Math 207, Section 31: Honors Analysis I
Autumn Quarter 2009
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Homework 6, Final Version
Due: Monday, November 9, 2009

1. (*) Read Kolmogorov and Fomin, Chapter 2, especially Section 8.
2. (*) Read Kolmogorov and Fomin, Chapter 3, especially Section 11.
3. (*) Read Sally, Chapter 4, especially Section 5.
4. (*) Prove that a metric space is sequentially compact if and only if it has the Bolzano-Weierstrass property.
5. Consider the interval $[0,1] \subset \mathbb{R}$ with the inherited metric. Consider the open cover consisting of $[0, \frac{1}{10})$, $(\frac{1}{2}, 1]$ and, for each $n \in \mathbb{N}$, the interval $(\frac{1}{n+2}, \frac{1}{n})$. Find the Lebesgue number of this cover.
6. Sally, Section 4.5, Exercises (*) 4.5.34, 4.5.35, 4.5.37, and 4.5.39.
(Oops!–you have already done 35 and 39. If you haven't, try them again!)
7. Let $\mathbb{R}^n$ have the usual metric. Show that if $A \subset \mathbb{R}^n$ is open and connected, then A is path-connected.
8. Let (X, d_1) and (Y, d_2) be metric spaces. We give the Cartesian product $X \times Y$ the distance function $d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$.
   (a) Show that $(X \times Y, d)$ is a metric space.
   (b) Show that a set $S \subset X \times Y$ is open if and only if, given any $x \in S$, there exist open sets $A \subset X$ and $B \subset Y$ such that $x \in A \times B \subset S$.
   (c) Show that if $X \times Y$ is connected, then X and Y are connected.
9. Show that the Topologist's Sine Curve given by
$$S = \{(x, 0) \mid x \leq 0\} \cup \{(x, \sin(1/x)) \mid x > 0\}$$
is connected but not path-connected as a subset of $X = \mathbb{R}^2$.
10. Let X be a metric space. The *connected component* of a point $x \in X$ is denoted $C(x)$ and is the union of all connected sets in X containing x. The space X is called *totally disconnected* if $C(x) = \{x\}$ for every $x \in X$. Show that $\mathbb{Q}$ is totally disconnected in the usual metric.
11. Do Sally, Project 4.6.3, on Topological Groups.
12. (*) Read Sally, Chapter 5, especially Section 1.
13. Sally, Section 5.1, Exercises (*) 5.1.3, 5.1.8, 5.1.10.
14. Let $K(x,y) \in C([a,b] \times [a,b])$ be such that $||K||_\infty = M$. Let $\lambda \in \mathbb{R}$ and $\phi(x) \in C[a,b]$. Let $\Gamma : C[a,b] \to C[a,b]$ take the function $f \in C[a,b]$ to the function $\Gamma(f) \in C[a,b]$ defined on each $x \in [a,b]$ by:
$$\Gamma(f)(x) = \phi(x) + \lambda \int_a^x K(x,y) f(y) dy.$$

(a) Show that for each $n \in \mathbb{N}$,

$$||\Gamma^n(f) - \Gamma^n(g)||_\infty \leq |\lambda|^n M^n \frac{(b-a)^n}{n!} ||f - g||_\infty.$$

Thus, for n large enough, Γ^n is contractive.

(b) Show that Γ has a unique fixed point. Thus, the integral equation

$$f(x) = \phi(x) + \lambda \int_a^x K(x, y) f(y) dy$$

has a unique solution in $C[a, b]$.

15. Let $X = [1, +\infty) \subset \mathbb{R}$, and define $f : X \to X$ by $f(x) = \frac{1}{2}(x + \frac{2}{x})$. Prove that f is contractive with contstant $\alpha = \frac{1}{2}$ and fixed point $x_0 = \sqrt{2}$.