussion and examples. Previous works of Arnol’d, Brieskorn, and Lehrer-Solomon gave various descriptions of $H^i(\text{PConf}_n(\mathbb{C}); \mathbb{Q})$ as representations of $S_n$ (see e.g. [14], and the references contained therein). In theory, these descriptions make it possible determine $\dim H^i(\text{Conf}_n(\mathbb{C}); W_n)$ for fixed $i$ and $n$ and $W_n$ via (1.1). However, to my knowledge, Theorem 1 (I) gives the first systematic formula of $\dim H^i(\text{Conf}_n(\mathbb{C}); W_n)$ for all $i$ and $n$ and $W_n$. Moreover, the formula implies structural properties such as stability and recurrence (see Corollary 2 below) for $\dim H^i(\text{Conf}_n(\mathbb{C}); W_n)$ as $n$ and $i$ change, which were not visible from the results of Arnol’d, Brieskorn, and Lehrer-Solomon.

Remark 2 (Twisted homological stability). Representation stability for $\text{PConf}_n(\mathbb{C})$ implies twisted homological stability for $\text{Conf}_n(\mathbb{C})$. Precisely, Church-Ellenberg-Farb (Theorems 1.9 in [8]) proved that for any character polynomial $P$ and for each fixed $i$, the twisted Betti numbers $\dim H^i(\text{Conf}_n(\mathbb{C}); P)$ stabilize when $n$ is sufficiently large. Later, Hersh-Reiner gave a different proof of the stability of $\dim H^i(\text{Conf}_n(\mathbb{C}); P)$ with an improved stable range in $n$ (Theorem 4.3 in [13]). We will give a third proof of this stability result in Corollary 2 using Theorem 1. The implied stable range is a small improvement of that obtained by Hersh-Reiner, and is optimal (see Remark 6 below). The three papers ([6], [13] and the present one) land at the same result from three totally different points of views respectively: topological, combinatorial, and arithmetic.
Linear recurrence of stable twisted Betti numbers in $i$. Besides finding new proofs of homological stability, we discover a new phenomenon: the stable cohomology of $\text{Conf}_n(\mathbb{C})$ and $\mathcal{T}_n(\mathbb{C})$ as $n \to \infty$ with twisted coefficients are linearly recurrent in $i$.

**Corollary 2 (Linear recurrence of stable twisted Betti numbers).** Fix an arbitrary character polynomial $P \in \mathbb{Q}[X_1, X_2, \cdots]$. Let $N = \deg P$.

(I) For each $i$, denote $\alpha_i := \lim_{n \to \infty} \dim H^i(\text{Conf}_n(\mathbb{C}); P)$. There exist integers $c_1, \cdots, c_N$ such that for all $i \geq N + 2$,

$$\alpha_i = c_1 \alpha_{i-1} + c_2 \alpha_{i-2} + \cdots + c_N \alpha_{i-N}.$$ 

(II) For each $i$, denote $\beta_i := \lim_{n \to \infty} \dim H^{2i}(\mathcal{T}_n(\mathbb{C}); P)$. There exist integers $d_1, \cdots, d_N$ such that for all $i \geq N$,

$$\beta_i = d_1 \beta_{i-1} + d_2 \beta_{i-2} + \cdots + d_N \beta_{i-N}.$$ 

For example, if we let $\alpha_i := \lim_{n \to \infty} \dim H^i(\text{Conf}_n(\mathbb{C}); \bigwedge^2 \mathbb{Q}^{n-1})$ where $\mathbb{Q}^{n-1}$ is the standard representation of $S_n$, then $\alpha_i$ satisfies the linear recurrence relation:

$$\alpha_i = 2\alpha_{i-1} - 2\alpha_{i-2} + 2\alpha_{i-3} - \alpha_{i-4}.$$  

See Section 2.8 for more details.

**Remark 3 (Topological proof?).** We deduce Corollary 2 from Theorem 1 by explicitly calculating the generating functions of $\alpha_i$ and $\beta_i$ as rational functions. The proof of Theorem 1 uses point-counting, hence crucially depends on the fact that $\text{Conf}_n(\mathbb{C})$ and $\mathcal{T}_n(\mathbb{C})$ are algebraic varieties. Is there any proof of Corollary 2 using only topology? Are there other examples of recurrent stable twisted Betti numbers in $i$?

**Method: point-counting over finite fields.** The method in this article combines ideas from two beautiful papers: one by Church-Ellenberg-Farb [7] and the other by Fulman [12]. Church-Ellenberg-Farb observed that there is a remarkable bridge, provided by the Grothendieck-Lefschetz fixed point theorem in étale cohomology, between cohomology in local coefficients (topology) and weighted point-counts on varieties over finite fields (arithmetic). Furthermore, they apply representation stability in topology to prove that certain weighted point-counts converge. Later, Fulman used a different method to improve the arithmetic calculations stated in [7] and obtained certain “finite $n$” formulas. In this paper, we will systematically extend Fulman’s calculations of weighted point-counts, and combine it with the approach of Church-Ellenberg-Farb but in the opposite direction: we use point-counting to compute cohomology.

The idea of using point-counting to study the topology of configuration spaces dates back at least to the work of Lehrer-Kisin [14], and is also used in Section 4.3 of [7]. Our results are continuations of the theme developed by Lehrer-Kisin and Church-Ellenberg-Farb: structures in the cohomology (e.g. stability and recurrence) are often reflected in the arithmetic of corresponding varieties, and *vice versa.*
1.2 Weighted point-counts on configuration spaces of smooth varieties.

Fulman’s method in [12] allows us to generalize a result of Church-Ellenberg-Farb as follows. Let Conf\(_n V\) be the configuration space of unordered \(n\)-tuples of distinct points on a smooth variety \(V\) defined over \(\mathbb{Z}\). When \(V\) is the affine line, Conf\(_n \mathbb{A}^1\) is just Conf\(_n\) as discussed above. For brevity we will consistently use Conf\(_n\) to abbreviate for Conf\(_n \mathbb{A}^1\) throughout the paper. In general, every class function of \(S_n\) gives a function Conf\(_n V(\mathbb{F}_q) \to \mathbb{Q}\), which can be viewed as a weighting (see Section 2.1 for more details). The following theorem computes weighted point-counts on Conf\(_n V(\mathbb{F}_q)\) in terms of the zeta function \(Z(V,t)\) of \(V\) over \(\mathbb{F}_q\).

