$\mathcal{E}$-FACTORS FOR GAUSS-MANIN DETERMINANTS

ALEXANDER BEILINSON, SPENCER BLOCH, AND HÉLÈNE ESNAULT

To Yuri Manin, with gratitude.

Abstract. We define ε-factors in the de Rham setting and calculate the determinant of the Gauß-Manin connection for a family of (affine) curves and a vector bundle equipped with a flat connection.

"Ordentliche Leute pflegten ihren Schatten mit sich zu nehmen, wenn sie in die Sonne gingen." A. v. Chamisso, from "Peter Schlemihls wundersame Geschichte"

Table of contents

1. Introduction

1.1. Let us first recall briefly the format of the classical theory of ε-factors, which grew out of Tate's thesis and the work of Dwork, Langlands, Deligne, and Laumon.

We consider only the case of function fields, so we fix a prime p, a finite field k of characteristic p, and look at classical local k-fields, i.e., topological k-fields isomorphic to $k'((t))$ where k' is a finite extension of k. For a local field F we denote by $\omega(F)$ the 1-dimensional F-vector space of differentials; set $\omega(F)^\times := \omega(F) \smallsetminus \{0\}$. Galois modules and local systems have coefficients in $\bar{\mathbb{Q}}_\ell$ for a prime $\ell \neq p$.

A classical theory of ε-factors is a rule which assigns to every pair (F, V), F is a local k-field, V a Galois module for F, a continuous function $\varepsilon(F, V) = \varepsilon(V) : \omega(F)^\times \to \bar{\mathbb{Q}}_\ell^\times$, $\nu \mapsto \varepsilon(V)_\nu$, which satisfies the following properties:

Date: May 10, 2002.
1991 *Mathematics Subject Classification.* Primary 14C40 19E20 14C99.

(i) *Multiplicativity with respect to V:* For a short exact sequence $0 \to V_1 \to V \to V_2 \to 0$ one has

$$\varepsilon(F,V)_\nu = \varepsilon(F,V_1)_\nu \varepsilon(F,V_2)_\nu. \tag{1.1.1}$$

Thus $\varepsilon(F,V)_\nu$ makes sense for virtual Galois modules.

(ii) *Induction:* Let F'/F be a finite separable extension, V' a Galois module for F', $\nu \in \omega(F)^\times \subset \omega(F')^\times$. Then

$$\varepsilon(F',V')_\nu = \varepsilon(F,\mathrm{Ind}V')_\nu \tag{1.1.2}$$

if V' is a virtual Galois module of rank 0.

(iii) *Product formula:* Let X be a smooth projective curve over k, $D \subset X$ a divisor, V a local system on $U := X \smallsetminus D$, ν an invertible 1-form on U. For $x \in D$ let F_x be the local field at x, and V_x, ν_x the restrictions of V, ν to F_x. Then[1]

$$\prod_{x\in D} \varepsilon(F_x,V_x)_{\nu_x} = \prod_i \det(-Fr_k, H^i(U_{\bar k},V))^{(-1)^{i+1}}. \tag{1.1.3}$$

It follows from a deep theorem of Laumon [L] 3.2 that a theory of ε-factors does exist. Namely, for fixed non-trivial character $\psi : k \to \bar{\mathbb{Q}}_\ell^\times$ and $p^{1/2} \in \bar{\mathbb{Q}}_\ell^\times$ the function $\varepsilon(F,V)_\nu := q_F^{-rkV\cdot v_F(\nu)/2}\varepsilon_\psi(O_F, Rj_*V,\nu)$ is a theory of ε-factors in the above sense. Here q_F is the number of elements of the residue field k_F, $v_F(\nu)$ the valuation of ν, O_F the ring of integers in F, and ε_ψ is as in [L] (3.1.5.4).[2]

Remarks. (i) Assume that our (F,V,ν) is such that V is unramified and $v_F(\nu)=0$. Then[3] (here (-1) is the Tate twist)

$$\varepsilon(F,V)_\nu = \det(-Fr_{k_F}, V(-1)). \tag{1.1.4}$$

(ii) To determine ε one invokes the Gabber-Katz theorem [GK] according to which every Galois module for $k((t))$ can be extended to a local system on $\mathbb{G}_m = \mathrm{Spec}\, k[t,t^{-1}]$ having tame ramification at $t=\infty$. Therefore, by the product formula, we know $\varepsilon(V)$ for arbitrary V if the global Frobenius determinants and $\varepsilon(V)$ for tame V's happen to be known.

(iii) The r.h.s. in (1.1.3) is the (super) trace of the Frobenius acting on the determinant super line of the complex $R\Gamma(U_{\bar k},V)[1]$.

[1]Here Fr_k is the geometric Frobenius for k, $\bar k$ is an algebraic closure of k.

[2]In notation of [D2] one has $\varepsilon(F,V)_\nu = \epsilon(V,\psi_\nu,\mu_\nu,1)\det(-Fr_{k_F},V^I(-1))$ where $\psi_\nu : F \to \bar{\mathbb{Q}}_\ell$ is an additive character $f \mapsto \psi\mathrm{TrRes}(f\nu)$, μ_ν a Haar measure on F self-dual with respect to ψ_ν, and $I \subset Gal(\bar F/F)$ the inertia subgroup.

[3]To see this compare (1.1.3) for U and U with one point removed.

(iv) For a theory ε of ε-factors and a fixed $a \in k^\times$ the function $\nu \mapsto \varepsilon(V)_{a\nu}$ is again a theory of ε-factors. So the set $\mathbb{E}(k)$ of theories of ε-factors carries a $k^\times$-action.

1.2. Unfortunately, the above story, being global-to-local, does not tell much about the nature of ε-factors. Therefore one is tempted to look for a direct, purely local, construction of ε-factors. If available, such construction would make it possible to use the product formula to compute the global Frobenius determinant.

A related dream is to have a *geometric* theory of ε-factors. One considers local fields over an arbitrary (not necessary finite) base field k, and we look for a rule $\mathcal{E}$ that assigns to every (F, V) as above an invertible (super) local system $\mathcal{E}(V)$ on the k-(ind-)scheme $\omega(F)^\times$. The compatibilities 1.1(i)–(iii) now mean canonical isomorphisms of invertible super local systems. The classical ε is recovered as the "trace of Frobenius" function of $\mathcal{E}$.[4]

1.3. The aim of this article is to present such geometric theory in the de Rham setting. So our k is now a field of characteristic 0, and instead of étale local systems we consider local systems in the de Rham sense, that is vector bundles equipped with flat connections. Recall that there is a well-known, yet mysterious, analogy between the phenomena of wild ramification (étale setting) and irregularity (de Rham setting). The role of local Galois modules is played by de Rham local systems $V = (V, \nabla)$ on a *formal* punctured disc[5] $\operatorname{Spec} F$, where F is a k-algebra isomorphic to $k'((t))$, k' is a finite extension of k.

So for every such V we define a super line bundle (its degree is the irregularity of ∇) with flat connection $\mathcal{E}(V)$ on $\omega(F)^\times$ together with data of canonical isomorphisms parallel to 1.1 (i)–(iii). For example, the product formula looks as follows. Consider X, U, D, V as in 1.1(iv), so V now is a local system on U. Let $\mathcal{E}(F_D, V)$ be the exterior tensor product of $\mathcal{E}(F_x, V_x)$ for $x \in D$. Then one has a canonical isomorphism of local systems on $\omega(U)^\times$

$$\mathcal{E}(F_D, V)|_{\omega(U)^\times} \overset{\sim}{\to} \bigotimes_i (\det H^i_{dR}(U, V))^{\otimes(-1)^{i+1}}_{\omega(U)^\times}. \tag{1.3.1}$$

[4]To be more precise, $\mathcal{E}(V)$ should rather be a twisted local system, an object of an appropriate "ε-gerbe". A choice of trivialization of the ε-gerbe identifies then $\mathcal{E}(V)$ with a plain local system. One can further speculate that in the gerbe setting compatibility (1.1.2) holds for arbitrary V', and the reason behind the rank 0 condition in (1.1.2) is the absence of a nice trivialization of the ε-gerbe. We are grateful to M. Kapranov for this suggestion.

[5]Notice that the Stokes structure, which is necessary to determine the *analytic* local structure of a connection, does not play any role here.

Here the l.h.s. is the restriction of $\mathcal{E}(F_D, V)$ to $\omega(U)^\times \hookrightarrow \prod \omega(F_x)^\times$, the r.h.s. is a constant local system.

Remark. Our ε-factors differ from de Rham counterparts of ε-factors from [L]. For example:

(i) For our de Rham ε-factors compatibility (1.1.2) holds for arbitrary V' (regardless of its rank).[6]

(ii) If V is a trivial local system of rank 1 on $\operatorname{Spec} F$ then the local system $\mathcal{E}(V)$ has a non-trivial ± 1 monodromy on the connected components of $\omega(F)^\times$ which consist of ν with odd order of zero.[7]

If our datum varies nicely[8] with respect to parameters S (this means, in particular, that irregularity does not jump) then $\mathcal{E}(F_s, V_s)$ form a super line bundle $\mathcal{E}(F/S, V)$ on $\omega(F/S)^\times$ equipped with an S-relative connection. If the vector bundle V on the space of the family carries an absolute flat connection extending the S-relative one, then our S-relative connection on $\mathcal{E}(F/S, V)$ extends canonically to an absolute flat connection, i.e., $\mathcal{E}(F/S)$ becomes a local system. The product formula isomorphism (1.3.1) is compatible with the absolute connections (the l.h.s. carries the Gauß-Manin one). So (1.3.1) yields a formula for the Gauß-Manin determinant, once we are able to compute the ε-connections.

1.4. The ε-line $\mathcal{E}(V)_\nu$ is defined as follows. Let τ_ν be the vector field ν^{-1}. Then $\nabla(\tau_\nu)$ is a Fredholm operator on the infinite-dimensional k-vector space V.[9] Let $V = V^+ \oplus V^-$ be any decomposition of the V into a sum of Fourier "positive" and "negative" parts.[10] Consider the component of $\nabla(\tau_\nu)$ acting on V_-. Our $\mathcal{E}(V)_\nu$ is the determinant line of this Fredholm operator. The auxiliary choice of $V^\pm$ is irrelevant.[11]

To see the product formula isomorphism notice that the composition $\Gamma(U, V) \to \oplus V_x \to \oplus V_x^-$ is Fredholm. Now (1.3.1) is the corresponding identification of the determinant lines for τ_ν on $\Gamma(U, V)$ and $\oplus V_x^-$.

[6] So in the de Rham setting the hoped-for ε-gerbe from ftn. 4 is canonically trivialized.

[7] In accordance with the fermionic nature of $\mathcal{E}$.

[8] Every variation is nice at the generic point; for more details see 4.4.

[9] Its index vanishes and the determinant line is canonically trivialized, see e.g. 5.9(a)(iv).

[10] Precisely, for $F = k((t))$ this means that for certain F-basis $\{e_i\}$ of V our V^+ contains $\Sigma k[[t]] e_i$ as a k-subspace of finite codimension.

[11] The determinant line of a Fredholm operator $F : U \to U$ does not change if U is modified by a finite-dimensional vector space, or the operator is perturbed by an operator with finite-dimensional image.

The ε-connection on $\mathcal{E}(V)$ is defined as follows. First, the multiplicativity of determinant lines with respect to product of operators shows that the action of $F^\times$ on $\omega(F)^\times$ lifts to an action on $\mathcal{E}(V)$ of a Heisenberg group controlled by $\det(V)$. The connection on $\det V$ identifies the corresponding Heisenberg Lie algebra with the standard one (for trivialized V). Finally, every $\nu \in \omega(F)^\times$ defines on F a non-degenerate scalar product which yields a "self-adjoint" splitting of the standard Heisenberg Lie algebra; this splitting specifies the horizontal directions for the ε-connection at ν. In fact, for all directions but one (namely, along the fibers of the map $\nu \mapsto \operatorname{Res}\nu$) the ε-connection comes from the action of the group of infinitesimal automorphisms of F on our picture. This provides the absolute ε-connection refered to in 1.3.

The above argument for the product formula is essentially Tate's proof [T2] of the residue formula. Replacing the first order differential operator $\nabla(\tau_\nu)$ by a zero order one you get a proof of Weil's reciprocity (cf. [ACK]); its infinitesimal version is the residue formula.

1.5. The idea that ε-factors have to do with fermionic determinants is due to Laumon: the introduction to [L] opens with a discussion of Witten's picture of classical Morse theory as a WKB approximation to quantum mechanics of a super particle. This passage was considered by some as *nezabudki*[12,13] for Laumon's construction itself is based on the mere fact that the Fourier transform of a compactly supported distribution f on a line is smooth, and its asymptotics at ∞ are controlled by the singularities of f.

1.6. The formula for the ε-connection looks as follows. We assume that $S = \operatorname{Spec} K$, K is a field. The group of isomorphism classes of line bundles with connection on S equals $\Omega^1_{K/k}/d\log(K^\times)$, so our formula should determine an element in this group.

An absolute connection (V, ∇) on $K((t))$ is said to be *admissible* if it admits a lattice $\mathcal{V}_O = \oplus K[[t]] \subset V$ with respect to which the connection has the form (for some m) $\nabla = d + \mathcal{A}(t) = d + g(t)dt/t^m + \eta/t^{m-1}$, $g \in \mathrm{GL}_n(K[[t]])$, $\eta \in \mathrm{Mat}_n(\Omega^1_K) \otimes K[[t]]$. When (V, ∇) is admissible and irregular, we find[14] that the class of the connection on $\mathcal{E}_{t^{-m}dt}$ equals

$$\operatorname{Res}_t \mathrm{Tr}(g^{-1}dg \wedge \eta/t^{m-1}) - \frac{m}{2} d\log\det(g(0)). \tag{1.6.1}$$

[12] Forget-me-nots from a fable *"Nezabudki i Zapyatki"* of Koz'ma Proutkoff[13] beloved by I. M. Gelfand.

[13] A famous writer and philosopher (1803–1863) whose project "On introducing unanimity in Russia" determined the Russian Way for XX–XXI centuries.

[14] See 5.6 for a more precise statement for general ν.

The corresponding formula for the global Gauß-Manin determinant was found (in case of admissible singularities and a genus 0 curve) in [BE3]; it was the starting point of the present article.

The Levelt-Turritin theorem helps to reduce (in non-effective way) computation of the ε-connection for arbitrary ∇ to the rank 1 situation (which is always admissible).

1.7. A list of natural questions:

(a) How the above picture is related to the rank 1 story of [BE2]?

(b) Can ε-factors be seen microlocally? What is the relation between the de Rham counterpart of Laumon's Fourier approach [L] and our picture?

(c) What could be a higher dimensional generalization?

(d) Is there a de Rham version of Tate's Thesis [T1]? More generally, what about the automorphic counterpart of the whole story?

(e) Can one treat families of connections with jumping irregularity?

1.8. The article is organized as follows. Section 2 reviews the basic formalism of determinant lines for families of Fredholm operators in the algebraic setting. The ideas here go back to [KM] and [T2]. Much of the subject is considered from the analytic point of view in the book [PS] (or in the earlier articles [DKJM], [SW]), but we were not able to find a convenient reference for the algebraic situation. Our exposition (which makes no pretense to originality) is based on localization statements 2.3(ii) and 2.12 borrowed from [Dr].[15] We tried to make signs take care of themselves (cf. [ACK]) using the language of "super extensions";[16] the appendix to sect. 2 (which is a variation on theme of SGA 4 XVII 1.4) stores the required general nonsense (for a thorough discussion of super subject we refer the reader to [BDM]). In the next section we consider the Heisenberg (super) extension of the group ind-scheme $k((t))^\times$. Its commutator pairing is the parametric version of the tame symbol from [C].[17] According to [T2] or [ACK], this format yields automatically the Weil reciprocity. The Heisenberg action controls the dependence of our ε-factors on ν and is responsible for the existence of the ε-connection. The key fact here (see 3.10) is that on the $k((t))^\times$-torsor of invertible forms $k((t))^\times dt$ every bundle equivariant with respect to the action of the Heisenberg group twisted

[15] A typical object we meet is a projective $R((t))$-module of finite rank, and 2.12 describes it as a topological R-module étale locally on $\operatorname{Spec} R$.

[16] Borrowed from [BD].

[17] The Heisenberg group approach to the construction of the Contou-Carrère symbol is a group-theoretic version of Tate's construction of residue [T2], and it is as simple and efficient as the latter, see 3.2–3.4.

by a local system on $\operatorname{Spec} k((t))$ acquires automatically a plain flat connection.[18] In the fourth section we define ε-factors and study their basic properties. The final section deals with explicit calculations of ε-connections. We explain how $\mathcal{E}_\nu$ depends on ν in the general situation. Then we treat the case of regular singularities (cf. [BE1]), the one of irregular admissible connections (cf. [BE3]), and show that the general computation can be reduced, in principle, to the rank 1 case.

Acknowledgements: It is a pleasure to thank Takeshi Saito for discussions on the analogy of the formula as written in [BE3] with the existing formulae for the ε-factors of L-functions. We also thank Gerd Faltings for explaining to us his own version of a variation of the global method of [BE3]. We are most greatful to Vladimir Drinfeld for advice and help in understanding the subject. We want to thank the referee for a number of corrections and illuminating remarks.