**Theorem 3 (Weighted point-counts on Conf\(_n V\)).** Let \(V\) be a smooth, connected variety over \(\mathbb{Z}\) of positive dimension, and let \(q\) be an odd prime power. Let \(\mu\) be the Möbius function, and define \(M_k(V,q) := \frac{1}{k} \sum_{m|k} \mu \left( \frac{k}{m} \right) |V(\mathbb{F}_{q^m})|\) for each \(k\). For any integer partition \(\lambda = 1^{\lambda_1} 2^{\lambda_2} \cdots l^{\lambda_l}\), we have the following equality of formal power series in \(t\):

\[
\sum_{n=0}^\infty \left[ \sum_{C \in \text{Conf}_n V(\mathbb{F}_q)} \left( \frac{X}{\lambda} \right)(\sigma_C) \right] t^n = \frac{Z(V,t)}{Z(V,t^2)} \prod_{k=1}^l \left( \frac{M_k(V,q)}{\lambda_k} \right) \left( \frac{t^k}{1 + t^k} \right)^{\lambda_k} \quad \text{(1.2)}
\]

Thanks to Weil conjectures (proved by Dwork, Grothendieck, Deligne et al.), \(Z(V,t)\) is a rational function in \(t\) with a simple pole at \(t = q^{-\dim V}\), which is of the smallest absolute value among all other poles or zeros of \(Z(V,t)\). By examining the location of poles in the generating function (1.2), we see that any point-count on Conf\(_n V(\mathbb{F}_q)\) weighted by a character polynomial converges as \(n \to \infty\) in the following sense.

**Corollary 4 (Convergence of weighted point-counts).** With the same assumptions as in Theorem 3 and letting \(d\) be the dimension of the variety \(V\), we have:

(a) Define \(\tilde{Z}(V,t)\) to be the rational function \(Z(V,t) \cdot (1 - q^d t)\) in \(t\). Then

\[
\lim_{n \to \infty} \frac{1}{q^{nd}} \sum_{C \in \text{Conf}_n V(\mathbb{F}_q)} \left( \frac{X}{\lambda} \right)(\sigma_C) = \frac{\tilde{Z}(V,q^{-d})}{\left( Z(V,q^{-2d}) \right)} \prod_{k=1}^l \left( \frac{M_k(V,q)}{\lambda_k} \right) \left( \frac{1}{1 + q^{kd}} \right)^{\lambda_k} \quad \text{(1.3)}
\]

In particular, for any character polynomial \(P\) the following limit exists:

\[
\lim_{n \to \infty} \frac{1}{q^{nd}} \sum_{C \in \text{Conf}_n V(\mathbb{F}_q)} P(\sigma_C). \quad \text{(1.4)}
\]

(b) The expected value of \(\left( \frac{X}{\lambda} \right)\) as a random variable on Conf\(_n V(\mathbb{F}_q)\) converges:

\[
\lim_{n \to \infty} \frac{1}{|\text{Conf}_n V(\mathbb{F}_q)|} \sum_{C \in \text{Conf}_n V(\mathbb{F}_q)} \left( \frac{X}{\lambda} \right)(\sigma_C) = \prod_{k=1}^l \left( \frac{M_k(V,q)}{\lambda_k} \right) \left( \frac{1}{1 + q^{kd}} \right)^{\lambda_k} \quad \text{(1.5)}
\]

**Remark 4 (Related works).** The convergence of (1.4) in the special case when \(V = \mathbb{A}^1\) was first proved by Church-Ellenberg-Farb (Theorem I in [7]). Part (a) generalizes their result to a general smooth variety. It concurs with the recent work of Farb-Wolfson, where they extend the topological approach of [7] and gives a different formula for the left-hand-side of (1.3) in terms of the étale cohomology of PConf\(_n V\)(Theorem B in [11]). Our proof,
inspired by the work of Fulman, is different from the topological approach in [7] and [11]. We obtain not only the asymptotic formula as \( n \to \infty \) (Corollary 4), but also a generating function for all \( n \) (Theorem 3).

Remark 5 (Probabilistic interpretation and analogs in number theory). Part (b) of Corollary 4 has the following probabilistic interpretation: the functions \( X_1, X_2, X_3, \ldots \), viewed as random variables on \( \text{Conf}_n V(\mathbb{F}_q) \), tends to independent random variables with binomial distribution as \( n \to \infty \). This is a geometric analog of the following fact in number theory: the \( p \)-adic orders, for \( p \) any prime number, of a random integer chosen uniformly from \( \{1, 2, \cdots, n\} \) tend to be independent random variables with geometric distributions as \( n \to \infty \). More results about weighted point-counts on \( \text{Conf}_n V(\mathbb{F}_q) \) (and other related spaces) motivated by this probabilistic point of view will be presented in the forthcoming work of the author [2].

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2 Cohomology of configuration spaces via point counting

In this section, we will first prove Theorem 3 and Corollary 4 about weighted point-counts on \( \text{Conf}_n V(\mathbb{F}_q) \). The main ideas of the proofs were already contained in Fulman’s paper [12], though he only proved the formulas in the special case when \( V = \mathbb{A}^1 \) and when \( \lambda = (0), (1) \) and \( (0, 1) \). We systematically extend Fulman’s result to all \( V \) and all \( \lambda \), using some technical input from the Weil conjectures. We then apply the general formula in the case when \( V = \mathbb{A}^1 \) to prove part (I) of Theorem 1 and of Corollary 2 about \( \text{Conf}_n (\mathbb{C}) \).

2.1 General set-up

Throughout this section, we will fix \( V \) to be a smooth and connected variety over \( \mathbb{Z} \) of dimension \( d \geq 1 \). Define the configuration space of \( V \) to be the (scheme-theoretic) quotient

\[
\text{Conf}_n V := \{ (x_1, \cdots, x_n) \in V^n : x_i \neq x_j, \forall i \neq j \}/S_n.
\]

where \( S_n \) acts on \( V^n \) by permuting the coordinates. \( \text{Conf}_n V \) is also a variety over \( \mathbb{Z} \) (by [17], page 66). So we can study its \( \mathbb{F}_q \)-points \( \text{Conf}_n V(\mathbb{F}_q) \). An element in \( \text{Conf}_n V(\mathbb{F}_q) \) is a set of distinct points \( C = \{x_1, \cdots, x_n\} \subseteq V(\mathbb{F}_q) \) such that the Frobenius map \( \text{Frob}_q : V(\mathbb{F}_q) \to V(\mathbb{F}_q) \) preserves the set. The action of \( \text{Frob}_q \) on \( C \) gives a permutation \( \sigma_C \in S_n \), well-defined and unique up to conjugacy. Therefore, any class function \( \chi \) of \( S_n \) gives a well-defined function \( \text{Conf}_n V(\mathbb{F}_q) \to \mathbb{Q} \) by \( C \mapsto \chi(\sigma_C) \).