The authors were partially supported by, respectively, NSF grants DMS-0100108 and DMS-0103765, and DFG-Schwerpunkt "komplexe Mannigfaltigkeiten".

2. Determinant lines of Fredholm operators

In this section we establish the basic structure of Fredholm determinants, working insofar as possible in the category of functors on algebras. Of particular importance is the material on Clifford algebras in (2.14)–(2.18): it will be used in section 3 to define a μ_2-structure on a certain determinant line over $\omega(F)^\times$ which, in turn, leads to a connection on the epsilon line.

We write "$P \in \mathcal{C}$" for "P is an object of a category $\mathcal{C}$."

2.1. **Super lines and super extensions.** Let R be a commutative ring. We denote by $\mathcal{M}_R^s$ the tensor category of *super R-modules.* So an object of $\mathcal{M}_R^s$ is a $\mathbb{Z}/2$-graded R-module $M = M^{\bar{0}} \oplus M^{\bar{1}}$. The tensor product as well as the associativity constraint, is the usual tensor product of $\mathbb{Z}/2$-graded modules, and the commutativity constraint is $a \otimes b = (-1)^{p(a)p(b)} b \otimes a$ where $p(a) \in \mathbb{Z}/2$ is the degree (or parity) of a, i.e., $p(a) = i$ for $a \in M^i$. See e.g. ch.1 of [BDM] for details.

A *super R-line* (or simply *super line*) is an invertible object of $\mathcal{M}_R^s$. Super lines form a Picard groupoid[19] $\mathcal{P}ic_R^s$. Explicitly, a super line is a pair $(\mathcal{L}, p)$ where $\mathcal{L}$ is an invertible R-module and p (the parity of our super line) is a locally constant function $\operatorname{Spec} R \to \mathbb{Z}/2$. In notation

[18] Here the Clifford module picture plays the key role.

[19] See appendix to this section, A1.

we usually abbreviate $(\mathcal{L}, p)$ to $\mathcal{L}$ and write $p = p(\mathcal{L})$; we also write $\mathcal{L} \cdot \mathcal{L}' := \mathcal{L} \otimes \mathcal{L}'$, $\mathcal{L}/\mathcal{L}' := \mathcal{L} \otimes \mathcal{L}'^{-1}$. The unit object $(R, 0)$ of $\mathcal{P}ic^s_R$ is denoted by 1_R.

The group of isomorphism classes of super lines $\pi_0(\mathcal{P}ic^s_R)$ is $Pic(R) \times \mathbb{Z}/2_R$ where $\mathbb{Z}/2_R$ is the group of locally constant functions $p : \operatorname{Spec} R \to \mathbb{Z}/2$. One has $\pi_1(\mathcal{P}ic^s_R) := \operatorname{Aut} 1_R = R^\times$.

The above picture is functorial with respect to morphisms of rings: every $f : R \to R'$ yields a base change morphism of tensor categories $f^* : \mathcal{M}^s_R \to \mathcal{M}^s_{R'}$, $M_R \mapsto M_{R'} := M_R \underset{R}{\otimes} R'$, hence a morphism of Picard groupoids $f^* : \mathcal{P}ic^s_R \to \mathcal{P}ic^s_{R'}$.

Variants: (a) Replacing line bundles in the above definition by μ_2-torsors on $\operatorname{Spec} R$ we get the Picard groupoid μ_2-$tors^s_R$ of *super* μ_2-*torsors.* So a super μ_2-torsor is the same as a super line $\mathcal{L}$ equipped with an isomorphism $\sigma : \mathcal{L}^{\otimes 2} \xrightarrow{\sim} 1_R$ (we identify $(\mathcal{L}, \sigma)$ with the μ_2-torsor of sections ℓ of $\mathcal{L}$ that satisfy $\sigma(\ell \otimes \ell) = 1$, its parity is the parity of $\mathcal{L}$).[20]

(b) Replacing $\mathbb{Z}/2$ in the above definition by $\mathbb{Z}$ we get the Picard groupoid $\mathcal{P}ic^{\mathbb{Z}}_R$ of $\mathbb{Z}$-*graded super lines.* So a $\mathbb{Z}$-graded super line is the same as a pair $(\mathcal{L}, v)$ where $\mathcal{L}$ is a super line and $v : \operatorname{Spec} R \to \mathbb{Z}$ is a locally constant function such that $p(\mathcal{L}) = v \mod 2$. As above, we usually abbreviate $(\mathcal{L}, v)$ to $\mathcal{L}$; we call $v = v(\mathcal{L})$ the *degree* or *index* of the $\mathbb{Z}$-graded super line.

We will consider $\mathcal{P}ic^s_R$-extensions (see A2) of various groupoids Γ and refer to them as *super* $\mathcal{O}^\times$-*extensions* or simply *super extensions.* Every super extension $\Gamma^\flat$ yields a homomorphism $\Gamma \to \pi_0(\mathcal{P}ic^s_R)$ (see A2, Remark (ii)), hence we have the parity homomorphism $p : \Gamma \to \mathbb{Z}/2_R$.

Notice that $\mathcal{P}ic^s_R$ coincides as a *monoidal* category[21] with the product of $\mathcal{P}ic_R$ and the discrete groupoid $\mathbb{Z}/2_R$. By A4, a $\mathcal{P}ic_R$-extension of Γ is the same as a sheaf of central extensions of Γ by $\mathcal{O}^\times$ on $\operatorname{Spec} R$; we refer to it as a *plain* $\mathcal{O}^\times$-*extension.* Thus[22] a super $\mathcal{O}^\times$-extension amounts to a plain $\mathcal{O}^\times$-extension of Γ together with a homomorphism $p : \Gamma \to \mathbb{Z}/2_R$.

If Γ is a group and we have its super extension $\Gamma^\flat$, then every pair of commuting elements $\gamma, \gamma' \in \Gamma$ yields $\{\gamma, \gamma'\}^\flat \in R^\times$ (see A5). Looking at the corresponding plain extension we get $\{\gamma, \gamma'\}^{plain} \in R^\times$. One has

$$\{\gamma, \gamma'\}^\flat = (-1)^{p(\gamma)p(\gamma')} \{\gamma, \gamma'\}^{plain}. \tag{2.1.1}$$

[20]The tensor product in this format is $(\mathcal{L}, \sigma) \otimes (\mathcal{L}', \sigma') = (\mathcal{L} \otimes \mathcal{L}', \sigma \cdot \sigma')$ where $(\sigma \cdot \sigma')((\ell_1 \otimes \ell_1') \otimes (\ell_2 \otimes \ell_2')) := \sigma(\ell_1 \otimes \ell_2)\sigma'(\ell_1' \otimes \ell_2')$.

[21]i.e., we forget about the commutativity constraint.

[22]Use A3 Remark (ii) and A2 Remark (ii).

We leave it to the reader to define the notion of *super* $\mathcal{O}^\times$*-extension* of a group valued functor on the category of commutative R-algebras.

Below we refer to $\mathcal{P}ic^s_R$-torsors (see A6) as (neutral) *super* $\mathcal{O}^\times$*-gerbes* or simply *super gerbes* on $\operatorname{Spec} R$.[23] As always, a *trivialization* of a super gerbe is its identification with $\mathcal{P}ic^s_R$. We also have the notion of *super pre-gerbe* = pre $\mathcal{P}ic^s_R$-torsor (see Remark in A6).

Replacing in $\mathcal{P}ic^s$ by $\mathcal{P}ic^{\mathbb{Z}}$ or μ_2-$tors^s$ we get the notions of $\mathbb{Z}$-graded super $\mathcal{O}^\times$-extension, resp. super μ_2-extension. Same for $\mathbb{Z}$-graded super $\mathcal{O}^\times$-gerbes, super μ_2-gerbes, etc.

2.2. **Determinant lines.** We denote by $\mathcal{V}_R$ the category of projective R-modules; $\mathcal{V}^{\mathrm{f}}_R \subset \mathcal{V}_R$ is the subcategory of modules of finite rank.

For $M \in \mathcal{V}^{\mathrm{f}}_R$ let $\det M = \det_R M \in \mathcal{P}ic^{\mathbb{Z}}_R$ be the top exterior power of M placed in degree $\mathrm{rk} M$. This $\mathbb{Z}$-graded super line is functorial with respect to isomorphisms of M's, so for $f : N \xrightarrow{\sim} M$ we have $\det f : \det N \xrightarrow{\sim} \det M$ or $\det f : 1_R \xrightarrow{\sim} \det M/\det N$.

The following standard compatibilities hold:

(i) For every finite family $\{M_\alpha\}$ there is a canonical isomorphism

$$\det(\oplus M_\alpha) = \otimes \det M_\alpha. \tag{2.2.1}$$

Recall that it is this compatibility that forces us to consider $\det M$ as a *super* line.

(ii) For M equipped with a finite filtration with projective subquotients, there is a canonical isomorphism

$$\det M = \det(\mathrm{gr} M). \tag{2.2.2}$$

(iii) There is a canonical isomorphism of super lines

$$\det(M^*) = (\det M)^{-1}, \tag{2.2.3}$$

where $M^* := \mathrm{Hom}(M, R)$ is the dual module, defined by a pairing $e_M : \det M \cdot \det M^* \to R$ such that for a base[24] $m_1, .., m_n$ of M and the dual base $m_1^*, .., m_n^*$ of M^* one has $e((m_1 \wedge .. \wedge m_n) \cdot (m_n^* \wedge .. \wedge m_1^*)) = 1$.

Notice that (2.2.3) changes the $\mathbb{Z}$-grading to the opposite one, and one has $e_{M^*} = e_M^m$ (see A1 for the notation).

The following two useful facts about projective modules of infinite rank are due, respectively, to Kaplansky and Drinfeld:

[23] We skip the word neutral since the subject of this article is R-local, so we can assume that all gerbes are neutral.

[24] We consider our picture locally on $\operatorname{Spec} R$.

2.3. **Proposition.** (i) If R is a local ring then every projective R-module is free.

(ii) Let M be a projective R-module and $K \subset M$ a finitely generated submodule. Then Zariski locally on $\operatorname{Spec} R$ one can find a finitely generated submodule $P \subset M$ which contains K such that M/P is projective.

Proof. (i) See [K].

(ii) See [Dr] 4.2; we reproduce the proof for completeness sake. Take any $x \in \operatorname{Spec} R$; let $R_{(x)}$ be the corresponding local ring. Then $M_{(x)} := R_{(x)} \otimes M$ is a free $R_{(x)}$-module by (i). So there is an embedding $i_{(x)} : R^n_{(x)} \hookrightarrow M_{(x)}$ with projective cokernel whose image contains $K_{(x)} \subset M_{(x)}$. Replacing $\operatorname{Spec} R$ by an open affine neighbourhood of x we can assume that $i_{(x)}$ comes from $i : R^n \to M$ whose image contains K, and then that i is an embedding with a projective cokernel (to see this represent M as a direct summand of a free module to reduce the statement first to the case when M is free, then to the case when M is free of finite rank). $\square$

Remark. If R is a Noetherian ring such that $\operatorname{Spec} R$ is connected then, according to [Ba], every projective R-module of infinite rank is free. We will not use this fact.

2.4. **Asymptotic morphisms and Fredholm morphisms.** For projective modules $M, N \in \mathcal{V}_R$ let $\operatorname{Hom}^{\mathfrak{f}}_R(N, M) \subset \operatorname{Hom}_R(N, M)$ be the R-submodule of morphisms whose image lies in a finitely generated R-submodule of M. Set

$$\operatorname{Hom}^{\infty}_R(N, M) := \operatorname{Hom}_R(N, M)/\operatorname{Hom}^{\mathfrak{f}}_R(N, M). \tag{2.4.1}$$

Elements of $\operatorname{Hom}^{\infty}$ are *asymptotic morphisms*; for $f : N \to M$ the corresponding asymptotic morphism is denoted by f^{∞}. The composition of asymptotic morphisms is well-defined.[25] So we have an R-category $\mathcal{V}^{\infty}_R$ of projective R-modules and asymptotic morphisms together with the obvious functor $\mathcal{V}_R \to \mathcal{V}^{\infty}_R$ which is the identity on objects. Notice that $M \in \mathcal{V}_R$ lies in $\mathcal{V}^{\mathfrak{f}}_R$ if and only if $id^{\infty}_M = 0$.

A morphism f is said to be *Fredholm* if f^{∞} is invertible. Denote by $\mathcal{V}^{\mathfrak{Fr}}_R$ the category of projective modules and Fredholm morphisms. Let $\mathcal{V}^{\times}_R$, $\mathcal{V}^{\infty\times}_R$ be groupoids of invertible morphisms in $\mathcal{V}_R$, $\mathcal{V}^{\infty}_R$. Thus $\mathcal{V}^{\times}_R \subset \mathcal{V}^{\mathfrak{Fr}}_R$ and the morphism of groupoids $\mathcal{V}^{\times}_R \to \mathcal{V}^{\infty\times}_R$ extends to the functor $\mathcal{V}^{\mathfrak{Fr}}_R \to \mathcal{V}^{\infty\times}_R$.

[25] The composition of a morphism from $\operatorname{Hom}^{\mathfrak{f}}$ with any morphism lies in $\operatorname{Hom}^{\mathfrak{f}}$.

Any base change of a Fredholm morphism f is Fredholm. So for every geometric point x of $\mathrm{Spec}\,R$ we have a Fredholm operator $f_x : N_x \to M_x$.[26] Denote by $i(f)_x = i(f_x)$ its index, $i(f)_x := \dim \mathrm{Coker} f_x - \dim \mathrm{Ker} f_x$. The index depends only on the corresponding asymptotic morphism, so we write $i(f)_x = i(f^\infty)_x$.

2.5. **Lemma.** For a morphism $f : N \to M$ of projective modules the following conditions are equivalent:

(i) f is Fredholm,

(ii) $\mathrm{Coker} f$ is finitely generated, for every geometric point x of $\mathrm{Spec}\,R$ the corresponding operator $f_x : N_x \to M_x$ is Fredholm, and the function $i(f) : x \mapsto i(f_x)$ on $\mathrm{Spec}\,R$ is locally constant.

(iii) f can be written as composition

$$N \xrightarrow{i} N \oplus Q \xrightarrow{\tilde{f}} M \oplus P \xrightarrow{p} M \tag{2.5.1}$$

where $P, Q \in \mathcal{V}^{\mathfrak{f}}$, i is the embedding, p the projection, and $\tilde{f}$ is an isomorphism.

Proof. It is clear that (iii) implies (i) and (ii).

(i)$\Rightarrow$(ii): Since f^∞ admits a right inverse $\mathrm{Coker} f$ is finitely generated. So there exists $Q \in \mathcal{V}^{\mathfrak{f}}$ and a morphism $\alpha : Q \to M$ such that $\pi := (f, \alpha) : N \oplus Q \to M$ is surjective. Then $P := \mathrm{Ker}\pi \in \mathcal{V}$, and (i) assures that it has finite rank. Since $i(f) = rk(Q) - rk(P)$, it is a locally constant function.

(i)$\Rightarrow$(iii): Define α, Q, P as above. A splitting of π yields a projector $\beta : N \oplus Q \to P$. Then $\tilde{f} := (f, \alpha; \beta) : N \oplus Q \xrightarrow{\sim} M \oplus P$ defines (2.5.1).

(ii)$\Rightarrow$(iii): We define $\tilde{f}$ as above. Our P is projective, so it remains to check that the conditions of (ii) imply that it is finitely generated. We know that every fiber P_x is of finite dimension, and this dimension is locally constant with respect to x. Take any $x \in \mathrm{Spec}\,R$; we want to find a Zariski open $x \in U \subset \mathrm{Spec}\,R$ such that $P|_U$ is finitely generated. By 2.3(ii) we can find U such that $P|_U$ can be written as $P' \oplus P''$ where P' is finitely generated and $P'_x = P_x$, i.e., $P''_y = 0$. Shrinking U if necessary to assure that $\dim P'_y$ is constant on U, we see that $P''_y = 0$ for every $y \in U$. By 2.3(i) one has $P'' = 0$. □

Remarks. (a) As follows from 2.5(ii), Fredholm morphisms have local nature with respect to the flat topology.

(b) If $I \subset R$ is a nilpotent ideal then f is Fredholm if and only if $f_{R/I} : N/IN \to M/IM$ is Fredholm (use 2.5(ii)).

[26]Here the k_x-vector space $M_x := k_x \otimes M$ is the fiber of M at x.

(c) If R is Noetherian then the last condition in 2.5(ii) is superfluous. Indeed, the proof of (ii)$\Rightarrow$(iii) proceeds as above, but we just notice that $P''_x = 0$ implies, after shrinking U if necessary, that $P''_y = 0$ for every generic point of U. So $P'' = 0$ by 2.3(i).

2.6. **Relative determinant lines.** Of course, there is no way to assign a determinant line to a projective module of infinite rank. However if one has two such modules M, N and an asymptotic isomorphism f^∞ between them then one can use f^∞ to mutually cancel infinities in $\det M$, $\det N$ so that the ratio $\det M / \det N$ (the relative determinant line) is well-defined. Let us explain this Dostoevskian ansatz[27,28] in more details.

Here is a list of data we look for:

(i) *A $\mathbb{Z}$-graded super extension $\mathcal{V}_R^{\infty\flat}$ of the groupoid $\mathcal{V}_R^{\infty\times}$.* Thus for every triple (M, N, f^∞), where $M, N \in \mathcal{V}_R$, f^∞ is an invertible asymptotic morphism $N \to M$, we want to have a $\mathbb{Z}$-graded super line $\det(M, N, f^\infty)$ (the relative determinant line), together with data of composition isomorphisms

$$c : \det(L, M, g^\infty) \cdot \det(M, N, f^\infty) \xrightarrow{\sim} \det(L, N, g^\infty f^\infty) \tag{2.6.1}$$

satisfying the obvious associativity property.

(ii) *A splitting of the pull-back of $\mathcal{V}_R^{\infty\flat}$ to $\mathcal{V}_R^\times$.* This means that for every isomorphism $f : N \xrightarrow{\sim} M$ in $\mathcal{V}$ we have a canonical trivialization

$$\det f : 1_R \xrightarrow{\sim} \det(M, N, f^\infty) \tag{2.6.2}$$

compatible with composition of f's (via (2.6.1)).

(iii) *Compatibility with sums.* For every (M_1, N_1, f_1^∞), (M_2, N_2, f_2^∞) we have a canonical isomorphism $\det(M_1 \oplus M_2, N_1 \oplus N_2, f_1^\infty \oplus f_2^\infty) = \det(M_1, N_1, f_1^\infty) \cdot \det(M_2, N_2, f_2^\infty)$ compatible with the associativity and commutativity constraints. It is nicer to write it then as a canonical identification of $\mathbb{Z}$-graded super lines

$$\det(\oplus M_\alpha, \oplus N_\alpha, \oplus f_\alpha^\infty) = \otimes \det(M_\alpha, N_\alpha, f_\alpha^\infty) \tag{2.6.3}$$

defined for any finite family $\{(M_\alpha, N_\alpha, f_\alpha^\infty)\}$. We want it to be compatible with the canonical isomorphisms from (i), (ii).

---

[27] "Let a reptile gobble up another of its kind," a comment of a hero of "The brothers Karamazov" on his father and a younger brother.

[28] One may prefer to think of two sumo wrestlers, each of infinite girth, locked in a Fredholm grip, with the match determined by a mere finite inbalance of forces.

(iv) *Compatibility with finite determinants.* For every $M, N \in \mathcal{V}^{\mathfrak{f}}$ there is a canonical identification[29]

$$\det(M,N) = \det M/\det N \tag{2.6.4}$$

such that (2.6.1)–(2.6.3) become the usual compatibilities from 2.2.

(v) *Base change.* Our structure should be compatible with the base change in the obvious manner.

Notice that (iv) and (ii) imply that the degree of the super line $\det(M, N, f^\infty)$ is the index $i(f^\infty)$ (see 2.4).

2.7. **Proposition.** Such a structure exists and is unique (up to a unique isomorphism).

Proof. Here is an explicit construction. We will not use the uniqueness statement, so its proof is left to the reader.[30]

Let us construct the determinant line $\det(M, N, f^\infty)$. Assume for a moment that M satisfies the following assumption, which is fulfilled, e.g., if M is a free R-module:

$(*)$ For every finitely generated R-submodule $T \subset M$ there exists a finitely generated R-submodule $P \subset M$ with M/P projective such that $T \subset P$.

Let $f : N \to M$ be any lifting of f^∞. Since $\mathrm{Coker} f$ is finitely generated (see 2.5.(ii)), we can choose $P \subset M$ as in $(*)$ such that $P + f(N) = M$. Set $Q := f^{-1}(P)$. Then $P, Q \in \mathcal{V}^{\mathfrak{f}}$.

Lemma-definition. The $\mathbb{Z}$-graded super lines $\det P/\det Q$ for all choices of f, P are canonically identified, so they can be considered as a single super line. This is our $\det(M, N, f^\infty)$.

Proof of Lemma. We fix f for a moment. Let P' be another submodule as above, $Q' := f^{-1}(P')$. Let us define the canonical identification $\phi = \phi_{PP'} : \det P/\det Q \xrightarrow{\sim} \det P'/\det Q'$.

If $P' \supset P$ then $P'/P \in \mathcal{V}^{\mathfrak{f}}$ and f yields an isomorphism $\bar{f} : Q'/Q \xrightarrow{\sim} P'/P$. Our ϕ is the image of $\det \bar{f}$ by the canonical identification $\det(P'/P)/\det(Q'/Q) = (\det P'/\det Q')/(\det P/\det Q)$ (see (2.2.2)).

If P' is arbitrary then we can choose P'' as in $(*)$ such that $P'' \supset P + P'$, and set $\phi_{PP'} := \phi^{-1}_{P'P''}\phi_{PP''}$. One checks in a moment that this definition does not depend on the choice of P'' and ϕ so defined satisfy the transitivity property. Thus all super lines for fixed lifting f of f^∞ are canonically identified.

[29] We omitted f^∞ in the l.h.s. of (2.6.4) for it bears no information: in the present situation one has $\mathrm{Hom}^\infty(N, M) = \mathrm{Isom}^\infty(N, M) = 0$.

[30] Hint: use 2.9.

Now let f' be another lifting of f^∞. Choose P as in $(*)$ so that $P + f(N) = M$ and $P \supset (f - f')(N)$. Then $P + f'(N) = M$ and $f^{-1}(P) = f'^{-1}(P)$. So the super lines for f and f' computed by means of such P are simply equal. If we change P the identifications ϕ for f and f' are also equal. These identifications of super lines are transitive with respect to f. We are done. □

In order to eliminate condition $(*)$ notice that super lines are local objects, so it suffices to define det locally on $\operatorname{Spec} R$. Now one can use 2.3(ii) instead of $(*)$.

Remark. One can also reduce the general situation to the situation of $(*)$ directly using the following trick: Choose $M' \in \mathcal{V}$ such that $M \oplus M'$ is free (and so satisfies $(*)$); then $\det(M, N, f^\infty) = \det(M \oplus M', N \oplus N', f^\infty \oplus id^\infty_{M'})$.

The canonical isomorphisms (2.6.1)–(2.6.3) are defined as follows. We use condition $(*)$ at will; in truth, one should work locally on $\operatorname{Spec} R$ and use 2.3(ii).

To define the composition isomorphism c of (2.6.1) we choose some liftings f, g and compute the determinant lines by means of finitely generated $S \subset L$ such that L/S is projective and $S + gf(N) = L$ and $P := g^{-1}(S)$, $Q := (gf)^{-1}(S) = f^{-1}(P)$. Our c is the standard isomorphism $(\det S/\det P)\cdot(\det P/\det Q) \xrightarrow{\sim} \det S/\det Q$. The independence of choices is immediate.

To define $\det f$ of (2.6.2) one computes $\det(M, N, f^\infty)$ using f and $P = 0$.

To define (2.6.3) you compute the r.h.s. using some f_α and P_α and then compute the l.h.s. using $f = \oplus f_\alpha$ and $\oplus P_\alpha$; then use (2.2.1).

2.6(iv), 2.6(v) are inherent in the construction. The mutual compatibilities of the above canonical isomorphisms are obvious. □

2.8. **Remark.** There is a more general construction (which is not needed in this article) that assigns a determinant line to every pair $(M^\cdot, d^\infty)$ where $M^\cdot$ is a finitely $\mathbb{Z}$-graded projective module and d^∞ is a differential of degree 1 on M^∞ such that $(M^\cdot, d^\infty)$ is a homotopically trivial complex. The construction above corresponds to $M^\cdot$ supported in degrees $0, -1$.

2.9. **Compatibility with filtrations.** Let M, N be projective modules equipped with finite (increasing) filtrations $M.$, $N.$ such that the associated graded modules $\mathrm{gr}M$, $\mathrm{gr}N$ are projective (i.e., the filtrations can be split). Let $f^\infty : N \to M$ be an asymptotic morphism compatible with filtrations such that $\mathrm{gr}f^\infty : \mathrm{gr}N \to \mathrm{gr}M$ is invertible. Then f^∞ is also invertible.

Lemma. There is a canonical isomorphism

$$\det(M, N, f^\infty) = \det(\mathrm{gr}M, \mathrm{gr}N, \mathrm{gr}f^\infty) \tag{2.9.1}$$

compatible with base change and such that (2.9.1) is the identity map provided that our filtered picture came from a graded one. Such data of isomorphisms is unique.

Proof. Here is a construction of (2.9.1) that makes use of general functoriality from 2.6 only.

Consider the Rees $R[t]$-module $M^{\mathfrak{r}} := \oplus M_i t^i$. This is a projective $R[t]$-module equivariant with respect to homotheties of t whose fibers at $t = 1$ and $t = 0$ are, respectively, M and $\mathrm{gr}M$. Similarly, we have $N^{\mathfrak{r}}$ and an invertible asymptotic $R[t]$-morphism $f^{\infty\mathfrak{r}} : N^{\mathfrak{r}} \to M^{\mathfrak{r}}$. The corresponding super $R[t]$-line $D := \det(M^{\mathfrak{r}}, N^{\mathfrak{r}}, f^{\infty\mathfrak{r}})$ is equivariant with respect to homotheties of t (by base change), i.e., D is a graded $R[t]$-module. Its fibers at $t = 1$ and $t = 0$ are, respectively, $\det(M, N, f^\infty)$ and $\det(\mathrm{gr}M, \mathrm{gr}N, \mathrm{gr}f^\infty)$. The fiber at 0 has degree $i = i(\mathrm{gr}f^\infty) = i(f^\infty)$ with respect to the action of homotheties. Let D^i be i^{th} component of D. This is a super R-line, and $D = R[t] \otimes_R D^i$. The restriction to $t = 1, 0$ identifies D^i with $\det(M, N, f^\infty)$ and $\det(\mathrm{gr}M, \mathrm{gr}N, \mathrm{gr}f^\infty)$. The composition of these identifications is (2.9.1).

The properties mentioned in the statement of Lemma, as well as the uniqueness, are immediate. □

Remarks. (i) Here is another, direct, construction of (2.9.1). Let us construct $\det(M, N, f^\infty)$ as in the proof of 2.7 by means of some f and $P \subset M$. Choose f compatible with filtrations and P such that $\mathrm{gr}P \subset \mathrm{gr}M$ is such that $\mathrm{gr}M/\mathrm{gr}P$ is projective and $\mathrm{gr}P + \mathrm{gr}(f)(\mathrm{gr}N) = \mathrm{gr}M$; then use (2.2.2) to define (2.9.1). The independence of auxiliary choices is immediate, as well as properties mentioned in the statement of Lemma.

(ii) Isomorphisms (2.9.1) are compatible with (2.6.1) and (2.6.2); if M, N are of finite rank then (2.6.4) reduces them to (2.2.2).

2.10. **The Tate extension.** Below we will consider set-valued functors on the category of commutative R-algebras which commute with finite inverse limits. The "category" of such functors (called *"R-spaces"*) is denoted by $\mathcal{S}_R$; it is closed under finite inverse limits. For an R-algebra B the corresponding representable functor is denoted by $\operatorname{Spec} B \in \mathcal{S}_R$. We have a ring object $\mathcal{O} = \operatorname{Spec} R[t] \in \mathcal{S}_R$, $\mathcal{O}(R') = R'$, so one can consider $\mathcal{O}$-modules, $\mathcal{O}$-algebras in $\mathcal{S}_R$, etc. For $X \in \mathcal{S}_R$ and an R-flat commutative R-algebra B, $\operatorname{Spec} B =: Y$, we have an R-space $\mathcal{H}om(Y, X)$, $\mathcal{H}om(Y, X)(R') := X(R' \underset{R}{\otimes} B)$. For $B = R[h]/h^2$

we get the *tangent bundle* Θ_X of X. Since $R[h]/h^2$ is naturally an R-module in the category of commutative R-algebras A equipped with a morphism $A \to R$, the tangent bundle is an $\mathcal{O}$-module over X, so for any $x \in X(R)$ the tangent space $\Theta_{X,x}$ (:= the fiber of Θ_X at x) is an $\mathcal{O}$-module in $\mathcal{S}_R$. For a group R-space G its tangent space at $1 \in G(R)$ is denoted by Lie$G$; this is a Lie $\mathcal{O}$-algebra in $\mathcal{S}_R$.[31]

Every projective R-module M defines an $\mathcal{O}$-module in $\mathcal{S}_R$ which we denote also by M, $M(R') := M_{R'} := M \underset{R}{\otimes} R'$. For $M, N \in \mathcal{V}_R$ we have $\mathcal{O}$-modules $\mathrm{Hom}(N,M), \mathrm{Hom}^\infty(N,M)$ in $\mathcal{S}_R$, $\mathrm{Hom}(N,M)(R') := \mathrm{Hom}_{R'}(N_{R'}, M_{R'})$, $\mathrm{Hom}(N,M)^\infty(R') := \mathrm{Hom}^\infty_{R'}(N_{R'}, M_{R'})$. So for $M \in \mathcal{V}_R$ we have the corresponding $\mathcal{O}$-algebras of endomorphisms and the groups of invertible endomorphisms $\mathrm{GL}(M), \mathrm{GL}^\infty(M) = \mathrm{GL}(M^\infty)$. There is a canonical homomorphism $\mathrm{GL}(M) \to \mathrm{GL}(M^\infty)$; denote by $\mathrm{GL}^{\mathfrak{f}}(M)$ its kernel.

By 2.6(v) the $\mathbb{Z}$-graded super extensions $\mathrm{Aut}^{\infty\flat}_{R'}(M_{R'})$ form the *Tate* super $\mathcal{O}^\times$-extension $\mathrm{GL}^\flat(M^\infty)$ of $\mathrm{GL}(M^\infty)$. For $g^\infty \in \mathrm{GL}(M^\infty)(R')$ we denote the corresponding $\mathbb{Z}$-graded super line in $\mathrm{GL}^\flat(M^\infty)(R')$ by $\det(M^\infty, g^\infty)$ or simply λ_{g^∞}.

Here is a list of basic properties of the Tate super extension:

(i) According to Remark (iii) in A3, the Tate extension depends functorially on M *considered as an object of* $\mathcal{V}_R^{\infty\times}$, i.e., every asymptotic isomorphism $f^\infty : N \to M$ yields a canonical identification of $\mathbb{Z}$-graded super extensions

$$\mathrm{Ad}^\flat_{f^\infty} : \mathrm{GL}^\flat(N^\infty) \xrightarrow{\sim} \mathrm{GL}^\flat(M^\infty). \tag{2.10.1}$$

To write it down explicitly, consider $\mathcal{V}_R^{\infty\flat}$ as a plain $\mathcal{O}^\times$-extension of $\mathcal{V}_R^{\infty\times}$ (see 2.1) and choose (locally on $\operatorname{Spec} R$) a generator $\tilde{f}$ of the line $\det(M, N, f^\infty)$. Now for $g^\infty \in \mathrm{GL}(N^\infty)$ the map $\mathrm{Ad}^\flat_{f^\infty} : \lambda_{g^\infty} \to \lambda_{\mathrm{Ad}_{f^\infty}(g^\infty)}$ is the adjoint action of $\tilde{f}$ (in our plain $\mathcal{O}^\times$-extension) multiplied by $(-1)^{p(f^\infty)p(g^\infty)}$ where p is parity (i.e., the index mod 2).

In particular, the adjoint action of $\mathrm{GL}(M^\infty)$ lifts canonically to an action of $\mathrm{GL}(M^\infty)$ on the Tate extension $\mathrm{GL}^\flat(M^\infty)$.

[31] See SGA3 t.1 exp.II, pp.26-27. Here is a sketch of an argument: (a) Check that the addition law on Lie$G$ comes from the product on G. (b) To define the Lie bracket of $\alpha, \beta \in (\mathrm{Lie}G)(R)$ consider them as elements of, respectively, $G(R[s]/s^2), G(R[t]/t^2) \subset G(R[s,t]/(s^2,t^2))$; then the commutator $(\alpha,\beta) := \alpha\beta\alpha^{-1}\beta^{-1}$ belongs to $G(R[h]/h^2) \subset G(R[s,t]/(s^2,t^2))$ where $h := st$; this is $[\alpha,\beta]$. (c) The Lie bracket is obviously skew-symmetric. To see Jacobi consider $\alpha,\beta,\gamma \in (\mathrm{Lie}G)(R)$ as elements of, respectively, $G(R[s]/s^2), G(R[t]/t^2), G(R[u]/u^2) \subset G(R[s,t,u]/(s^2,t^2,u^2))$. Then $[\alpha,[\beta,\gamma]]$ corresponds to $(\alpha,(\beta,\gamma)) \in G(R[v]/v^2)$ where $v := stu$, and Jacobi follows since (α,β) commutes with (α,γ), α, β.

Remark. The groupoid $\mathcal{V}^{\infty\times}$ (as opposed to $\mathcal{V}^{\times}$) does not satisfy the flat descent property (see [Dr] for a discussion of this subject). One can sheafify it formally, and the above property assures that automorphism groups of objects of this stack have a canonical super $\mathcal{O}^{\times}$-extension.

(ii) The map $\mathrm{GL}(M) \to \mathrm{GL}(M^\infty)$, $g \mapsto g^\infty$, lifts canonically to a homomorphism

$$\mathrm{GL}(M) \to \mathrm{GL}^\flat(M^\infty), \quad g \mapsto \det(M,g) \in \det(M^\infty, g^\infty). \tag{2.10.2}$$

Namely, for $g \in \mathrm{GL}(M)$ one has $\det(M^\infty, g^\infty) = \det(M, M, g^\infty)$, and $\det(M,g)$ is $\det g$ from (2.6.2).