Example: \( V = \mathbb{A}^1 \). When \( V \) is the affine line \( \mathbb{A}^1 \), we use \( \text{Conf}_n \) to abbreviate for \( \text{Conf}_n \mathbb{A}^1 \). Elements \( C \in \text{Conf}_n (\mathbb{F}_q) \) are in bijection with monic, square-free, degree-\( n \) polynomials in \( \mathbb{F}_q[x] \) via the map

\[
C = \{x_1, \cdots, x_n\} \mapsto f_C(x) := (x - x_1) \cdots (x - x_n).
\]
Under this bijection, \( X_k(\sigma_C) \), defined as the number of \( k \)-cycles in \( \sigma_C \), equals to the number of degree-\( k \) factors in the irreducible factorization of \( f_C(x) \) over \( \mathbb{F}_q \).

### 2.2 Proof of Theorem

First we recall some basic facts about the zeta function of a variety \( V \) over \( \mathbb{F}_q \):

\[
Z(V, t) := \prod_x (1 - q^{\deg x})^{-1}
\]

where the product is taken over all closed points \( x \) on \( V \) over \( \mathbb{F}_q \). Weil conjectures give that \( Z(V, t) \) is a rational function in \( t \). Let \( M_k(V, q) \) denote the number of closed points on \( V \) of degree \( k \), which is equivalently the number of orbits of \( \text{Frob}_q \) acting on \( V(\mathbb{F}_q) \) of size \( k \). We have

\[
Z(V, t) = \prod_{k=1}^{\infty} (1 - q^k)^{-M_k(V, q)}. \tag{2.1}
\]

Note that the fixed points of \( \text{Frob}_q \) on \( V(\mathbb{F}_q) \) are precisely \( V(\mathbb{F}_q) \). Similarly, for each \( k \) we have

\[
|V(\mathbb{F}_q^*)| = \sum_{m|k} m M_m(V, q).
\]

By Möbius inversion,

\[
M_k(V, t) = \frac{1}{k} \sum_{m|k} \mu\left(\frac{k}{m}\right) |V(\mathbb{F}_q^m)|.
\]

**Proof of Theorem**

Define a formal power series in \( x_1, \ldots, x_l \) and \( t \):

\[
F(x_1, \ldots, x_l, t) := \sum_{n=0}^{\infty} \left[ \sum_{C \in \text{Conf}_n V(\mathbb{F}_q)} x_1^{X_1(\sigma_C)} x_2^{X_2(\sigma_C)} \cdots x_l^{X_l(\sigma_C)} \right] t^n. \tag{2.2}
\]

Recall that an element \( C \in \text{Conf}_n V(\mathbb{F}_q) \) is just a subset of \( V(\mathbb{F}_q) \) of size \( n \) that is preserved by \( \text{Frob}_q \). Thus, every \( C \in \text{Conf}_n V(\mathbb{F}_q) \) can be decomposed uniquely into a disjoint union of distinct orbits of \( \text{Frob}_q \) acting on \( V(\mathbb{F}_q) \). The number of \( \text{Frob}_q \)-orbits in \( C \) of size \( k \) is \( X_k(\sigma_c) \). The unique decomposition of \( C \in \text{Conf}_n V(\mathbb{F}_q) \) into disjoint union of distinct \( \text{Frob}_q \)-orbits gives the following product formula, in analogy to the Euler product formula of Riemann’s zeta function given by the unique factorization of integers:

\[
F(x_1, \ldots, x_l, t) = \prod_{k>l} (1 + t^k)^{M_k(V, q)} \prod_{k \leq l} (1 + x_k t^k)^{M_k(V, q)}
\]

\[
= \prod_{k=1}^{\infty} (1 + t^k)^{M_k(V, q)} \prod_{k \leq l} \left( \frac{1 + x_k t^k}{1 + t^k} \right)^{M_k(V, q)}
\]

\[
= \prod_{k=1}^{\infty} \left( \frac{1 - t^{2k}}{1 - t^k} \right)^{M_k(V, q)} \prod_{k \leq l} \left( \frac{1 + x_k t^k}{1 + t^k} \right)^{M_k(V, q)}.
\]
By the product formula (2.1), we obtain
\[ F(x_1, \cdots, x_l, t) = \frac{Z(V, t)}{Z(V, t^2)} \prod_{k \leq l} \left( \frac{1 + x_k t^k}{1 + t^k} \right)^{M_k(V,q)}. \tag{2.3} \]

Next we apply the formal differential operator
\[ (\frac{\partial}{\partial x})^\lambda := (\frac{\partial}{\partial x_1})^{\lambda_1} \left( \frac{\partial}{\partial x_2} \right)^{\lambda_2} \cdots \left( \frac{\partial}{\partial x_l} \right)^{\lambda_l} \]
to the series \( F(x_1, \cdots, x_l, t) \) and then evaluate at \( (x_1, \cdots, x_l) = (1, \cdots, 1) \), obtaining the following equalities. The symbol \( \lambda! \) is an abbreviation for \( (\lambda_1)! (\lambda_2)! \cdots (\lambda_l)! \). Differentiating (2.2) gives
\[ (\frac{\partial}{\partial x})^\lambda F(1, \cdots, 1, t) = \lambda! \sum_{n=0}^{\infty} \left( \sum_{C \in \text{Conf}_n V(\mathbb{F}_q)} \left( \frac{X}{\lambda} \right)(\sigma_C) \right) t^n. \]

Differentiating (2.3) gives
\[ (\frac{\partial}{\partial x})^\lambda F(1, \cdots, 1, t) = \lambda! \frac{Z(V, t)}{Z(V, t^2)} \prod_{k=1}^{l} \left( \frac{M_k(V,q)}{\lambda_k} \right) \left( \frac{t^k}{1 + t^k} \right)^{\lambda_k}. \]

Theorem 3 follows by equating these two expressions for \( (\frac{\partial}{\partial x})^\lambda F(1, \cdots, 1, t) \).

\[ \square \]

2.3 Proof of Corollary 4

First we recall the following basic fact from calculus.

\textbf{Lemma 5.} \textit{Given} \( A(t) = \sum_{n=0}^{\infty} a_n t^n \) \textit{where} \( a_n \) \textit{are real numbers. Suppose} \( A(t) = H(t)/(1-ct) \) \textit{where} \( c \) \textit{is a constant, and the radius of convergence of} \( H(t) \) \textit{is strictly greater than} \( |c^{-1}| \). \textit{Then} \( \lim_{n \to \infty} \frac{a_n}{n!} \) \textit{exists and is equal to} \( H(c^{-1}) \).

Define
\[ A(t) := \frac{Z(V, t)}{Z(V, t^2)} \prod_{k=1}^{l} \left( \frac{M_k(V,q)}{\lambda_k} \right) \left( \frac{t^k}{1 + t^k} \right)^{\lambda_k} \]

The Riemann Hypothesis over finite fields (proved by Deligne [9]) says that \( Z(V,t) \) has a simple pole at \( t = q^{-d} \) where \( d = \dim V \). Moreover, each other zero or pole of \( Z(V,t) \) has absolute value \( q^{-d} \) for some \( j \leq 2d - 1 \). Thus,
\[ Z(V,t) := Z(V,t)(1-q^d t) \]

has no pole at \( |t| < q^{-d+\frac{1}{2}} \); while \( 1/Z(V; t^2) \) has no pole at \( |t| < q^{-2d} < q^{-d} \) (recall that \( d = \dim V > 0 \)). Hence \( A(t) \) and \( c = q^{-d} \) satisfy the hypothesis of Lemma 5 by which we conclude
\[ \lim_{n \to \infty} \frac{1}{q^{na}} \sum_{C \in \text{Conf}_n V(\mathbb{F}_q)} \left( \frac{X}{\lambda} \right)(\sigma_C) = \left[ A(t)(1-q^d t) \right]_{t=q^{-d}} \]

This establishes (1.3).

Every character polynomial \( P \) is a \( \mathbb{Q} \)-linear combination of \( \left( \frac{X}{\lambda} \right) \) for different \( \lambda \). Thus, the limit (1.4) converges for all \( P \). Part (a) is proved.
In the case when $\lambda = (0)$, part (a) gives
\[
\lim_{n \to \infty} \frac{|\text{Conf}_n V(F_q)|}{q^n d} = \frac{\tilde{Z}(V, q^{-d})}{Z(V, q^{-2d})}.
\] (2.4)
Part (b) follows by taking the ratio of (1.3) and (2.4).

### 2.4 Connecting arithmetic and topology of $\text{Conf}_n$

For the rest of this paper, we will focus on the case when $V = \mathbb{A}^1$. Recall that we use $\text{Conf}_n$ to abbreviate for $\text{Conf}_n \mathbb{A}^1$. Let $W$ be a representation of $S_n$, with character $\chi_W$. Church-Ellenberg-Farb proved the following equation connecting arithmetic of $\text{Conf}_n(F_q)$ and topology of $\text{Conf}_n(\mathbb{C})$: (Proposition 4.1 in [7])
\[
\sum_{C \in \text{Conf}_n(\mathbb{F}_q)} \chi_W(\sigma_C) = q^n \sum_i (-1)^i \dim H^i(\text{Conf}_n(\mathbb{C}); W) q^{-i}.
\] (2.5)
By additivity, same formula holds if we replace $W$ by a virtual representation. See Section 4 in [7] for how (2.5) is obtained from the Grothendieck-Lefschetz fixed point theorem in étale cohomology. Results from the previous section (in the case when $V = \mathbb{A}^1$) give us access to the left-hand-side of (2.5), from which we can prove results about $H^i(\text{Conf}_n(\mathbb{C}); W)$.

### 2.5 Proof of Theorem 1 (I)

We abbreviate the twisted Betti number as
\[
\alpha_i(n) := \dim H^i(\text{Conf}_n(\mathbb{C}); \left(\frac{X}{\lambda}\right))
\] (2.6)
for each $i$ and $n$. Define the double generating function for $\alpha_i(n)$ as the formal power series in two variables $z$ and $t$
\[
\Phi_\lambda(z, t) := \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \alpha_i(n)(-z)^i t^n.
\] (2.7)
We want to compute $\Phi_\lambda(z, t)$ as a rational function. We will need the following lemma.

**Lemma 6.** Suppose $\Phi(z, t)$ and $\Psi(z, t)$ are two power series in two formal variables $z$ and $t$. If for every prime power $q$, we have $\Phi(q^{-1}, t) = \Psi(q^{-1}, t)$ as formal power series in $t$, then $\Phi(z, t) = \Psi(z, t)$ as formal series in $z$ and $t$.

**Proof of Lemma.** Suppose $\Phi_\lambda(t, z) = \sum_{n=0}^{\infty} \phi_n(z)t^n$ and $\Psi(t, z) = \sum_{n=0}^{\infty} \psi_n(z)t^n$, where $\phi_n(z)$ and $\psi_n(z)$ are formal series in $z$ for each $n$. By hypothesis, for every prime power $q$, we have $\phi_n(q^{-1}) = \psi_n(q^{-1})$. Recall the following fact from calculus:

- If an infinite series $h(z) = \sum_{i=0}^{\infty} a_i z^i$ converges at $z = z_0$, then it converges absolutely at all $z$ with $|z| < |z_0|$.

Hence, both $\phi_n(z)$ and $\psi_n(z)$ are holomorphic functions on a disk with a positive radius centered at 0. Since $\phi_n(z) = \psi_n(z)$ for all $z \in \{q^{-1} | q$ is a prime power$\}$ which accumulates at 0, it must be $\phi_n(z) = \psi_n(z)$ as holomorphic functions. By the uniqueness of power series expansion, $\phi_n(z) = \psi_n(z)$ as formal series in $z$. Thus $\Phi(z, t) = \Psi(z, t)$ as formal series in $z$ and $t$. \qed
Now we evaluate the double generating function $\Phi_\lambda(z, t)$ at $z = q^{-1}$.

$$
\Phi_\lambda(q^{-1}, t) = \sum_{n=0}^\infty \sum_{i=0}^\infty (-1)^i b_i(n) q^{-i} t^n
= \sum_{n=0}^\infty \left[ \sum_{C \in \text{Conf}_n(\mathbb{F}_q)} \left( \frac{X}{\lambda} \right)(\sigma_C) \right] (q^{-1} t)^n
= \frac{Z(A^1, (tq^{-1})^2)}{Z(A^1, tq^{-1})} \prod_{k=1}^n \left( M_k(A^1, q) \right)^{\lambda_k} \left( \frac{(tq^{-1})^k}{1 + (tq^{-1})^k} \right)^{\lambda_k}.
$$

By Theorem 3

The $k$-th necklace polynomial in $x$ is

$$
M_k(x) = \frac{1}{k} \sum_{m|k} \mu\left( \frac{k}{m} \right) x^m.
$$

A standard calculation gives that $Z(A^1, t) = \frac{1}{1 - qt}$, and that $M_k(A^1, q) = M_k(q)$. Thus, we simplify the above:

$$\Phi_\lambda(q^{-1}, t) = \frac{1 - t q^{-1}}{1 - t} \prod_{k=1}^n \left( M_k(q) \right)^{\lambda_k} \left( \frac{(tq^{-1})^k}{1 + (tq^{-1})^k} \right)^{\lambda_k}. \tag{2.8}$$