Exercises. (a) Show that the restriction of (2.10.2) to the subgroup $\mathrm{GL}^{\mathfrak{f}}(M)$ is the usual determinant $\det : \mathrm{GL}^{\mathfrak{f}}(M) \to \mathcal{O}^{\times}$.

(b) Assume that $f^\infty, g^\infty \in \mathrm{Aut}^\infty(M)$ commute and f^∞ can be lifted to $f \in \mathrm{Aut}(M)$. Then f acts by functoriality on the determinant line $\det(M, M, g^\infty)$. Show that this action is multiplication by $\{f^\infty, g^\infty\}^\flat$.[32]

(iii) For a finite collection $\{M_\alpha\}$ the restriction of $\mathrm{GL}^\flat(\oplus M_\alpha^\infty)$ to the subgroup $\Pi \mathrm{GL}(M_\alpha^\infty) \subset \mathrm{GL}(\oplus M_\alpha^\infty)$ identifies canonically with the Baer product of the Tate super extensions $\mathrm{GL}^\flat(M_\alpha^\infty)$ (see 2.6(iii)). In other words, for every α the embedding $\mathrm{GL}(M_\alpha^\infty) \hookrightarrow \mathrm{GL}(\oplus M_\alpha^\infty)$ lifts canonically to an embedding of super extensions $\mathrm{GL}^\flat(M_\alpha^\infty) \hookrightarrow \mathrm{GL}^\flat(\oplus M_\alpha^\infty)$, and for different α's the images of these embeddings commute.[33]

Remark. This property is no longer true if we consider our super extensions as plain $\mathcal{O}^{\times}$-extensions (see 2.1).

(iv) Assume that an object $M^\infty \in \mathcal{V}_R^\infty$ has filtration which admits a splitting; let $B \subset \mathrm{GL}(M^\infty)$ be the subgroup preserving this filtration. Then the restriction $B^\flat$ of $\mathrm{GL}^\flat(M^\infty)$ to B is canonically isomorphic to the pull-back of $\mathrm{GL}^\flat(\mathrm{gr} M^\infty)$ by the map $B \to \mathrm{GL}(\mathrm{gr} M^\infty)$ by 2.9.

Consider the Lie algebras $\mathfrak{gl}(M), \mathfrak{gl}^{\mathfrak{f}}(M), \mathfrak{gl}(M^\infty), \mathfrak{gl}^\flat(M^\infty)$ of our groups $\mathrm{GL}(M)$, etc. The first three of them are just the corresponding algebras of endomorphisms considered as Lie algebras, so $\mathfrak{gl}(M^\infty) = \mathfrak{gl}(M)/\mathfrak{gl}^{\mathfrak{f}}(M)$, and $\mathfrak{gl}^\flat(M^\infty)$ is a central extension of $\mathfrak{gl}(M^\infty)$ by $\mathcal{O}$. By (ii) the projection $\mathfrak{gl}(M) \to \mathfrak{gl}(M^\infty)$ lifts canonically to a morphism of Lie algebras $\mathfrak{gl}(M) \to \mathfrak{gl}^\flat(M^\infty)$; on $\mathfrak{gl}^{\mathfrak{f}}(M)$ this morphism is the trace map $tr : \mathfrak{gl}^{\mathfrak{f}}(M) \to \mathcal{O}$.[34] Therefore one has:

(v) $\mathfrak{gl}^\flat(M^\infty)$ is the push-forward of the extension $0 \to \mathfrak{gl}^{\mathfrak{f}}(M) \to \mathfrak{gl}(M) \to \mathfrak{gl}(M^\infty) \to 0$ by the ad-invariant morphism tr.

[32] See A5 for notation.

[33] i.e., for $\gamma \in \mathrm{GL}^\flat(M_\alpha^\infty)$, $\gamma' \in \mathrm{GL}^\flat(M_{\alpha'}^\infty)$, $\alpha \neq \alpha'$, one has $\{\gamma, \gamma'\}^\flat = 1$, see A5.

[34] Indeed, tr is tangent to det from Exercise (a) above.

(vi) Since $\mathfrak{gl}^\flat(M^\infty)$ is a central extension of $\mathfrak{gl}(M^\infty)$ we can rewrite its bracket as a pairing $[\,,\,]^\flat : \mathfrak{gl}(M^\infty)\times\mathfrak{gl}(M^\infty) \to \mathfrak{gl}^\flat(M^\infty)$. It satisfies the *cyclic identity*: for every $a, b, c \in \mathfrak{gl}(M^\infty)$ one has

$$[ab, c]^\flat + [ca, b]^\flat + [bc, a]^\flat = 0. \tag{2.10.3}$$

Here ab is the product of a, b as endomorphisms of M^∞.[35]

2.11. **The topological setting: Tate R-modules.** An unpleasant feature of the above formalism is the absence of duality: the dual of a projective module of an infinite rank is not there. A way to recover the duality is to consider the setting of Tate modules.

If R is a commutative ring then a *topological R-module* is an R-module equipped with a topology[36] which has a base formed by R-submodules; we always assume that the topology is complete and separated. Topological R-modules form an additive R-category. A morphism $R \to R'$ defines an obvious functor from the category of topological R'-modules to that of R-modules; its left adjoint is the base change functor $F_R \mapsto F_{R'} := F_R \hat{\otimes}_R R'$.

We can consider every plain R-module N_R as a topological module with discrete topology. If M_R is a *projective* R-module then its dual $M_R^* = \mathrm{Hom}_R(M_R, R)$ equipped with the weak topology (formed by annihilators of finite subsets in M_R) is a topological R-module. The canonical map $M_R \to (M_R^*)^* := \mathrm{Hom}^{cont}(M_R^*, R)$ is an isomorphism, and the functor $M_R \mapsto M_R^*$ is fully faithful. For every $R \to R'$ one has $M_{R'}^* = (M_R^*)_{R'}$.[37]

Remark. For any N_R, M_R as above the image of every continuous morphism $M_R^* \to N_R$ is a finitely generated R-module.[38]

We call a projective R-module considered as discrete topological R-module a *discrete Tate* R-module. The corresponding duals are called *compact Tate* R-modules. A topological R-module F_R is *special Tate* R-module if it can be represented as the direct sum of a discrete and a compact Tate R-modules. Following Drinfeld [Dr], we define a *Tate* R-module as a topological R-module which is a direct summand of a special Tate R-module. Notice that a direct summand of a compact Tate module is a compact Tate module; same is true for discrete Tate

---

[35] Lift a, b, c to $\tilde a, \tilde b, \tilde c \in \mathfrak{gl}(M)$. By (v), it suffices to show that $[\tilde a\tilde b, \tilde c] + [\tilde c\tilde a, \tilde b] + [\tilde b\tilde c, \tilde a] = 0$. This is immediate.

[36] The topologies we consider are always compatible with the additive structure; "base" means "base of neighbourhoods of 0."

[37] To see this represent M_R as a direct summand of a free R-module.

[38] To see this notice that we can assume that M_R is a free R-module.

modules. The base change sends Tate modules to Tate modules. All base changes of a given Tate R-module considered simultaneously (see 2.10) form a *Tate $\mathcal{O}$-module* in $\mathcal{S}_R$.

Examples. (i) $R((t))$ equipped with topology with base $t^n R[[t]]$, $n \geq 0$, is a special Tate R-module.

(ii) Let F be a finitely generated projective $R((t))$-module. Then F carries a canonical topology whose base is formed by $R[[t]]$-submodules of F which generate F as an $R((t))$-module. F is a Tate R-module.

(iii) Let k be a commutative ring and $R \subset k[x]$ the subalgebra of polynomials $f = f(x)$ such that $f(1) = f(0)$. Let $F_R \subset k[x]((t))$ be the R-submodule of elements $g = g(x,t)$ such that $g(1,t) = tg(0,t) \in k((t))$. It carries a topology with base $F \cap t^n k[x]((t))$, $n \geq 0$. Then F is a Tate R-module which is not special. Moreover, it is not special Zariski locally on $\operatorname{Spec} R$.

The following result is due to Drinfeld [Dr] 8.1, 4.1:

2.12. **Proposition.** (i) Let M be a projective R-module. Then every asymptotic projector $\pi^\infty \in \operatorname{End}_R^\infty(M)$, $(\pi^\infty)^2 = \pi^\infty$, can be lifted to a true projector $\pi \in \operatorname{End}_R M$ étale locally on $\operatorname{Spec} R$.

(ii) Every Tate R-module is special étale locally on $\operatorname{Spec} R$.

Remark. The proof actually shows that the statements hold Nisnevich locally.

Proof. (i) Choose any lifting of π^∞ to $\operatorname{End}_R M$. Let us consider M as an $R[t]$-module where t acts on M by this lifting. We will find (after an appropriate localization of R) a polynomial $p = p(t) \in R[t]$ such that (a) $p(p-1)$ kills M, and (b) $p(0) = 0$, $p(1) = 1$. Let π be the action of p on M. This is a projector by (a) which lifts π^∞ because of (b).[39]

Choose a finitely generated R-submodule $L \subset M$ which contains $t(t-1)(M)$. There is a monic $f(t) \in R[t]$ which kills L, so $t(t-1)f(t)$ kills M. Localizing $\operatorname{Spec} R$ with respect to étale[40] topology we can assume that $f(t) = g(t)h(t)$ where $g(1)$ and $h(0)$ are invertible, and $g(t), h(t)$ generate the unit ideal. Then M is supported on the union of non-intersecting subschemes $tg(t) = 0$ and $(t-1)h(t) = 0$. Our $p(t)$ is any polynomial which vanishes on the first subscheme and equals to 1 on the second one.

[39]Indeed, (b) means that $p(t) - t$ is divisible by $t(t-1)$, so the corresponding asymptotic endomorphism of M vanishes (since the one for $t(t-1)$ does).

[40]In fact, Nisnevich

(ii) Let F be a Tate R-module. Choose a special Tate R-module G such that F is a direct summand of G. We will find, after an appropriate localization of R, an open submodule $L \subset G$ such that (a) L is a compact Tate module, and (b) $P := F/F \cap L$ is a projective R-module. Then F is a special Tate module. Indeed, a section $\gamma : P \to F$ yields $F \tilde{\to} P \oplus F \cap L$. Now $F \cap L$ is a direct summand of L,[41] hence it is a compact Tate module.

Let $\Pi \in \mathrm{End}G$ be a projector such that $\Pi(G) = F$, and $G = M \oplus N^*$ be any decomposition of G into a sum of a discrete and compact Tate modules. Let ϕ be the composition $M \hookrightarrow G \overset{\Pi}{\to} G \twoheadrightarrow M$. Then ϕ^∞ is an asymptotic projector (see Remark in 2.11). By (i), after an appropriate localization of $\mathrm{Spec}\, R$, one can find a projector $\pi \in \mathrm{End}_R M$ such that $\phi^\infty = \pi^\infty$. The images of the composition $N^* \hookrightarrow G \overset{\Pi}{\to} G \twoheadrightarrow M$ and of $\pi - \phi$ are contained in a finitly generated submodule $K \subset M$ (see Remark in 2.11). By 2.3(ii) applied to the images of K in $\mathrm{Ker}\pi$ and in $\mathrm{Im}\pi$ one can find finitely generated R-submodules $L' \subset \mathrm{Ker}\pi$ and $L'' \subset \mathrm{Im}\pi$ such that $\mathrm{Ker}\pi/L'$ and $\mathrm{Im}\pi/L''$ are projective R-modules. Set $L := N^* \oplus L' \oplus L'' \subset G$. Since L', L'' are projective L satisfies (a) above; since $F/F \cap L = \mathrm{Im}\pi/L''$ it satisfies (b). We are done. □

All constructions of this article are local with respect to the flat topology of $\mathrm{Spec}\, R$, so we will often tacitly assume that the Tate modules we deal with are special.

Remark. According to [Dr]8.3, Tate R-modules, as opposed to special Tate R-modules, satisfy the flat descent property.

2.13. **Duality, lattices, and the Tate extension.** Let F be a Tate R-module. We say that an R-submodule $L \subset F$ is *bounded* if for every discrete R-module P and a continuous morphism $F \to P$ the image of L in N is contained in a finitely generated R-submodule of P. If F is realized as a direct summand in a special Tate module $G = M \oplus N^*$, $M, N \in \mathcal{V}_R$, then L is bounded if and only if the image of L in $M = G/N^*$ is contained in a finitely generated R-submodule.[42] Notice that bounded open submodules of F form a base of the topology of F.

For a Tate R-module F its *dual* F^* is $\mathrm{Hom}_R^{cont}(F, R)$ equipped with a topology whose base is formed by orthogonal complements to bounded R-submodules of F. Then F^* is again Tate module, and the canonical morphism $F \to (F^*)^*$ is an isomorphism. To see this notice that duality

[41] A projector $\Pi : G \twoheadrightarrow F$ yields a projector $L \twoheadrightarrow F \cap L$, $\ell \mapsto \Pi(\ell) - \gamma(\Pi(\ell) \mathrm{mod} L)$.

[42] This follows from Remark in 2.11.

commutes with (finite) direct sums, and for a special Tate module $F = M \oplus N^*$, $M, N \in \mathcal{V}_R$, one has $F^* = N \oplus M^*$. So the duality functor is an anti-equivalence of the category of Tate R-modules. It commutes with the base change.

For a Tate R-module F a *c-lattice* in F is an open R-submodule $L \subset F$ which satisfies either of the following conditions (to check their equivalence is an exercise for the reader):

(i) L is a compact Tate module and the projection $F \twoheadrightarrow F/L$ admits an R-linear section,

(ii) L is a bounded submodule of F such that F/L is a projective R-module.

Of course, a c-lattice exists if and only if F is a special Tate module.

A *d-lattice* in F is an R-submodule $M \subset F$ complementary to some c-lattice.

Remarks. (i) c-lattices need not form a base of the topology of F. However, by 2.12(ii) and 2.3(ii), they form a base Nisnevich locally.

(ii) If $L \supset L'$ are c-lattices then $L/L' \in \mathcal{V}_R^{\mathrm{f}}$.

(iii) Every d-lattice is a projective R-module. If $M \supset M'$ are d-lattices then $M/M' \in \mathcal{V}_R^{\mathrm{f}}$.

Let L, L' be c-lattices in a special Tate module F. Every splitting $F/L \hookrightarrow F$ yields a morphism $F/L \to F/L'$. The corresponding asymptotic morphism (see 2.4) is an isomorphism that does not depend on the choice of splitting. Therefore the objects $(F/L)^\infty \in \mathcal{V}_R^\infty$, L is any c-lattice in F, are canonically identified. They form a *canonically defined object* of the category $\mathcal{V}_R^\infty$ which we denote by F^∞. Our F^∞ depends on F in a functorial way. This construction is compatible with the base change.

For L, L' as above set

$$\det(L : L') := \det(F/L', F/L, id_{F^\infty}). \tag{2.13.1}$$

One has canonical composition isomorphisms of $\mathbb{Z}$-graded super lines $\det(L : L') \cdot \det(L' : L'') \xrightarrow{\sim} \det(L : L'')$, so we have defined a super pre-gerbe structure on the set of c-lattices. The corresponding $\mathbb{Z}$-graded super gerbe is called the *gerbe of c-lattices.*

For every $L \supset L'$ there is a canonical identification $\iota : \det(L : L') \xrightarrow{\sim} \det(L/L')$ such that for $L \supset L' \supset L''$ the composition becomes the standard isomorphism $\det(L/L') \cdot \det(L'/L'') \xrightarrow{\sim} \det(L/L'')$.

Remark. A super pre-gerbe structure on the set of c-lattices together with identifications ι and compaitibility with base change is unique up to a unique isomorphism (use Remark (i) above).

Consider the semigroup of endomorphisms g of F such that g^∞ is invertible. Pulling back the Tate super extension by the homomorphism $g \to g^\infty$ to $\mathrm{Aut}(F^\infty)$ one gets a super extension of our semigroup. In particular, we have a super extension $\mathrm{Aut}^\flat(F)$ of $\mathrm{Aut}(F)$.

Notice that if L is a c-lattice such that $L + g(F) = F$ then

$$\lambda_g := \lambda_{g^\infty} = \det(L : g^{-1}(L)). \tag{2.13.2}$$

If $g \in \mathrm{Aut}(F)$ then any L will do, and, acting by g on the r.h.s., we get a canonical isomorphism $\lambda_g = \det(g(L) : L)$.

Remark. The group $\mathrm{Aut}(F)$ acts on the set of c-lattices and on the datum of super lines $\det(L : L')$, hence on the gerbe of c-lattices. The above formula means that $\mathrm{Aut}^\flat(F)$ is the super extension defined by this action, see A6.

If F is an arbitrary Tate R-module then the object F^∞ is well-defined only after certain étale (or Nisnevich) localization of $\mathrm{Spec}\, R$ (see 2.12(ii)). Since super lines have étale local nature this suffices to define the Tate super extension of the above semigroup hence of $\mathrm{Aut}(F)$.