Since (2.8) holds at $z = q^{-1}$ for any prime power $q$. By Lemma 6, the same equation holds when $q^{-1}$ is replaced by a formal variable $z$.

\begin{proof}
It suffices to consider when $P = (\lambda)$ for some partition $\lambda = 1^{\lambda_1} 2^{\lambda_2} \cdots l^{\lambda_l}$. In this case $\text{deg} (\lambda) = \sum_k k \lambda_k$. Let $\alpha_i(n)$ be as in (2.6), and let $\Phi_\lambda(z, t)$ be as in (2.7).

$$
(1 - t) \Phi_\lambda(t, z) = 1 + t \sum_{n=0}^\infty \sum_{i=0}^\infty [\alpha_i(n + 1) - \alpha_i(n)] z^i t^n
$$

\end{proof}

2.6 Stability of Betti numbers

It was known by the general theory of representation stability developed by Church-Ellenberg-Farb that for any character polynomial $P$, the twisted Betti numbers $\dim H^i(\text{Conf}_n(\mathbb{C}); P)$ will be independent of $n$ when $n$ is sufficiently large. We will give a different proof of this result with an improved stability range for $n$.

Corollary 7. For every character polynomial $P$ and for every $i$, we have

$$
\dim H^i(\text{Conf}_n(\mathbb{C}); P) = \dim H^i(\text{Conf}_{n+1}(\mathbb{C}); P) \tag{2.9}
$$
when $n \geq i + \text{deg } P + \text{deg } P + 1$.

Remark 6. Church-Ellenberg-Farb first proved (2.9) when $n \geq 2i + \text{deg } P$ (Theorem 1 [7]). Later, Hersh-Reiner gave a different proof of (2.9) with a better stable range: $n \geq \max\{2 \text{deg } P, \text{deg } P + i + 1\}$ (Theorem 4.3 in [13]). The stable range in Corollary 7 is a small improvement of the range obtained by Hersh-Reiner, and is sharp, as we will show it in Section 2.8.

Proof. It suffices to consider when $P = (\lambda)$ for some partition $\lambda = 1^{\lambda_1} 2^{\lambda_2} \cdots l^{\lambda_l}$. In this case $\text{deg} (\lambda) = \sum_k k \lambda_k$. Let $\alpha_i(n)$ be as in (2.6), and let $\Phi_\lambda(z, t)$ be as in (2.7).
It suffices to check that \((1 - t)\Phi_{\lambda}(t, z)\) is a sum of monomials of the form \(z^i t^n\) where \(n - i \leq \sum_{k=0}^{l} k\lambda_k + 1\).

We will say an infinite series in \(z\) and \(t\) has slope \(\leq m\) if it is a sum of monomials \(z^i t^n\) where \(n - i \leq m\). We want to show that the series given by Theorem 1

\[
(1 - t)\Phi_{\lambda}(t, z) = (1 - zt^2) \prod_{k=1}^{l} \left( M_k(z^{-1}) \right) \left( \frac{(tz)^k}{1 + (z)^k} \right)^{\lambda_k}
\]

(2.10)

has slope \(\leq \sum_{k=0}^{l} k\lambda_k + 1\). We analyze each factor.

- \((1 - zt^2)\) has slope \(\leq 2 - 1 = 1\).

- For each \(k\), the factor \(M_k(z^{-1})\) has slope \(\leq k\). Thus, \(\left( M_k(z^{-1}) \right)\) has slope \(\leq k\lambda_k\).

- For each \(k\), \(\left( \frac{tz^k}{1 + (z)^k} \right)^{\lambda_k}\) has slope \(\leq 0\).

Therefore, the product in (2.10) has slope \(\leq 1 + \sum_{k=0}^{l} k\lambda_k\). This establishes the corollary. 

2.7 Proof of Corollary 2, (I).

Let \(\alpha_i\) be \(\alpha_i(n)\) when \(n \geq i + |\lambda| + 1\) in the stable range. Define the generating function

\[
\Phi_{\lambda}^\infty(z) := \sum_{i=0}^{\infty} \alpha_i(-z)^i
\]

By Lemma 5 we can calculate \(\Phi_{\lambda}^\infty(z)\) using \(\Phi_{\lambda}(z, t)\):

\[
\Phi_{\lambda}^\infty(z) = \left[ (1 - t)\Phi_{\lambda}(z, t) \right]_{t=1} \text{ by Theorem 1} = (1 - z) \prod_{k=1}^{l} \left( M_k(z^{-1}) \right) \left( \frac{z^k}{1 + z^k} \right)^{\lambda_k}
\]

(2.11)

In particular, \(\Phi_{\lambda}^\infty(z)\) in (2.11) is a rational function in \(z\). The denominator is a polynomial in \(z\) of degree \(\sum_{k=1}^{l} k\lambda_k = |\lambda|\). The numerator has degree at most \(1 + |\lambda|\). This implies that \(\alpha_i\) satisfies a linear recurrence relation of length \(|\lambda|\) once \(i > |\lambda| + 1\).

2.8 Examples

Recall that irreducible representations of \(S_n\) are in bijection with partitions of \(n\). For a fixed partition \(\mu \vdash n\) where \(\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r)\), we will denote by \(V(\mu)_n\) the representation of \(S_n\) corresponding to the partition \(n = (n - \sum_{i=1}^{r} \mu_i) + \mu_1 + \cdots + \mu_r\) for all \(n\) sufficiently large, i.e. for \(n - \sum_{i=1}^{r} \mu_i \geq \mu_1\). Sometimes we will suppress \(n\) and use \(V(\mu)\) to represent \(V(\mu)_n\) for all \(n\) large enough. Going from \(V(\mu)_n\) to \(V(\mu)_{n+1}\) corresponds to adding one block in the first row of the corresponding Young diagram. Church-Ellenberg-Farb proved
that $H^i(\text{PConf}_n(\mathbb{C});\mathbb{Q})$ is multiplicity stable (Theorem 1.9 in [8]): for each $i$, there is a finite set $Q_i$ of partitions such that

$$H^i(\text{PConf}_n(\mathbb{C});\mathbb{Q}) \cong \bigoplus_{\mu \in Q_i} V(\mu)^{\otimes d_i(\mu)}$$

for all $n$ sufficiently large. In particular, the sum over $Q_i$ is independent of $n$. Farb proposed the problem of computing $d_i(\mu)$ for each $i$ and each $\mu$ (Problem 3.5 in [10]). Macdonald proved that for all partition $\mu$, the character of $V(\mu)_n$ is given by a unique character polynomial $P_{\mu}$ for all $n$ sufficiently large (Example I.7.14 in [16]). Therefore, by transfer (1.1), computing $d_i(\mu)$ is equivalent to computing the stable cohomology of $H^i(\text{Conf}_n(\mathbb{C}); P_{\mu})$.

We will demonstrate the case of computing these using Theorem 1 (I) in three examples where $\mu$ is the partition 1 = 1, or 2 = 1 + 1, or 2 = 2.

**Example 1: $W_n = V(1)_n$.** When $n \geq 2$, the irreducible representation $V(1)_n$ corresponds to the Young diagram $(n - 1, 1)$. It is also known as the standard representation:

$$V(1)_n \cong \{(x_1, \cdots, x_n) \mid \sum x_i = 0\} \cong \mathbb{Q}^{n-1}$$

where $S_n$ acts by permuting the coordinates.

The $S_n$-character of $W$ is given by the character polynomial $X_1 - 1$. If we abbreviate the Betti number as

$$\alpha_i(n) = \dim H^i(\text{Conf}_n(\mathbb{C}); V(1)_n),$$

then Theorem [11] gives that the double generating function of $\alpha_i(n)$ is

$$\sum_{n=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i \alpha_i(n) z^i t^n = \frac{1 - t^2 z}{1 - t} \left[ \frac{t}{1 + tz} - 1 \right]$$

$$= (z + z^2) t^3 + (-z + 2z^2 - z^3) t^4 + (-z + 2z^2 - 2z^3 + z^4) t^5$$

$$+ (-z + 2z^2 - 2z^3 + 2z^4 + z^5) t^6 + \cdots$$

Thus, we conclude that when $n \geq 3$,

$$\alpha_i(n) = \begin{cases} 
0 & i = 0 \\
1 & i = 1 \\
2 & 0 < i < n - 1 \\
1 & i = n - 1
\end{cases}$$

**Remark 7.** A computation of $\dim H^i(\text{Conf}_n(\mathbb{C}); V(1)_n)$ from Lehrer-Solomon’s description of $H^i(\text{PConf}_n(\mathbb{C}); \mathbb{Q})$ was presented in Proposition 4.5 of [7]. It took about one and half pages. The computation above using generating function is a faster procedure.

The stable Betti numbers are:

$$\alpha_i := \lim_{n \to \infty} \dim H^i(\text{Conf}_n(\mathbb{C}); V(1)) = \begin{cases} 
0 & i = 0 \\
1 & i = 1 \\
2 & i > 1
\end{cases}$$

When $i \geq 2$, the stable Betti numbers $\alpha_i$ are the same, which in particular satisfy a recurrence relation of length 1. From this example we see that the bounds in Corollary [2] (I) and
Corollary 7 are sharp.

**Example 2:** \( W_n = V(1, 1)_n \). When \( n \geq 3 \), the irreducible representation \( V(1, 1)_n \) corresponds to the Young diagram \( (n - 2, 1, 1) \). The dimension of \( V(1, 1) \) is \((n^2 - 3n + 2)/2\). In fact, we have \( V(1, 1) \cong \bigwedge^2 \mathbb{Q}^{n-1} \) where \( \mathbb{Q}^{n-1} \) is the standard representation \( V(1) \). The character of \( V(1, 1) \) is given by the following character polynomial:

\[
\left( \frac{X_1}{2} \right) - X_1 - X_2 + 1
\]

If we abbreviate the Betti numbers \( \alpha_i(n) = \dim H^i(\text{Conf}_n(\mathbb{C}); V(1, 1)_n) \), then Theorem 1 gives that

\[
\sum_{n=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i \alpha_i(n) \; z^i t^n = \Phi_{(2)}(z, t) - \Phi_{(1)}(z, t) - \Phi_{(0,1)}(z, t) + \Phi_{(0)}(z, t)
\]

\[
= \frac{1 - t^2 z}{1 - t} \left[ \frac{(1 - z) t^2}{2(1 + t z)^2} - \frac{t}{1 + t z} - \frac{(1 - z) t^2}{2(1 + (t z)^2)} + 1 \right]
\]

By expanding the generating function, we have the following table of the Betti numbers:

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<th>( (i, n) )</th>
<th>( n = 3 )</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
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</tr>
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<tbody>
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</tbody>
</table>

The bold entries lie on the line \( n = i + 3 \). In each row, the Betti number stabilizes as \( n \geq i + 3 \). This agrees with the stability bound as predicted in Corollary 2.9: \( n > i + \text{deg}(X^i_1) - X_1 - X_2 + 1 = i + 2 \). Moreover, we can see from the table that the bound is sharp.

Furthermore, from (2.11), we have the following formula for the generating function of the stable Betti numbers \( \alpha_i := \lim_{n \to \infty} \dim H^i(\text{Conf}_n(\mathbb{C}); V(1, 1)_n) \):

\[
\sum_{i=0}^{\infty} (-1)^i \alpha_i z^i = \Phi_{(2)}(z) - \Phi_{(1)}(z) - \Phi_{(0,1)}(z) + \Phi_{(0)}(z) = (1 - z) \left[ \frac{1 - z}{2(1 + z)^2} - \frac{1}{1 + z} - \frac{1 - z}{2(1 + z)^2} + 1 \right]
\]

\[
= 2z^2 - 5z^3 + 6z^4 - 7z^5 + 10z^6 - 13z^7 + 14z^8 - 15z^9 + 18z^{10} - 21z^{11} + \cdots
\]
The stable Betti numbers satisfy the linear recurrence relation:

\[ \alpha_i = 2\alpha_i - 2\alpha_{i-2} + 2\alpha_{i-3} - \alpha_{i-4}. \]

By explicitly solving the recurrence relation, we have \( \alpha_0 = 0, \alpha_1 = 2, \) and when \( i \geq 3, \)

\[ \alpha_i = \begin{cases} 
2i - 2 & i = 0 \mod 4 \\
2i - 3 & i = 1 \mod 4 \\
2i - 2 & i = 2 \mod 4 \\
2i - 1 & i = 3 \mod 4 
\end{cases} \]

Remark 8. In Section 4.4 of [7], Church-Ellenberg-Farb used \( L \)-functions to compute the stable cohomology of \( H^i(\text{Conf}_n(C); \Lambda^2\mathbb{Q}^n). \) Since \( \Lambda^2\mathbb{Q}^n \cong \Lambda^2\mathbb{Q}^n \oplus \mathbb{Q}^n, \) we recover their computation. Moreover, we also obtained unstable cohomology.