This construction is compatible with base change, so we have a group object $\mathrm{GL}(F)$ of $\mathcal{S}_R$, $\mathrm{GL}(F)(R') := \mathrm{Aut}(F_{R'})$, and its Tate super extension $\mathrm{GL}^\flat(F)$. Let $\mathfrak{gl}(F)$ be the Lie algebra of $\mathrm{GL}(F)$, so $\mathfrak{gl}(F)(R')$ is the Lie algebra of endomorphisms of the Tate R'-module $F_{R'}$; the Lie algebra $\mathfrak{gl}^\flat(F)$ of $\mathrm{GL}^\flat(F)$ is a central extension of $\mathfrak{gl}(F)$ by $\mathcal{O}$.

Let $L, M \subset F$ be, respectively, a c- and a d-lattice in a (special) Tate module F. Denote by $\mathrm{GL}(F, L)$, $\mathrm{GL}(F, M) \subset \mathrm{GL}(F)$ the parabolic subgroups of transformations preserving L, M. One has the *standard splittings*

$$s^c_L : \mathrm{GL}(F, L) \to \mathrm{GL}^\flat(F), \quad s^d_M : \mathrm{GL}(F, M) \to \mathrm{GL}^\flat(F), \tag{2.13.3}$$

defined as $s^c_L(g) := \det(F/L, g)$, $s^d_M(g') := \det(M, g')$, see (2.10.2). If $L \oplus M \xrightarrow{\sim} F$ then $s^c_L(g) = s^d_M(g)$ for $g \in \mathrm{GL}(F, L) \cap \mathrm{GL}(F, M)$.

The central extension $\mathfrak{gl}^\flat(F)$ can be described as follows. Set $\mathfrak{gl}_c(F) := \mathrm{Ker}(\mathfrak{gl}(F) \to \mathfrak{gl}(F^\infty))$ and let $\mathfrak{gl}_d(F) \subset \mathfrak{gl}(F)$ be the submodule of endomorphisms with open kernel. Both $\mathfrak{gl}_c(F), \mathfrak{gl}_d(F)$ are ideals in $\mathfrak{gl}(F)$,[43] and their sum equals $\mathfrak{gl}(F)$. Set $\mathfrak{gl}^\mathfrak{f}(F) := \mathfrak{gl}_c(F) \cap \mathfrak{gl}_d(F)$. There is a canonical Ad-invariant trace functional $tr : \mathfrak{gl}^\mathfrak{f}(F) \to \mathcal{O}$. Namely, to compute $tr(a)$ for $a \in \mathfrak{gl}^\mathfrak{f}(F)$ you find (after possible Zariski localization of R) c-lattices $L \subset L'$ such that $a(L) = 0$ and $a(F) \subset L'$; then $tr(a)$ is the trace of the induced endomorphism of L'/L.

[43]Moreover, they are stable with respect to the adjoint action of $\mathrm{GL}(F)$.

One has $\mathfrak{gl}(F^\infty) = \mathfrak{gl}(F)/\mathfrak{gl}_c(F) = \mathfrak{gl}_d(F)/\mathfrak{gl}^{\mathfrak{f}}(F)$. As follows from 2.10(v), the central extension $\mathfrak{gl}^\flat(F^\infty)$ is the push-out of $0 \to \mathfrak{gl}^{\mathfrak{f}}(F) \to \mathfrak{gl}_d(F) \to \mathfrak{gl}(F^\infty) \to 0$ by tr. Our $\mathfrak{gl}^\flat(F)$ is the pull-back of $\mathfrak{gl}^\flat(F^\infty)$ by the projection $\mathfrak{gl}(F) \to \mathfrak{gl}(F^\infty)$.

Here is a c-d-symmetric description. As follows from the above construction, $\mathfrak{gl}^\flat(F)$ admits canonical ad-invariant splittings s_c, s_d over the ideals $\mathfrak{gl}_c(F), \mathfrak{gl}_d(F) \subset \mathfrak{gl}(F)$ such that on $\mathfrak{gl}^{\mathfrak{f}}(F)$ one has $s_d - s_c = tr$. This structure determines $\mathfrak{gl}^\flat(F)$ uniquely.

One has a canonical bijection $L \mapsto L^\perp$ between the sets of c-lattices in F and its dual F^*. There is a canonical isomorphism of super lines

$$\det(L : L') \xrightarrow{\sim} \det(L^\perp : L'^\perp) \tag{2.13.4}$$

compatible with composition, i.e., the pre-gerbes of c-lattices in F and F^* are canonically identified. To define (2.13.4) it suffices, by Remark after (2.13.1), to consider the case $L \supset L'$. Here l.h.s. is $\det(L/L')$, r.h.s. is inverse to $\det((L/L')^*)$, and (2.13.4) is (2.2.3) for $M = (L/L')^*$. The compatibility with composition is immediate.

The isomorphism $\mathrm{GL}(F) \xrightarrow{\sim} \mathrm{GL}(F^*)$, $g \mapsto {}^t g^{-1}$, is compatible with the actions on the pre-gerbes of c-lattices. So, by Remark after (2.13.2), it lifts canonically to super extensions

$$\mathrm{GL}^\flat(F) \xrightarrow{\sim} \mathrm{GL}^\flat(F^*). \tag{2.13.5}$$

Notice that (2.13.5) changes the sign of the $\mathbb{Z}$-grading to the opposite. It interchanges the ideals $\mathfrak{gl}_c, \mathfrak{gl}_d \subset \mathfrak{gl}$ and the splittings s_c, s_d.

Remark. The above compatibility with duality can be extended to a larger semi-group of Fredholm endomorphisms as follows. We say that a morphism $g : N \to M$ of Tate R-modules is *Fredholm* if both $g^\infty : N^\infty \to M^\infty$, ${}^t g^\infty : M^{*\infty} \to N^{*\infty}$ are invertible. Equivalently, this means that (locally on $\operatorname{Spec} R$) one can find c-lattices $L \subset M, P \subset N$ such that $g(N) + L = M$, $P \cap \operatorname{Ker} g = 0$ and $g^{-1}(L) \subset N$, $g(P) \subset M$ are again c-lattices. Notice that any pair of isomorphisms $N^\infty \xrightarrow{\sim} M^\infty$, $M^{*\infty} \xrightarrow{\sim} N^{*\infty}$ can be lifted to a Fredholm morphism $N \to M$.

A Fredholm g defines a $\mathbb{Z}$-graded super line $\det(M, N, g)$. To define it choose L, P as above such that $L \supset g(P)$; then $\det(M, N, g) := \det(L/g(P))/\det(g^{-1}(L)/P)$. The arguments of Lemma-Definition in 2.7 show that our super line does not depend on the auxiliary choice of L, P and actually depends only on $(g^\infty, g^{*\infty})$. It is compatible with composition, and there is a canonical isomorphism $\det(M, N, g) = \det(N^*, M^*, {}^t g)^{-1}$ coming from (2.2.3). If g is an isomorphism then our super line is canonically trivialized by $\det g \in \det(M, N, g)$.

Let g be a Fredholm endomorphism of F; set $\det(F,g) := \det(F,F,g)$. There is a canonical isomorphism

$$\lambda_g \cdot \lambda_{{}^t g} = \det(F,g). \tag{2.13.6}$$

Namely, for c-lattices L, P as above one has $\lambda_g = \det(L : g^{-1}(L))$, $\lambda_{{}^t g} = \det(P^\perp : ({}^t g)^{-1}(P^\perp)) = \det(P : g(P))$ (see (2.13.2) and (2.13.4)), and (2.13.6) is the product of these isomorphisms. If g is invertible then the r.h.s. of (2.13.6) is trivialized by $\det g$, and (2.13.6) amounts to (2.13.5)

2.14. **Clifford modules.** Here is another description of the Tate extension which is inherently self-dual.

Fix a Tate module F. The Tate module $F \oplus F^*$ carries a standard hyperbolic symmetric bilinear form. Denote by $W = W_F$ the same Tate module considered as a super object placed in *odd* degree. The above form considered as a bilinear form on W is *skew*-symmetric; we denote it by $<,>$. So $\mathcal{C} := W \oplus R$ is a Lie super R-algebra with respect to a bracket whose only non-zero component is $<,>$.

A *Clifford module* for W is a $\mathcal{C}$-module[44] $\mathcal{Q}$ such that the $\mathcal{C}$-action is continuous with respect to the discrete topology of $\mathcal{Q}$ and $1 \in R = \mathcal{C}^0$ acts as id_Q. The category of Clifford modules is denoted by $\mathcal{M}^s_{\mathcal{C}}$. For morphisms of R's one has the obvious base change functors. It is clear that Clifford modules are local objects with respect to the flat topology of $\operatorname{Spec} R$.

For a super R-module M and a Clifford module $\mathcal{Q}$ the tensor product $M \otimes \mathcal{Q}$ is a Clifford module in the obvious way. A Clifford module $\mathcal{P}$ is *invertible* if the functor $\mathcal{M}^s_R \to \mathcal{M}^s_{\mathcal{C}}$, $M \mapsto M \otimes \mathcal{P}$, is an equivalence of categories. Denote by $\mathcal{P}ic^s_{\mathcal{C}}$ the groupoid of invertible $\mathcal{C}$-modules. For $\mathcal{L} \in \mathcal{P}ic^s_R$, $\mathcal{P} \in \mathcal{P}ic^s_{\mathcal{C}}$ one has $\mathcal{L} \otimes \mathcal{P} \in \mathcal{P}ic^s_{\mathcal{C}}$, so $\mathcal{P}ic^s_{\mathcal{C}}$ carries a canonical action of the Picard groupoid $\mathcal{P}ic^s_R$.

Assume that F is a special Tate R-module. Let $L_W \subset W$ be a c-lattice, $L_W^\perp \subset W$ its $<,>$-orthogonal complement; this is again a c-lattice. If $L_W \subset L_W^\perp$ then $<,>$ yields a non-degenerate form on $\bar{W} := L_W^\perp / L_W \in \mathcal{V}^{\mathfrak{f}}$. Let $\bar{\mathcal{C}} = \bar{W} \oplus R$ be the corresponding Clifford Lie super algebra. In other words, $L_W^\perp \oplus R$ is the normalizer of $L_W \subset \mathcal{C}$, and $\bar{\mathcal{C}}$ is the subquotient algebra of $\mathcal{C}$. Let $\mathcal{M}^s_{\bar{\mathcal{C}}}$ be the category of the corresponding Clifford modules.[45] Now the functor

$$\mathcal{M}^s_{\mathcal{C}} \to \mathcal{M}^s_{\bar{\mathcal{C}}}, \quad \mathcal{Q} \mapsto \bar{\mathcal{Q}} = \mathcal{Q}^{L_W}, \tag{2.14.1}$$

[44] in the tensor category $\mathcal{M}^s_R$ of super R-modules, see 2.1.

[45] i.e., an object of $\mathcal{M}^s_{\bar{\mathcal{C}}}$ is a $\bar{\mathcal{C}}$-module such that $1 \in R \subset \bar{\mathcal{C}}$ acts as identity.

is an equivalence of categories[46] (its inverse is the induction functor). In particular, if $L_W^\perp = L_W$ then $\bar{\mathcal{C}} = R$, hence $\mathcal{M}^s_{\mathcal{C}} \xrightarrow{\sim} \mathcal{M}^s_R$. This happens when $L_W = L \oplus L^\perp$ where $L \subset F$ is any c-lattice and $L^\perp := (F/L)^* \subset F^*$ is its orthogonal complement.

Equivalences (2.14.1) commute with functors $M \otimes \cdot$. The groupoid $\mathcal{P}ic^s_{\mathcal{C}}$ is a $\mathcal{P}ic^s_R$-torsor, i.e., a super $\mathcal{O}^\times$-gerbe. Indeed, for every L_W as above such that $L_W^\perp = L_W$ (2.14.1) identifies $\mathcal{P}ic^s_{\mathcal{C}}$ with $\mathcal{P}ic^s_R$.

Let $\mathrm{O}(W) \subset \mathrm{GL}(W)$ be the subgroup of automorphisms preserving $<,>$. It acts on $\mathcal{C}$ in the obvious way, hence $\mathrm{O}(W)$ acts on the category of Clifford modules. Explicitly, for $g \in \mathrm{O}(W)$ and $\mathcal{P} \in \mathcal{M}^s_{\mathcal{C}}$ the Clifford module $g\mathcal{P}$ equals $\mathcal{P}$ as a super R-module, and the $\mathcal{C}$-action on it is $w, p \mapsto g^{-1}(w)p$. This action obviously commutes with functors $M \otimes \cdot$. Therefore $\mathrm{O}(W)$ acts on $\mathcal{P}ic^s_{\mathcal{C}}$ as on a super $\mathcal{O}^\times$-gerbe. Let $\mathrm{O}^\flat(W)$ be the super $\mathcal{O}^\times$-extension *opposite* to the one defined by this action (see A6), or, equivalently, the super extension defined by the action on the opposite super gerbe. So for $g \in \mathrm{O}(W)$ its super line in $\mathrm{O}^\flat(W)$ is $\mathcal{P}/g\mathcal{P}$, $\mathcal{P} \in \mathcal{P}ic^s_{\mathcal{C}}$. Equivalently, $\mathrm{O}^\flat(W)$ consists of pairs $(g, g^\flat)$ where $g \in \mathrm{O}(W)$ and $g^\flat$ is an automorphism of $\mathcal{P}$, considered as a plain R-module, such that for $w \in W, p \in \mathcal{P}$ one has $g^\flat(wp) = g(w)g^\flat(p)$.

If F is an arbitrary (not necessary special) Tate module then we define $\mathrm{O}^\flat(W)$ using 2.12(ii).[47]

Remarks. (i) For L_W as above set $\mathrm{O}(W, L_W) := \mathrm{O}(W) \cap \mathrm{GL}(W, L_W)$; there is an obvious projection $\mathrm{O}(W, L_W) \to \mathrm{O}(\bar{W})$. Then (2.14.1) identifies the restriction of $\mathrm{O}^\flat(W)$ to $\mathrm{O}(W, L_W)$ with the pull-back of the extension $\mathrm{O}^\flat(\bar{W})$. In particular, if $L_W = L_W^\perp$ then there is a canonical splitting $s^c_{L_W} : \mathrm{O}(W, L_W) \to \mathrm{O}^\flat(W)$.

(ii) The above discussion for c-lattices has an immediate d-version. Namely, let $M_W \subset W$ be a d-lattice (see 2.13) such that $<,>$ vanishes on M_W. Then $M_W^\perp \subset W$ is a d-lattice and $\bar{W} := M_W^\perp / M_W \in \mathcal{V}^{\mathfrak{f}}$ carries the induced non-degenerate form; let $\bar{\mathcal{C}}$ be the corresponding Clifford Lie super algebra. Thus $M_W^\perp \oplus R$ is the normalizer of $M_W \subset \mathcal{C}$, and $\bar{\mathcal{C}}$ is the subquotient Lie algebra. We have an equivalence of categories

$$\mathcal{M}^s_{\mathcal{C}} \xrightarrow{\sim} \mathcal{M}^s_{\bar{\mathcal{C}}}, \quad \mathcal{Q} \mapsto \mathcal{Q}_{M_W}. \tag{2.14.2}$$

It identifies the restriction of $\mathrm{O}^\flat(W)$ to a parabolic subgroup $\mathrm{O}(W, M_W)$ $:= \mathrm{O}(W) \cap \mathrm{GL}(W, M_W)$ with the pull-back of the extension $\mathrm{O}^\flat(\bar{W})$ by

[46] If F is a finitely generated projective R-module then (2.14.1) is the usual Morita equivalence. The general case reduces to this one by 2.3(ii) (see Remark (i) in 2.13) since Clifford modules are Zariski local objects.

[47] As we did to define $\mathrm{GL}^\flat(F)$.

the projection $\mathrm{O}(W, M_W) \to \mathrm{O}(\bar{W})$. In particular, if $M_W = M_W^\perp$ then there is a canonical splitting $s^d_{M_W} : \mathrm{O}(W, M_W) \to \mathrm{O}^\flat(W)$.

2.15. **Proposition.** The embedding $\mathrm{GL}(F) \hookrightarrow \mathrm{O}(W)$ which assigns to $g \in \mathrm{GL}(F)$ the diagonal matrix g^o with components $g, {}^t g^{-1}$ lifts canonically to a morphism of super extensions

$$\mathrm{GL}^\flat(F) \hookrightarrow \mathrm{O}^\flat(W). \tag{2.15.1}$$

Proof. By Remark after (2.13.2), $\mathrm{GL}^\flat F)$ is the super extension defined by the action of $\mathrm{GL}(F)$ on the gerbe of c-lattices. The super extension induced from $\mathrm{O}^\flat(W)$ is defined by the action of $\mathrm{GL}(F)$ on the gerbe opposite to $\mathcal{P}ic^s_{\mathcal{C}}$. It remains to identify these two gerbes in a way compatible with the $\mathrm{GL}(F)$-actions.

By Remark in A6 we have to assign to every c-lattice L an invertible Clifford module $\mathcal{Q}_L$ and define identifications $\det(L : L') = \mathcal{Q}_{L'}/\mathcal{Q}_L$ which satisfy the transitivity property. By Remark after (2.13.1) it suffices to establish the latter identifications for $L \supset L'$.

We define $\mathcal{Q}_L$ as the Clifford module such that $(\mathcal{Q}_L)^{L_W} = R$ where $L_W := L \oplus L^\perp$ (see (2.14.1)). Then $\mathcal{Q}_{L'}/\mathcal{Q}_L = (\mathcal{Q}_{L'})^{L_W}$, so the promised identification can be rewritten as a canonical isomorphism

$$\det(L : L') = \mathcal{Q}^{L_W}/\mathcal{Q}^{L'_W} \tag{2.15.2}$$

valid for any $\mathcal{Q} \in \mathcal{P}ic^s_{\mathcal{C}}$. If $L \supset L'$ then the super lines $(\mathcal{Q}_{L'})^{L_W}, \mathcal{Q}^{L'_W}_{L'}$ lie in $\mathcal{Q}^P_{L'}$, where $P := L' \oplus L^\perp \subset W$, which is a module for the Clifford algebra $\bar{\mathcal{C}}$ for $\bar{W} = P^\perp/P = L/L' \oplus L'^\perp/L^\perp$. Now $\det(L : L') = \det(L/L')$ lies in $\bar{\mathcal{C}}$, and its action transforms $\mathcal{Q}^{L'_W}$ to $\mathcal{Q}^{L_W}$. This is (2.15.2). The transitivity property and compatibility with the $\mathrm{GL}(F)$-action are obvious. □

Remarks. (i) We see that $\mathrm{GL}^\flat(F)$ acts canonically on every Clifford module $\mathcal{P}$. The action of $\mathfrak{gl}^\flat(F)$ can be described explicitly as follows. For $a^\flat \in \mathfrak{gl}^\flat(F)$ over $a \in \mathfrak{gl}(F)$ its action on $\mathcal{P}$ is compatible with the action of $W \subset \mathcal{C}$: one has $a^\flat(wp) - w(a^\flat p) = a(w)p$ for $w \in W$, $p \in \mathcal{P}$. This condition alone does not determine the action of $a^\flat$; an extra normalization is needed. Since $\mathfrak{gl}(F) = \mathfrak{gl}_c(F) + \mathfrak{gl}_d(F)$[48] it suffices to consider the cases $a^\flat = s_c(a)$, $a \in \mathfrak{gl}_c(F)$, and $a^\flat = s_d(a)$, $a \in \mathfrak{gl}_d(F)$. The action of $s_c(a)$ is uniquely determined by property that it kills $\mathcal{P}^L$ for a sufficiently large c-lattice $L \subset F$; the action of $s_d(a)$ is uniquely determined by property that it kills $\mathcal{P}^{L^\perp}$ for a sufficiently small c-lattice $L \subset F$.

[48]See the discussion at the end of 2.13.

(ii) Let $L, M \subset F$ be a c- and d-lattice. Set $L_W := L \oplus L^\perp, M_W := M \oplus M^\perp \subset W$; these are c- and d-lattices in W such that $L_W^\perp = L_W$, $M_W^\perp = M_W$, and (2.15.1) identifies sections s_L^c, s_M^d of (2.13.3) with restrictions of sections $s^c_{L_W}$, $s^d_{M_W}$ from Remarks (i), (ii) of 2.14.

(iii) Recall that $\mathrm{GL}^\flat(F)$ is a $\mathbb{Z}$-*graded* super extension. This $\mathbb{Z}$-grading can be described in Clifford terms as follows. Consider a $\mathbb{Z}$-grading on W with F in degree 1 and F^* in degree -1; then $\mathrm{GL}(F) \subset \mathrm{O}(W)$ is the group of orthogonal automorphisms preserving this $\mathbb{Z}$-grading. One defines $\mathbb{Z}$-graded Clifford modules in the obvious way. The picture of 2.14 remains valid in the $\mathbb{Z}$-graded setting, so we have a $\mathbb{Z}$-graded super $\mathcal{O}^\times$-gerbe $\mathcal{P}ic_{\mathcal{C}}^{\mathbb{Z}}$ equipped with an action of $\mathrm{GL}(F)$ and the corresponding $\mathbb{Z}$-graded super extension of $\mathrm{GL}(F)$. This defines a canonical $\mathbb{Z}$-grading on the pull-back of $\mathrm{O}^\flat(W)$ to $\mathrm{GL}(F) \subset \mathrm{O}(W)$. It is easy to see that (2.15.1) is compatible with the $\mathbb{Z}$-gradings.

(iv) The Clifford algebras for F and F^* coincide, hence the corresponding groups $\mathrm{O}(W)$ and their super extensions $\mathrm{O}^\flat(W)$ are equal. On the subgroups $\mathrm{GL}(F)$, $\mathrm{GL}(F^*)$ this identification is the standard isomorphism $g \mapsto {}^t g^{-1}$; its lifting to the Tate super extensions via (2.15.1) is (2.13.5).

2.16. **Scalar products and super μ_2-torsors.** We suggest reading this subsection simultaneously with 3.6.

Let $\Phi = \Phi(F) \in \mathcal{S}_R$ be the object of all symmetric non-degenerate bilinear forms on F, or, equivalently, that of symmetric isomorphisms $\phi : F \xrightarrow{\sim} F^*$ (the form corresponding to ϕ is $(a,b)_\phi := (\phi a, b)$).

For $\phi \in \Phi$ let $\phi^o \in \mathrm{O}(W)$ be the anti-diagonal matrix with components ϕ, ϕ^{-1}. Let λ_ϕ be the super line of ϕ^o in $\mathrm{O}^\flat(W)$. One has $\phi^{o2} = 1$, so the corresponding identification $\lambda_\phi^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_\phi$ yields a super μ_2-torsor μ_ϕ (see 2.1). We have defined a canonical super μ_2-torsor μ_Φ on Φ.

For $\phi, \phi' \in \Phi(F)$ set $g_{\phi',\phi} := {\phi'}^{-1}\phi \in \mathrm{GL}(F)$. One has $\phi'^o = g^o_{\phi',\phi}\phi^o \in \mathrm{O}(W)$. By (2.15.1) this provides a canonical isomorphism of super lines

$$\lambda_{g_{\phi',\phi}} \cdot \lambda_\phi \xrightarrow{\sim} \lambda_{\phi'}. \tag{2.16.1}$$

Notice that $g_{\phi'',\phi} = g_{\phi'',\phi'} g_{\phi',\phi}$ and identifications (2.16.1) are transitive.

One can rephrase this as follows. Let $\mathcal{G}_\Phi$ be the simply transitive groupoid on Φ, so for every $\phi, \phi' \in \Phi$ there is a single arrow $\phi \to \phi'$ in $\mathcal{G}_\Phi$. We have a homomorphism of groupoids $g : \mathcal{G}_\Phi \to \mathrm{GL}(F)$ which sends the above arrow to $g_{\phi',\phi}$. Let $\mathcal{G}_\Phi^\flat$ be the g-pull-back of the Tate super extension. Now (2.16.1) is a $\mathcal{G}_\Phi^\flat$-action on λ_Φ, i.e., a λ_Φ-splitting of $\mathcal{G}_\Phi^\flat$ in the terminology of A2.

Therefore (see A2) $\mathcal{G}^\flat_\Phi$ comes from a μ_Φ-split super μ_2-extension $\mathcal{G}^\mu_\Phi$ of $\mathcal{G}_\Phi$. If $1/2 \in R$ then any μ_2-extension is étale, hence it is canonically trivialized over the formal completion $\hat{\mathcal{G}}_\Phi$ of $\mathcal{G}_\Phi$.[49] The corresponding trivialization of $\mathcal{G}^\flat_\Phi$ over $\hat{\mathcal{G}}_\Phi$ is a *canonical formal rigidification* of $\mathcal{G}^\flat_\Phi$.

Remarks. (i) The above constructions are compatible with direct sums: if $F = \oplus F_\alpha$, $\phi = \oplus \phi_\alpha$, then $\mu_\phi = \otimes \mu_{\phi_\alpha}$, and duality: $\mu_{\phi^{-1}} = \mu_\phi$.

(ii) Fix any $\phi \in \Phi(F)$. Then the morphism $g_\phi : \Phi(F) \to \mathrm{GL}(F)$, $\phi' \mapsto g_{\phi',\phi}$, is injective, and its image consists of all $g \in \mathrm{GL}(F)$ which are self-adjoint with respect to $(\,,\,)_\phi$. The involution $\mathfrak{s}_\phi := \mathrm{Ad}_{\phi^o}$ preserves $\mathrm{GL}(F), \Phi \subset \mathrm{O}(W)$, so it induces involutions of $\mathrm{GL}(F)$ and Φ which we denote again by $\mathfrak{s}_\phi$. For $g \in \mathrm{GL}(F)$, $\phi' \in \Phi$ one has $\mathfrak{s}_\phi(g) = \phi^{-1}({}^t g)^{-1}\phi$ (which is the adjunction with respect to $(\,,\,)$) and $\mathfrak{s}_\phi(\phi') = \phi\phi'^{-1}\phi$. Thus $g_\phi(\mathfrak{s}_\phi(\phi')) = (g_\phi(\phi'))^{-1}$. The action of $\mathfrak{s}_\phi$ on λ_Φ is compatible with the trivialization $\lambda_\Phi^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_\phi$, so $\mathfrak{s}_\phi$ acts on μ_Φ.

(iii) The action of $\mathfrak{s}_\phi$ on $\mathfrak{gl}(F)$ interchanges the ideals $\mathfrak{gl}_c(F), \mathfrak{gl}_d(F) \subset \mathfrak{gl}(F)$ and the sections s_c, s_d (see (2.13.5) and Remark (iv) in 2.15). Our g_ϕ identifies the tangent space to $\phi \in \Phi$ with the submodule $\mathfrak{gl}(F)^\phi \subset \mathfrak{gl}(F)$ of $(\,,\,)_\phi$-self-adjoint operators. If $1/2 \in R$ then the canonical formal rigidification of $\mathcal{G}^\flat_\phi$ yields a splitting $\nabla^\flat_\phi : \mathfrak{gl}(F)^\phi \to \mathfrak{gl}^\flat(F)$ which is $\mathfrak{s}_\phi$-invariant; it is uniquely defined by this property.

(iv) Let L, L_W, M, M_W be as in Remark (ii) of 2.15. Set

$$\text{(2.16.2)}\quad \Phi_L := \{\phi \in \Phi : \phi(L) = L^\perp\},\ \Phi_M := \{\phi \in \Phi : \phi(M) = M^\perp\}.$$

Since $\Phi^o_L \subset \mathrm{O}(F, L_W)$, $\Phi^o_M \subset \mathrm{O}(W, M_W)$ the sections $s^c_{L_W}$, $s^d_{M_W}$ from Remarks (i), (ii) of 2.14 trivialize the restrictions of our μ_2-torsor μ_Φ to Φ_{L_W} and Φ_{M_W}. We denote these trivializations also by s^c_L, s^d_M. Notice that for $\phi, \phi' \in \Phi_L$ the action (2.16.1) identifies $s^c_L(g_{\phi',\phi})s^c_L(\phi)$ with $s^c_L(\phi')$ (see Remark (ii) of 2.15). Same for L replaced by M.

2.17. Let us describe μ_ϕ explicitly assuming that R is a field.

(i) Case $p(\mu_\phi) = 0$.[50] Then one can find a c-lattice L such that the ϕ-orthogonal complement $L^\perp_\phi$ equals L, i.e., $\phi \in \Phi_L$. As above, it yields the trivialization $s^c_L(\phi)$ of μ_ϕ. If $F \neq 0$ and we live in characteristic $\neq 2$ then the Grassmannian of such L's has 2 components. We leave it to the reader to check that our trivialization changes sign as we switch the component. Thus μ_ϕ is the set of components considered as a μ_2-torsor.

(ii) Case $p(\mu_\phi) = 1$. Then one can find a c-lattice L such that $L^\perp_\phi \supset L$ and $\bar{L} := L^\perp_\phi / L$ has dimension 1. We have the induced $\bar\phi : \bar{L} \xrightarrow{\sim} \bar{L}^*$.

[49]For $\phi, \phi' \in \Phi(R')$ the arrow $\phi \to \phi'$ lies in $\hat{\mathcal{G}}_\Phi \subset \mathcal{G}_\Phi$ if and only ϕ equals ϕ' modulo some nilpotent ideal $I \subset R'$.

[50]Here p is the parity of our super torsor, see 2.1.

Let us define a canonical identification

$$\mu_\phi = \{\bar{l} \in \bar{L} : (\bar{l},\bar{l})_{\bar{\phi}} = 1\}. \tag{2.17.1}$$

Indeed, ϕ^o preserves $L_W := L \oplus \phi(L)$ and $\bar{W} := L_W^\perp / L_W = \bar{L} \oplus \bar{L}^*$, so, by Remark (i) in 2.14, we have $\mu_\phi = \mu_{\bar{\phi}}$. Take any invertible Clifford module $\bar{\mathcal{P}}$ for $\bar{W}$. An element $\sigma \in \mu_{\bar{\phi}}$ is an automorphism of $\bar{\mathcal{P}}$ as a k-vector space such that $\sigma^2 = 1$ and $\sigma(wp) = \bar{\phi}^o(w)\sigma(p)$ for $w \in \bar{W}$, $p \in \mathcal{P}$. The restriction of such σ to $\bar{\mathcal{P}}^{\bar{L}^*} \subset \bar{\mathcal{P}}$ is multiplication by $\bar{l} \in \bar{L}$ such that $(\bar{l},\bar{l})_{\bar{\phi}} = 1$. This provides (2.17.1).

Appendix

A1.[51] A *Picard groupoid* $\mathcal{P}$ is a symmetric monoidal category[52] such that every object in $\mathcal{P}$ is invertible, as well as every morphism. We denote the operation in $\mathcal{P}$ by $\cdot$ and the unit object by $1_{\mathcal{P}}$. The commutative group of isomorphism classes of objects in $\mathcal{P}$ is denoted by $\pi_0(\mathcal{P})$. Set $\pi_1(\mathcal{P}) := \mathrm{Aut}(1_{\mathcal{P}})$; this is a commutative group, and for every $P \in \mathcal{P}$ the identification $1 \cdot P = P$ yields a canonical isomorphism $\pi_1(\mathcal{P}) \xrightarrow{\sim} \mathrm{Aut}P$, $a \mapsto a \cdot id_P$.

For $P \in \mathcal{P}$ the corresponding element of $\pi_0(\mathcal{P})$ is denoted by $|P|$. For a finite family $\{F_\alpha\}$ of objects of $\mathcal{P}$ we usually denote their product by $\otimes F_\alpha$.

Remark. For any commutative group A the category A-*tors* of A-torsors is a Picard groupoid in the obvious way. For every Picard groupoid $\mathcal{P}$ its Picard subgroupoid $\mathcal{P}^0$ of objects isomorphic to $1_{\mathcal{P}}$ is canonically equivalent to $\mathcal{A}_{\mathcal{P}}$-*tors* via $\mathcal{P}^0 \xrightarrow{\sim} \pi_1(\mathcal{P})$-*tors*, $P \mapsto \mathrm{Hom}(1_{\mathcal{P}}, P)$.

For two Picard groupoids $\mathcal{P}$, $\mathcal{P}'$ a *morphism* $\phi : \mathcal{P} \to \mathcal{P}'$ is the same as a symmetric monoidal functor. All morphisms form a Picard groupoid $\mathrm{Hom}(\mathcal{P},\mathcal{P}')$: namely, the product of two morphisms ϕ, ψ is $(\phi\cdot\psi)(P) = \phi(P)\cdot\psi(P)$.

For $P \in \mathcal{P}$ its *inverse* P^{-1} is an object of $\mathcal{P}$ together with an identification $e : P \cdot P^{-1} \xrightarrow{\sim} 1_{\mathcal{P}}$; the inverse is determined uniquely up to a unique isomorphism. There are *two* natural identifications $P \xrightarrow{\sim} (P^{-1})^{-1}$ defined by pairings $e^c, e^m : P^{-1}\cdot P \xrightarrow{\sim} 1_{\mathcal{P}}$ where e^c is defined using commutativity, and e^m is determined by the condition that $e \cdot id_P = id_P \cdot e^m : P \cdot P^{-1} \cdot P \xrightarrow{\sim} P$ (or, equivalently, that $e^m \cdot id_{P^{-1}} = id_{P^{-1}} \cdot e : P^{-1}\cdot P \cdot P^{-1} \xrightarrow{\sim} P^{-1}$). Notice that the definition of e^m does *not* use commutativity of $\cdot$, and $e^m/e^c = \alpha$ where

[51] See SGA 4 XVIII 1.4.

[52] We always assume that $\mathcal{P}$ is equivalent to a small category.

$\alpha = \alpha(P) \in \pi_1(\mathcal{P})$ is the action of the commutativity symmetry on $P \cdot P$. *Unless stated explicitly otherwise, we use identification* e^c.

Using commutativity, one extends naturally the map $P \mapsto P^{-1}$ to an auto-equivalence of the Picard groupoid $\mathcal{P}$.

For $P, Q \in \mathcal{P}$ we write $P/Q := P \cdot Q^{-1}$.

A2. For a category Γ, a $\mathcal{P}$*-extension* $\Gamma^\flat$ is a rule that assigns to every arrow γ in Γ an object $P^\flat_\gamma \in \mathcal{P}$ and to every pair of composable arrows γ, γ' a composition isomorphism $c_{\gamma,\gamma'} : P^\flat_\gamma \cdot P^\flat_{\gamma'} \to P^\flat_{\gamma\gamma'}$. We demand that c be associative in the obvious sense.