Example 3: \( W_n = V(2)_n. \) When \( n \geq 4, \) the irreducible representation \( V(2)_n \) corresponds to the Young diagram \( (n-2, 2). \) The dimension of \( V(2) \) is \( (n^2 - 3n)/2. \) In fact, \( V(2) \) is a direct summand in the symmetric square of the standard representation \( \mathbb{Q}^{n-1}. \) More precisely, we have

\[ \text{Sym}^2(\mathbb{Q}^{n-1}) \cong \mathbb{Q}^n \oplus V(2) \]

The character of \( V(2) \) is given by the following character polynomial

\[ \left( \frac{X_1}{2} \right) + X_2 - X_1 \]

If we abbreviate the Betti numbers \( \alpha_i(n) := \dim H^i(\text{Conf}_n(C); V(2)_n), \) Theorem 1 gives

\[ \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i \alpha_i(n) z^i t^n = \Phi_1(2)(z, t) + \Phi_0(1)(z, t) - \Phi_1(1)(z, t) \]

\[ = \frac{1 - t^2 z}{1 - t} \left[ \frac{(1 - z)t^2}{2(1 + tz)^2} + \frac{(1 - z)t^2}{2(1 + (tz)^2)} - \frac{t}{1 + tz} \right] \]

By expanding the generating function, we have the following table of Betti numbers:

<table>
<thead>
<tr>
<th>( (i, n) )</th>
<th>( n = 4 )</th>
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<th>8</th>
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The bold entries lie on the line \( n = i + 3 \). In each row, the Betti number stabilizes when \( n \geq i + 3 \). This agrees with the stability bound as predicted in Corollary 2.9 \( n > i + \deg(\frac{X_1}{2}) + X_2 - X_1) = i + 2 \). We can see from the table that the bound is sharp.

Furthermore, from (2.11), we have the following formula for the generating function of the stable Betti numbers:

\[
\sum_{i=0}^{\infty} (-1)^i \alpha_i z^i = \Phi_{\infty}^{(2)}(z) - \Phi_{\infty}^{(1)}(z) - \Phi_{\infty}^{(0,1)}(z) + \Phi_{\infty}^{(0,0)}(z) = (1 - z) \left[ \frac{1 - z}{2(1+z)^2} + \frac{1 - z}{2(1+z^2)} - \frac{1}{1 + z} \right]
\]

\[= -z + 2z^2 - 3z^3 + 6z^4 - 9z^5 + 10z^6 - 11z^7 + 14z^8 - 17z^9 + 18z^{10} - 19z^{11} + \cdots\]

The stable Betti numbers satisfies the linear recurrence relation:

\[\alpha_i = 2\alpha_{i-1} - 2\alpha_{i-2} + 2\alpha_{i-3} - \alpha_{i-4}.\]

We can explicitly solve the recurrence relation and obtain that \( \alpha_0 = 0 \), and when \( i \geq 1 \),

\[\alpha_i := \lim_{n \to \infty} \dim H^i(\text{Conf}_n(\mathbb{C}); V(2)) = \begin{cases} 2i - 2 & i = 0 \mod 4 \\ 2i - 1 & i = 1 \mod 4 \\ 2i - 2 & i = 2 \mod 4 \\ 2i - 3 & i = 3 \mod 4 \end{cases}\]

\section{Cohomology of \( \mathcal{T}_n(\mathbb{C}) \) via point counting}

In this section we prove part (II) of Theorem 1 and Corollary 2. Our analysis of \( \mathcal{T}_n \) closely parallels that of Conf\(_n \) before.

\subsection{General set-up}

\( \mathcal{T}_n = \overline{\mathcal{T}}_n / S_n \) is a scheme over \( \mathbb{Z} \) (again, see page 66 in [17]). The \( \mathbb{F}_q \)-points \( \mathcal{T}_n(\mathbb{F}_q) \) consists of sets \( L = \{L_1, \ldots, L_n\} \) of \( n \) linearly independent lines in \( \mathbb{P}^{n-1}(\mathbb{F}_q) \) such that the Frobenius map \( \text{Frob}_q : \mathbb{P}^{n-1}(\mathbb{F}_q) \to \mathbb{P}^{n-1}(\mathbb{F}_q) \) preserves the set \( T \).

Let \( F \) abbreviate the Frobenius map. An \( F \)-stable torus in \( \text{GL}_n(\mathbb{F}_q) \) is an algebraic subgroup which becomes diagonalizable over \( \mathbb{F}_q \). An \( F \)-stable torus is \textit{maximal} if it is not properly contained in any larger one. Given any \( F \)-stable maximal torus \( T \), its \( n \) eigenvectors in \( \mathbb{F}_q^n \) defines a set \( L_T \) of \( n \) independent lines in \( \mathbb{F}_q^n \). Thus \( L_T \) is an element of \( \mathcal{T}_n(\mathbb{F}_q) \). The map \( T \mapsto L_T \) gives a bijection between \( F \)-stable maximal tori in \( \text{GL}_n(\mathbb{F}_q) \) and \( \mathbb{F}_q \)-points of \( \mathcal{T}_n \). Therefore, \( \mathcal{T}_n(\mathbb{F}_q) \) is precisely the set of \( F \)-stable maximal torus in \( \text{GL}_n(\mathbb{F}_q) \). See Section 5.1 of [7] for a proof.

For any \( T \in \mathcal{T}_n(\mathbb{F}_q) \), the action of \( \text{Frob}_q \) on \( L_T \), a set of \( n \) lines in \( \mathbb{F}_q^n \), gives a permutation \( \sigma_T \in S_n \), unique up to conjugacy. Church-Ellenberg-Farb proved the following equation using the Grothendieck-Lefschetz fixed point formula. Given any \( S_n \)-representation \( W \) with character \( \chi_W \),

\[
\sum_{T \in \mathcal{T}_n(\mathbb{F}_q)} \chi_W(\sigma_T) = q^n(n-1) \sum_{i=0}^{n(n-1)/2} \dim H^{2i}(\mathcal{T}_n(\mathbb{C}); W) q^{-i} \tag{3.1}
\]

This formula was stated in Theorem 5.3 in [7]. By additivity, the same formula holds when \( W \) is taken to be a virtual representation of \( S_n \).
3.2 Arithmetic statistics for $F$-stable maximal tori in $\text{GL}_n(F_q)$

In this subsection, we will compute the left-hand-side of (3.1) when $W$ is given by a character polynomial of the form $(X^\lambda)$. Our approach will be a systematic extension of Fulman’s method in [12]. All the ideas in this subsection were already in Fulman’s paper.