Assume we have a $\mathcal{P}$-bundle $\mathcal{L}$ on the set of objects of Γ which is a rule which assigns to every object x in Γ an object $\mathcal{L}_x$ of $\mathcal{P}$. An *action* of $\Gamma^\flat$ on $\mathcal{L}$ is a rule that assigns to every arrow $\gamma : x \to x'$ an isomorphism $P^\flat_\gamma \cdot \mathcal{L}_x \overset{\sim}{\to} \mathcal{L}_{x'}$ which satisfies an obvious transitivity property.

Notice that for a given $\mathcal{L}$ there is a unique (up to a unique isomorphism) $\mathcal{P}$-extension $\Gamma^\flat_\mathcal{L}$ of Γ acting on $\mathcal{L}$. Indeed, the action amounts to a datum of isomorphisms $P^\flat_{\mathcal{L}\gamma} := \xi_{x'}/\xi_x$ which identifies the composition in $\Gamma^\flat$ with the obvious product. We call $\Gamma^\flat_\mathcal{L}$ the $\mathcal{L}$*-split* $\mathcal{P}$-extension. So for an arbitrary $\mathcal{P}$-extension $\Gamma^\flat$ of Γ we also refer to its action on $\mathcal{L}$ as an $\mathcal{L}$*-splitting* of $\Gamma^\flat$.

Remarks. (i) For every object x in Γ there is a canonical identification $e_x : P^\flat_{id_x} \overset{\sim}{\to} 1_\mathcal{P}$ defined by $c_{id_x, id_x} : P^\flat_{id_x} \cdot P^\flat_{id_x} \overset{\sim}{\to} P^\flat_{id_x}$. This isomorphism identifies every $c_{id_x,\gamma'}$ and c_{γ, id_x} with the canonical isomorphisms $1_\mathcal{P} \cdot P^\flat_{\gamma'} \overset{\sim}{\to} P^\flat_{\gamma'}$, $P^\flat_\gamma \cdot 1_\mathcal{P} \overset{\sim}{\to} P^\flat_\gamma$. For every invertible arrow $\gamma : x \to x'$ in Γ there is a canonical identification $P^\flat_{\gamma^{-1}} \overset{\sim}{\to} (P^\flat_\gamma)^{-1}$ defined by the isomorphism $e_\gamma := e_{x'} c_{\gamma,\gamma^{-1}} : P^\flat_\gamma \cdot P^\flat_{\gamma^{-1}} \overset{\sim}{\to} 1_\mathcal{P}$. Notice that $e_{\gamma^{-1}} = e^m_\gamma$.

(ii) A $\mathcal{P}$-extension $\Gamma^\flat$ yields a homomorphism[53] $\Gamma \to \pi_0(\mathcal{P})$, $\gamma \mapsto |P^\flat_\gamma|$. If $\mathcal{P}$ is discrete, i.e., $\mathcal{A}_\mathcal{P} = 1$, then $\mathcal{P}$-extensions are the same as homomorphisms $\Gamma \to \pi_0(\mathcal{P})$.

A3. Let $\Gamma^\flat$, $\Gamma^{\flat'}$ be two $\mathcal{P}$-extensions of Γ. A *morphism* $\theta : \Gamma^\flat \to \Gamma^{\flat'}$ is a system of morphisms $\theta_\gamma : P^\flat_\gamma \to P^{\flat'}_\gamma$ compatible with composition; it is clear how to compose morphisms of $\mathcal{P}$-extensions. The *Baer product* of $\Gamma^\flat$, $\Gamma^{\flat'}$ is a $\mathcal{P}$-extension $\gamma \mapsto P^\flat_\gamma \cdot P^{\flat'}_\gamma$ with the obvious composition rule; the Baer product is associative and commutative in the obvious way. Therefore for small Γ its $\mathcal{P}$-extensions form a Picard groupoid which we denote by $\mathcal{E}xt(\Gamma, \mathcal{P})$. Its unit object $1_\mathcal{E}$ is the *trivial split extension* $\gamma \mapsto 1_\mathcal{P}$. One has $\pi_1(\mathcal{E}xt(\Gamma, \mathcal{P})) = \mathrm{Hom}(\Gamma, \pi_1(\mathcal{P}))$.

---

[53]For a group A a homomorphism $a : \Gamma \to A$ assigns to every arrow γ an element $a(\gamma) \in A$ so that $a(\gamma\gamma') = a(\gamma)a(\gamma')$ for every composable γ, γ'.

For a functor $\phi : \Gamma' \to \Gamma$ and a $\mathcal{P}$-extension $\Gamma^\flat$ it is clear what is the pull-back of $\Gamma^\flat$ by ϕ. Thus for small Γ, Γ' we have a morphism of Picard groupoids $\phi^* : \mathcal{E}xt(\Gamma, \mathcal{P}) \to \mathcal{E}xt(\Gamma', \mathcal{P})$.

Remarks. (i) The notion of $\mathcal{P}$-extension of Γ depends on the *isomorphism* class of Γ, and *not* on its equivalence class: if $\phi : \Gamma \to \Gamma'$ is an equivalence then ϕ^* need not be an equivalence.[54]

(ii) The notion of $\mathcal{P}$-extension depends only on *monoidal* structure of $\mathcal{P}$: the commutativity constraint is irrelevant. However, the commutativity constraint is used in the definition of the Baer product.

(iii) Let $\Gamma^\flat$ be a $\mathcal{P}$-extension of a *groupoid* Γ. Then for $x \in \Gamma$ we have a $\mathcal{P}$-extension $\mathrm{Aut}^\flat(x)$ of the group $\mathrm{Aut}(x)$. It depends *functorially* on x: for every $\gamma : x \to x'$ the identification $Ad_\gamma : \mathrm{Aut}(x) \tilde{\to} \mathrm{Aut}(x')$ lifts canonically to an isomorphism of $\mathcal{P}$-extensions $\mathrm{Ad}^\flat_\gamma : \mathrm{Aut}^\flat(x) \tilde{\to} \mathrm{Aut}^\flat(x')$. Namely, for $g \in \mathrm{Aut}(x)$ the corresponding identification $\mathrm{Ad}^\flat_\gamma : P^\flat_g \tilde{\to} P^\flat_{\mathrm{Ad}_\gamma(g)}$ is the composition

$$P^\flat_g \tilde{\to} P^\flat_g \cdot (P^\flat_\gamma \cdot P^\flat_{\gamma^{-1}}) \tilde{\to} P^\flat_\gamma \cdot P^\flat_g \cdot P^\flat_{\gamma^{-1}} \tilde{\to} P^\flat_{Ad_\gamma(g)}.$$

Here the first arrow is $id_{P^\flat_g} \cdot (e_\gamma)^{-1}$, the second one is the commutativity constraint, the third is the composition map.[55] For composable γ, γ' one has $\mathrm{Ad}^\flat_{\gamma\gamma'} = \mathrm{Ad}^\flat_\gamma \mathrm{Ad}^\flat_{\gamma'}$.

A4. Let A be a commutative group. A *central A-extension* of Γ is a category $\tilde{\Gamma}$ together with a functor $\pi : \tilde{\Gamma} \to \Gamma$ and a homomorphism from A to the automorphisms of the identity functor of $\tilde{\Gamma}$. We demand that π is bijective on objects, for every x, x' the projection $\pi : \mathrm{Hom}_{\tilde{\Gamma}}(x, x') \to \mathrm{Hom}_\Gamma(x, x')$ is surjective, and the action of A $a(\tilde{\gamma}) = a_{x'}\tilde{\gamma} = \tilde{\gamma} a_x$ is simply transitive along the fibers of this projection. If Γ is a group (i.e., a groupoid with single object), then its central A-extensions coincide with central extensions of Γ by A.

Any $\tilde{\Gamma}$ as above yields an A-$tors$-extension $\Gamma^\flat$ of Γ: namely, $P^\flat_\gamma$ is the A-torsor $\pi^{-1}(\gamma)$, the composition c is the composition of arrows in $\tilde{\Gamma}$.

This way we see that A-$tors$-extensions are the same as central A-extensions.

A5. Suppose that Γ is a group; set $A := \pi_1(\mathcal{P})$. As we have seen in Remark (iii) of A3, the adjoint action of Γ on itself lifts canonically to a Γ-action $\mathrm{Ad}^\flat$ on any $\mathcal{P}$-extension $\Gamma^\flat$. If $\gamma, \gamma' \in \Gamma$ commute then

[54] Consider the case when Γ is a trivial groupoid and Γ' is an equivalent groupoid with 2 objects.

[55] The compatibility of $\mathrm{Ad}^\flat_\gamma$ is with product (composition) follows from the fact that the composition map $P^\flat_{\gamma^{-1}} \cdot P^\flat_\gamma \to 1_\mathcal{P}$ is equal to e^m_γ, see Remark (i) in A2.

$\mathrm{Ad}^\flat_\gamma : P^\flat_{\gamma'} \xrightarrow{\sim} P^\flat_{\gamma'}$ is multiplication by an element of $\mathcal{A}$ which we denote by $\{\gamma,\gamma'\}^\flat$. Equivalently, $\{\gamma,\gamma'\}^\flat \in A$ is the composition

$$P^\flat_\gamma \cdot P^\flat_{\gamma'} \overset{c_{\gamma,\gamma'}}{\to} P^\flat_{\gamma\gamma'} = P^\flat_{\gamma'\gamma} \overset{c^{-1}_{\gamma',\gamma}}{\to} P^\flat_{\gamma'} \cdot P^\flat_\gamma \xrightarrow{\sim} P^\flat_\gamma \cdot P^\flat_{\gamma'}$$

where the last arrow is the commutativity constraint. If Γ is commutative then $\{\ ,\ \}^\flat : \Gamma \times \Gamma \to \mathcal{A}$ is a bimultiplicative skew-symmetric pairing. If $\mathcal{P} = A\text{-}tors$ then $\{\ ,\ \}^\flat$ is the usual commutator pairing for the corresponding central A-extension.

A6. A $\mathcal{P}$-*action* on a category $\mathcal{C}$ is a functor $\cdot : \mathcal{P} \times \mathcal{C} \to \mathcal{C}$ equipped with an associativity constraint $P\cdot(P'\cdot C) \xrightarrow{\sim} (P\cdot P')\cdot C$ and $1_{\mathcal{P}}\cdot C \xrightarrow{\sim} C$ which satisfies the obvious compatibilities. We say that $\mathcal{C}$ is a $\mathcal{P}$-*torsor* if $\mathcal{C}$ is non-empty and for every $C \in \mathcal{C}$ the functor $\mathcal{P} \to \mathcal{C}$, $P \mapsto P \cdot C$, is an equivalence of categories. For a $\mathcal{P}$-torsor $\mathcal{C}$ one has a canonical functor $\mathcal{C}\times\mathcal{C} \to \mathcal{P}$, $C, C' \mapsto C/C'$, together with natural identifications $(C/C')\cdot C' = C$; such functor is unique.

Remark. Let $\mathcal{C}$ be a $\mathcal{P}$-torsor. We have then a $\mathcal{P}$-extension of the simply transitive groupoid on its set of objects: for $C', C \in \mathcal{C}$ the object $P^\flat_{C,C'} \in \mathcal{P}$ corresponding to the (unique) arrow $C' \to C$ is C/C', the composition is obvious. Conversely, suppose we have a set K together with a $\mathcal{P}$-extension $P^\flat_K$ of the simply transitive groupoid on K.[56] Then there is a $\mathcal{P}$-torsor $\mathcal{C}$ together with a map from K to the objects of $\mathcal{C}$ and its lifting to a morphism of $\mathcal{P}$-extensions. Such datum is unique in the obvious sense. We call $\mathcal{C}$ the $\mathcal{P}$-torsor generated by *pre* $\mathcal{P}$-*torsor* $(K, P^\flat_K)$.

Let Γ be a group. Assume that it acts on a category $\mathcal{C}$, i.e., we have a rule that assigns to every $\gamma \in \Gamma$ a functor $\gamma : \mathcal{C} \to \mathcal{C}$, $C \mapsto \gamma C$, together with natural isomorphisms $\gamma(\gamma' C) = (\gamma\gamma')C$, $1C = C$; this datum should satisfy the usual compatibilities. If $\mathcal{C}$ is also equipped with a $\mathcal{P}$-action then we say that the Γ and $\mathcal{P}$-actions *commute* if we are given a system of natural isomorphisms $\gamma(P\cdot C) = P\cdot(\gamma C)$ that satisfy the obvious compatibilities.

For example, assume that $\mathcal{P}$ acts on $\mathcal{C}$ and we have a $\mathcal{P}$-extension $\Gamma^\flat$ of Γ. Then $\gamma C := P^\flat_\gamma \cdot C$ is a Γ-action on $\mathcal{C}$ which commutes with the $\mathcal{P}$-action in the obvious way.

If $\mathcal{C}$ is a $\mathcal{P}$-torsor then every Γ-action on $\mathcal{C}$ commuting with the $\mathcal{P}$-action (we say then that Γ acts on $\mathcal{C}$ *as on a* $\mathcal{P}$-*torsor*) arises as above. Precisely, there is a unique, up to a unique isomorphism, $\Gamma^\flat$

[56] Which is a rule that assigns to $k, k' \in K$ an object $P^\flat_{k,k'} \in \mathcal{P}$ and composition maps $P^\flat_{k,k'} \cdot P^\flat_{k',k''} \to P^\flat_{k,k''}$ which satisfy the associativity property.

together with an identification of the corresponding action on $\mathcal{C}$ with our action which is compatible with the constraints. Namely, one has $P^\flat_\gamma = (\gamma C)/C$ (for various $C \in \mathcal{C}$ these objects are canonically identified in the obvious way) and the composition law is $P^\flat_\gamma \cdot P^\flat_{\gamma'} = (\gamma'(\gamma C)/\gamma C)\cdot(\gamma C/C) = \gamma'\gamma C/C = P^\flat_{\gamma\gamma'}$. We call $\Gamma^\flat$ the $\mathcal{P}$-extension defined by the action of Γ on $\mathcal{C}$.

For example, suppose we have a pre $\mathcal{P}$-torsor $(K, P^\flat_K)$ as in Remark above equipped with a Γ-action (so Γ acts on the simple transitive groupoid on K,[57] and this action is lifted to $P^\flat_K$). Then Γ acts on the $\mathcal{P}$-torsor $\mathcal{C}$ generated by $(K, P^\flat_K)$, which yields a $\mathcal{P}$-extension $\Gamma^\flat$.

3. The Heisenberg group and its cousins

The purpose of this section is to study various Heisenberg central extensions which arise by pullback from the Tate central extension in 2.10. In 3.5, the superline $\lambda_{\omega(F)^\times}$ on $\omega(F)^\times$ is constructed and the structure group is reduced to μ_2. In particular, this line has a canonical connection. For V a projective F-module of rank n, a connection on $\det_F V$ leads in 3.8 to an identification of formal groupschemes $\widehat{F}^{\times n\flat} \cong \widehat{F}^{\times\flat}_{(V)}$. For $\mathcal{E}$ any superline bundle on $\omega(F)^\times$ with an action of $\widehat{F}^{\times\flat}_{(V)}$, this isomorphism enables us to transfer the connection from $\lambda_{\omega(F)^\times}$ to $\mathcal{E}$.

3.1. **The setting.** Let R be a commutative ring and F_R a commutative topological R-algebra which is a Tate R-module (see 2.11). Let $F_R^\times$ be the multiplicative group of F_R. By base change we have a group valued functor $F^\times \in \mathcal{S}_R$, $F^\times(R') := F^\times_{R'}$. The Lie algebra of $F^\times$ is F: its R'-points is the additive group of $F_{R'}$.

Similarly, let $\mathrm{Aut}(F_R)$ be the group of continuous R-automorphisms of the algebra F; we have a group functor $\mathrm{Aut}(F) \in \mathcal{S}_R$, $\mathrm{Aut}(F)(R') := \mathrm{Aut}(F_{R'})$, which acts on $F^\times$. The R-points of the Lie algebra of $\mathrm{Aut}(F)$ is the Lie algebra $\Theta(F_R)$ of continuous R-derivations of F_R.

Let F^τ_R be our F_R considered as a Tate R-module, so we have a group functor $\mathrm{GL}(F^\tau)$, $R' \mapsto \mathrm{GL}(F^\tau_{R'})$ (see 2.13). One has the obvious embeddings

$$F^\times,\ \mathrm{Aut}(F) \hookrightarrow \mathrm{GL}(F^\tau). \tag{3.1.1}$$

They identify the action of $\mathrm{Aut}(F)$ on $F^\times$ with the adjoint action of $\mathrm{GL}(F^\tau)$.

The pull-back of the Tate super extension of $\mathrm{GL}(F^\tau)$ from 2.13 to $F^\times$ is the *Heisenberg super extension* $F^{\times\flat}$ of $F^\times$. Its Lie algebra is a central extension $F^\flat$ of F by $\mathcal{O}$. One deduces an explicit description of $F^\flat$ from

[57] In all examples we consider Γ actually acts on K as on a mere set.

2.10(iv). Notice that the action of $\mathrm{Aut}(F)$ on $F^\times$ lifts canonically to the (adjoint) action on $F^{\times\flat}$.