**Proposition 8.** For each fixed integer partition $\lambda = \lambda_1 \lambda_2 \cdots \lambda_l$, let $z_\lambda := \prod_{k=1}^l \lambda_k!k^{\lambda_k}$. We have the following equation of formal power series in $t$.

$$
\sum_{n=0}^\infty \left[ \sum_{T \in T_n(F_q)} \left( X^\lambda \right)(\sigma_T) \right] \frac{t^n}{|\text{GL}_n(F_q)|} = \frac{1}{z_\lambda} \left[ \prod_{k=1}^l \left( \frac{q^{-k}t^k}{1-q^{-k}} \right)^{\lambda_k} \right] \cdot \left[ \prod_{i=1}^\infty \frac{1}{1-q^{-i}t} \right] \tag{3.2}
$$

**Proof.** We will use the following result of Fulman (stated as Theorem 3.2 in [12]).

**Theorem (Fulman).** With the notation as above,

$$
\sum_{n=0}^\infty \left[ \sum_{T \in T_n(F_q)} \prod_{i=1}^n x_i(\sigma_T) \right] \frac{t^n}{|\text{GL}_n(F_q)|} = \prod_{k=1}^\infty \exp \left[ \frac{x_k t^k}{(q^k-1)k} \right] \tag{3.3}
$$

Let $F(\vec{x},t)$ denote both sides of (3.3) as a formal power series in infinitely many variables $t$ and $x_1, x_2, \ldots$. We apply the formal differential operator

$$(\frac{\partial}{\partial x})^\lambda := (\frac{\partial}{\partial x_1})^{\lambda_1} (\frac{\partial}{\partial x_2})^{\lambda_2} \cdots (\frac{\partial}{\partial x_l})^{\lambda_l}$$

to the series $F(\vec{x},t)$ and then evaluate at $x_i = 1$ for all $i$. Let $\lambda!$ be an abbreviation for $(\lambda_1!)(\lambda_2!)(\lambda_l!)$. Then

$$
\lambda! \sum_{n=0}^\infty \left[ \sum_{T \in T_n(F_q)} \left( X^\lambda \right)(\sigma_T) \right] \frac{t^n}{|\text{GL}_n(F_q)|} = (\frac{\partial}{\partial x})^\lambda \left[ F(\vec{x},t) \right]_{x_i=1, \forall i} = (\frac{\partial}{\partial x})^\lambda \left[ \prod_{k=1}^l \exp \left[ \frac{x_k t^k}{(q^k-1)k} \right] \right]_{x_i=1, \forall i} = \left[ \prod_{k=1}^l \left( \frac{t^k}{(q^k-1)k} \right)^{\lambda_k} \right] \cdot \left[ \prod_{k=1}^\infty \exp \frac{t^k}{(q^k-1)k} \right] = \left[ \prod_{k=1}^l \left( \frac{t^k}{(q^k-1)k} \right)^{\lambda_k} \right] \cdot \left[ \prod_{i=1}^\infty \frac{1}{1-q^{-i}t} \right]
$$

where the last equality follows from

$$
\prod_{k=1}^\infty \exp \frac{t^k}{(q^k-1)k} = \prod_{i=1}^\infty \frac{1}{1-q^{-i}t}
$$

which can be proved by expanding both sides into power series. \qed

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3.3 Proof of Theorem 1 (II)

For each \( i \) and \( n \), we abbreviate the twisted Betti number as

\[
\beta_i(n) := \dim H^{2i}(T_n(C); \left( \frac{X}{\lambda} \right))
\]  

(3.4)

Define a formal power series in \( z \) and \( t \)

\[
\Psi_\lambda(z, t) := \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{\beta_i(n)}{(1 - z)(1 - z^2) \cdots (1 - z^n)} z^i t^n
\]  

(3.5)

We evaluate \( \Psi_\lambda(z, t) \) at \( z = q^{-1} \):

\[
\Psi_\lambda(q^{-1}, t) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{\beta_i(n) q^{-i} t^n}{(1 - q^{-1})(1 - q^{-2}) \cdots (1 - q^{-n})}
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{\left| \text{GL}_n(F_q) \right|} \sum_{n=0}^{\infty} \left[ \sum_{T \in T_n(F_q)} \left( \frac{X}{\lambda} \right)(\sigma_T) \right] (t q^n)^n
\]

(3.6)

On the other hand, by Lemma 5, we have

\[
\lim_{n \to \infty} \sum_{i=0}^{n(n-1)} \frac{\beta_i(n)}{(1 - z)(1 - z^2) \cdots (1 - z^n)} z^i = \sum_{i=0}^{\infty} \frac{\beta_i}{\prod_{j=1}^{\infty} (1 - z^j)} z^i
\]

(3.6)

Equating (3.6) and (3.7), we have

\[
\sum_{i=0}^{\infty} \beta_i z^i = \frac{1}{z^\lambda} \prod_{k=1}^{l} \left( \frac{1}{1 - z^k} \right)^{\lambda_k}
\]

(3.7)

3.4 Proof of Corollary 2 (II).

As before, it suffices to consider when \( P = \left( \frac{X}{\lambda} \right) \). Let \( \beta_i(n) \) be as in (3.4) and let \( \beta_i \) be \( \lim_{n \to \infty} \beta_i(n) \), then we have

\[
\lim_{n \to \infty} \sum_{i=0}^{n(n-1)} \frac{\beta_i(n)}{(1 - z)(1 - z^2) \cdots (1 - z^n)} z^i = \sum_{i=0}^{\infty} \frac{\beta_i}{\prod_{j=1}^{\infty} (1 - z^j)} z^i
\]

(3.6)

On the other hand, by Lemma 5 we have

\[
\lim_{n \to \infty} \sum_{i=0}^{n(n-1)} \frac{\beta_i(n)}{(1 - z)(1 - z^2) \cdots (1 - z^n)} z^i = \left[ (1 - t)\Psi_\lambda(z, t) \right]_{t=1}
\]

\[
= \frac{1}{z^\lambda} \prod_{k=1}^{l} \left( \frac{1}{1 - z^k} \right)^{\lambda_k} \prod_{j=1}^{\infty} \frac{1}{1 - z^j}
\]

(3.7)

Equating (3.6) and (3.7), we have

\[
\sum_{i=0}^{\infty} \beta_i z^i = \frac{1}{z^\lambda} \prod_{k=1}^{l} \left( \frac{1}{1 - z^k} \right)^{\lambda_k}
\]
The generating function for $\beta_i$ is a rational function in $z$ with denominator a polynomial of degree $|\lambda|$. Thus $\beta_i$ satisfies a linear recurrence relation of length $|\lambda|$.

References


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