The Heisenberg super extension defines the commutator pairing[58]

$$\{\, ,\, \}^\flat : F^\times \times F^\times \to \mathcal{O}^\times \tag{3.1.2}$$

and its Lie algebra version

$$[\, ,\,]^\flat : F \times F \to \mathcal{O}. \tag{3.1.3}$$

So the adjoint action of $f \in F^\times$ on $F^{\times\flat}$ and $F^\flat$ is[59]

$$\mathrm{Ad}_f(g^\flat) = \{f,g\}^\flat g^\flat, \quad \mathrm{Ad}_f(a^\flat) = [f, f^{-1}a]^\flat + a^\flat. \tag{3.1.4}$$

Consider the Tate dual $F_R^* := \mathrm{Hom}_R^{cont}(F_R^\tau, R)$ as an F_R-module. As follows from (2.10.3) the R-linear morphism

$$d : F \to F^*, \quad da(b) := [a,b]^\flat, \tag{3.1.5}$$

is a continuous derivation. It yields a canonical F_R-linear morphism $\omega_{F_R} \to F_R^*$ where $\omega_{F_R} = \omega_{F_R/R}$ is the module of continuous differentials relative to R. Let $\mathrm{Res}^\flat : \omega_{F_R} \to R$ be its composition with evaluation at $1 \in F_R$, i.e., $\mathrm{Res}^\flat(bda) := [a,b]^\flat$. Thus the adjoint action of $F^\times$ on $F^\flat$ is given by a cocycle

$$F^\times \times F \to \mathcal{O}, \quad f, a \mapsto \mathrm{Res}^\flat(a\, d\log f). \tag{3.1.6}$$

Exercises. (i) Let $A \to R$ be a morphism of commutative algebras such that R is finite and flat over A. Then F_R is also a Tate A-module, so we have the two pairings $\{\, ,\, \}_R^\flat : F_R^\times \times F_R^\times \to R^\times$ and $\{\, ,\, \}_A^\flat : F_R^\times \times F_R^\times \to A^\times$. Show that $\{\, ,\, \}_A^\flat = Nm_{R/A}\{\, ,\, \}_R^\flat$.[60]

(ii) Assume that F is a product of finitely many algebras F_α as above. Show that for $f = (f_\alpha)$, $g = (g_\alpha) \in F^\times = \Pi F_\alpha^\times$ one has $\{f,g\}_F^\flat = \Pi\{f_\alpha, g_\alpha\}_{F_\alpha}^\flat$.

3.2. **The case of $F_R = R((t))$.** An important example of the above F is the algebra $F_R = R((t))$ of Laurent power series equipped with the usual topology.[61] Let us discuss it in more detail.

The language of ind-schemes is convenient here. For us, an *ind-scheme* is a functor X on the category of commutative rings which can be presented as a filtering inductive limit of functors $R \mapsto X_\alpha(R)$

[58] See 2.1, A5.

[59] The second formula comes from the description 2.10(v) of $F^\flat$: if $\tilde{a}$ is an operator with open kernel which represents $a^\flat$ then $\mathrm{Ad}_f(a^\flat)$ is represented by $f\tilde{a}f^{-1}$, so $\mathrm{Ad}_f(a^\flat) - a^\flat = tr(f\tilde{a}f^{-1} - \tilde{a}) = tr([f, \tilde{a}f^{-1}]) = [f, af^{-1}]^\flat = [f, f^{-1}a]^\flat$.

[60] Here $Nm_{R/A} : A^\times \to R^\times$ is the norm map.

[61] A base of neighbourhood of 0 is $t^n R[[t]]$, $n \geq 0$.

where X_α are schemes and transition maps $X_\alpha \to X_\beta$ are closed embeddings. We say that X is *ind-affine* if all X_α are affine schemes, X is *formally smooth* if the usual Grothendieck property is satisfied; X is *reduced* if one can choose X_α to be reduced schemes. To every X there corresponds the reduced ind-scheme $X_{\text{red}} := \varinjlim (X_\alpha)_{\text{red}}$.

For a commutative ring R set $F_R = R((t))$. It is well-known that the multiplicative group $F_R^\times$ is a direct product of the following four subgroups: (i) the subgroup $\mathbb{Z}_R$ of elements t^d; (ii) $R^\times$; (iii) the subgroup of elements $1 + a_1 t + a_2 t^2 + ..$, $a_i \in R$; and (iv) the subgroup of elements $1 + b_1 t^{-1} + b_2 t^{-2} + ..$ where b_i are *nilpotent* elements of R and almost all $b_i = 0$.

The functor $F^\times : R \mapsto F^\times(R) := F_R^\times$ is a commutative group ind-scheme which is ind-affine and formally smooth. The subgroup (i) is represented by the discrete group $\mathbb{Z}$, (ii) by $\mathbb{G}_m$, (iii) by the group scheme $\mathbb{W}$ of (big) Witt vectors, and (iv) by the formal completion $\hat{\mathbb{W}}$ of $\mathbb{W}$ (via the change of variable $t \mapsto t^{-1}$). Therefore

$$F^\times = \mathbb{Z} \times \mathbb{G}_m \times \mathbb{W} \times \hat{\mathbb{W}}. \tag{3.2.1}$$

One has $F_{\text{red}}^\times = \mathbb{Z} \times \mathbb{G}_m \times \mathbb{W}$. The projection $F^\times \to \mathbb{Z}$ has connected kernel; for $d \in \mathbb{Z}$ let $F^{\times d} \subset F^\times$ be the preimage of d.

The functor $\mathrm{Aut}(F) : R \mapsto \mathrm{Aut}(F)(R) := \mathrm{Aut}(F_R)$ is a group ind-scheme. The map $\mathrm{Aut}(F) \to F^{\times 1}, \phi \mapsto \phi(t)$, is an isomorphism of ind-schemes.

The next proposition shows that (3.1.2) coincides in the present situation with the inverse to a "parametric" version of the tame symbol introduced by Contou-Carrère [C] (see also [D3] 2.9); the usual tame symbol is its restriction to $F_{\text{red}}^\times$.

3.3. **Proposition.** (i) $\{\, , \}^\flat$ is $\mathrm{Aut}(F)$-invariant.

(ii) With respect to decomposition (3.2.1) the non-zero components of $\{\, , \}^\flat$ are:

(a) the $\mathbb{Z} \times \mathbb{Z}$-component: one has $\{t^m, t^n\}^\flat = (-1)^{mn}$.

(b) the $\mathbb{G}_m \times \mathbb{Z}$-component (and its transpose): $\{r, t^m\}^\flat = r^{-m}$.

(c) the $\mathbb{W} \times \hat{\mathbb{W}}$-component (and its transpose): for $f(t) \in \mathbb{W}$ and $g(t) \in \hat{\mathbb{W}}$ one has $\{f(t), g(t^{-1})\}^\flat = r(f * g)$. Here $*$ is the product of Witt vectors[62] which sends $\mathbb{W} \times \hat{\mathbb{W}}$ to $\hat{\mathbb{W}}$, and $r : \hat{\mathbb{W}} \to \mathbb{G}_m$ is

[62]Recall (see e.g. [Mu] Lecture 26) that $\mathbb{W}$ is naturally a commutative ring scheme; the ring structure is uniquely characterized by property that every map $w_i : \mathbb{W}(R) \to R$, $\Sigma w_i t^i := -t\partial_t \log f$, is a ring homomorphism.

the morphism $h(t) \mapsto h(1)^{-1}$.[63]

(iii) The morphism $\mathrm{Res}^\flat$ is the usual residue Res at $t = 0$, so the canonical morphism $\omega_F \to F^*$ is an isomorphism.

Proof. (i) is clear since $\mathrm{Aut}(F)$ acts on $F^{\times\flat}$.

We will prove (ii), (iii) simultaneously.

(a) follows from (2.1.1) since $\{f, f\}^{plain} = 1$ for any f, and (b) follows from Exercise (b) in 2.10(ii).

For the remainder of the proof, we assume we are working in the category of $\mathbb{Q}$-algebras. We can do this since it suffices to check our statements for the universal R; it has no torsion, so we loose no information tensoring by $\mathbb{Q}$.

Let us check that $\mathbb{Z} \times \mathbb{G}_m$ is orthogonal to $\mathbb{W} \times \hat{\mathbb{W}}$. Since we are in char 0 and $\mathbb{W} \times \hat{\mathbb{W}}$ is connected we can replace this group by its Lie algebra which is the subspace $F^0 \subset F$ of formal power series $\Sigma a_i t^i$ with $a_0 = 0$. The action of homotheties $\subset \mathrm{Aut}(F)$ via $t \mapsto at$ preserves F^0 and has trivial coinvariants there. Since homotheties fix $\mathbb{G}_m \subset \mathbb{Z} \times \mathbb{G}_m$ we see, by (i), that $\mathbb{G}_m$ is orthogonal to F^0. The action of homotheties on the quotient $\mathbb{Z} = \mathbb{Z} \times \mathbb{G}_m / \mathbb{G}_m$ is trivial, so the same argument shows that $\mathbb{Z}$ is also orthogonal to F^0, q.e.d.

It remains to compute the restriction of $\{\, ,\, \}^\flat$ to $\mathbb{W} \times \hat{\mathbb{W}}$. Again, it suffices to check the formulas on the level of Lie algebras.

The group pairing $f(t), g(t^{-1}) \mapsto r(f * g)$ from (c) looks as follows. For $f(t) = \exp(\alpha_1 t + \alpha_2 t^2 + ..)$, $g(t) = \exp(\beta_1 t + \beta_2 t^2 + ..)$ one has $(f * g)(t) = \exp(-\sum i\alpha_i\beta_i t^i)$, so $r(f * g) = \exp(\sum i\alpha_i\beta_i)$. It is clear that the corresponding infinitesimal pairing is $a, b \mapsto \mathrm{Res}(bda)$.

So it remains to check (iii), i.e., to show that $[t^m, t^n]^\flat = m\delta_{m,-n}$, which is an easy calculation (use presentation 2.10(iv) for $F^\flat$). □

Remarks. (i) Here is another convenient formula.[64] Let $f \in R[[t]]$ be a formal power series with invertible constant term, and $g \in R[t, t^{-1}]$ a Laurent polynomial with one of coefficients invertible and other nilpotent. Then $\mathrm{div} g$ is a Cartier divisor supported at $t = 0$ and, as follows from Exercise (b) in 2.10(ii), one has

$$\{f, g\}^\flat = \mathrm{N}_{\mathrm{div} g}(f)^{-1}. \tag{3.3.1}$$

(ii) According to [C] the pairing $\{\, ,\, \}^\flat$ is non-degenerate, i.e., it identifies the group ind-scheme $F^\times$ with its Cartier dual. For an analytic version of this statement see [D3], sect. 4.

[63]Here $\hat{\mathbb{W}}(R)$ is identified with units in $1 + tR[t]$, i.e., with polynomials of the form $1 + a_1 t + ...$ where a_i are nilpotent.

[64]We are grateful to referee for pointing it out to us.

3.4. **The standard splittings and the Weil reciprocity.** Let R, F_R be as in 3.1. Suppose we have R-subalgebras $O_R, A_R \subset F_R$ which are, respectively, c- and d-lattices in F (see 2.13). By base change we can consider it as an embedding $O, A \hookrightarrow F$ of $\mathcal{O}$-algebras in $\mathcal{S}_R$ (see 2.10). Then $O^\times, A^\times \subset F^\times$ and (2.13.3) gives the standard splittings

$$s^c_O : O^\times \to F^{\times\flat}, \quad s^d_A : A^\times \to F^{\times\flat}. \tag{3.4.1}$$

In particular, the restriction of $\{\,,\}^\flat$ to $O^\times$ and $A^\times$ vanishes.

This situation occurs in geometry as follows. Let X be a smooth projective family of curves over $\operatorname{Spec} R$, $D \subset X$ a relative divisor such that $U = X \smallsetminus D$ is affine. Let O_D be the algebra of functions on the formal completion of X at D and F_D its localization with respect to an equation of D.[65] Then F_D is a Tate R-module, and $O_D \subset F_D$ is a c-lattice. We have the obvious embedding of R-algebras $\mathcal{O}(U) \hookrightarrow F_D$ which identifies $\mathcal{O}(U)$ with a d-lattice in F_D. So we have the standard splittings

$$s^c_{O_D} : O_D^\times \to F_D^{\times\flat}, \quad s^d_{\mathcal{O}(U)} : \mathcal{O}(U)^\times \to F_D^{\times\flat}. \tag{3.4.2}$$

Let $\{\,,\}^\flat_D$ be the symbol map (3.1.2) for F_D. If D is a disjoint union of R-points x_α then $F_D = \Pi F_{x_\alpha}$ and 3.3[66] identifies $\{\,,\}^\flat_D$ with the product of (the inverse of) local tame symbols at x_α. If D is étale over $\operatorname{Spec} R$ then one computes $\{\,,\}^\flat_D$ using Exercise (i) in 3.1. So the vanishing of $\{\,,\}^\flat_D$ on $\mathcal{O}^\times(U) \subset F_D^\times$ is the classical Weil reciprocity law. Passing to Lie algebras one gets, by 3.3(iii), the residue formula.

3.5. $\omega(F)^\times$ **and the extended Heisenberg.** Let R, F_R as in 3.1 and assume that locally in flat topology of $\operatorname{Spec} R$ our F_R is isomorphic to a product of several copies of $R((t))$. All our constructions are local, so we tacitly assume that we work flat locally on $\operatorname{Spec} R$.

So, localizing R if needed, we get, by 3.3(iii), a canonical isomorphism $\omega(F) := \omega_F \xrightarrow{\sim} F^*$ of free F-modules of rank 1. Denote by $\omega(F)^\times$ the set of generators, i.e., invertible elements, of $\omega(F)$.

The story of 2.16 in our special setting looks as follows.

As in 2.14 set $W := F \oplus F^* = F \oplus \omega(F)$, so we have the orthogonal group $\mathrm{O} = \mathrm{O}(W)$ and its super extension $\mathrm{O}^\flat$. One has an embedding $F^\times \hookrightarrow \mathrm{O}$, $f \mapsto f^o :=$ the diagonal matrix with entries f, f^{-1}. By 2.15 the restriction of $\mathrm{O}^\flat$ to $F^\times$ is the Heisenberg super extension $F^\flat$.

[65]So $O_D := \varprojlim \Gamma(X, \mathcal{O}/\mathcal{O}(-nD))$, $F_D := \varinjlim \varprojlim \Gamma(X, \mathcal{O}(mD)/\mathcal{O}(-nD))$.

[66]Together with Exercise (ii) in 3.1.

Consider an embedding[67] $\omega(F)^\times \hookrightarrow \mathrm{O}(W)$ which identifies ν with the anti-diagonal matrix ν^o with entries ν, ν^{-1}. The disjoint union of $F^\times$ and $\omega(F)^\times$ is a subgroup $\tilde{F}^\times$ in $\mathrm{O}(W)$ which contains $F^\times$ as a normal subgroup of index 2; the multiplication table is

$$\nu^o \nu'^o = (\nu'/\nu)^o, \quad f^o \nu^o = (f^{-1}\nu)^o = \nu^o f^o. \tag{3.5.1}$$

Let $\tilde{F}^{\times\flat}$ be the super extension of $\tilde{F}^\times$ induced from $\mathrm{O}^\flat(W)$; for $g \in \tilde{F}^\times$ we denote the corresponding super line by λ_g. In particular, we have a super line $\lambda_{\omega(F)^\times}$ on $\omega(F)^\times$.

Below we consider $\omega(F)^\times$ as an $F^\times$-torsor with respect to the action $f, \nu \mapsto f^{-1}\nu$. The multiplication in $F^{\times\flat}$ lifts this action to an action of $F^{\times\flat}$ on $\lambda_{\omega(F)^\times}$.

Notice that over $\omega(F)^\times$ the tensor square of λ is canonically trivialized: one has $\lambda_{\nu^o}^{\otimes 2} = \mathcal{O}_\nu$ since $\nu^{o2} = 1$. Therefore we have defined a super μ_2-torsor $\mu_{\omega(F)^\times}$ on $\omega(F)^\times$.

The groupoid that corresponds to the $F^{\times\flat}$-action on $\omega(F)^\times$ is a super extension $\mathcal{G}^\flat_{\omega(F)^\times}$ of the simple transitive groupoid $\mathcal{G}_{\omega(F)^\times}$. Its action on $\lambda_{\omega(F)^\times}$ identifies $\mathcal{G}^\flat_{\omega(F)^\times}$ with the $\lambda_{\omega(F)^\times}$-split super extension (see A2). Since $\lambda_{\omega(F)^\times}$ comes from the super μ_2-torsor $\mu_{\omega(F)^\times}$ we see that $\mathcal{G}^\flat_{\omega(F)^\times}$ comes from the $\mu_{\omega(F)^\times}$-split super μ_2-extension $\mathcal{G}^\mu_{\omega(F)^\times}$. Explicitly, $\mathcal{G}^\mu_{\omega(F)^\times} \subset \mathcal{G}^\flat_{\omega(F)^\times}$ is the super μ_2-subextension whose action preserves $\mu_{\omega(F)^\times} \subset \lambda_{\omega(F)^\times}$.

If $1/2 \in R$ then $\mu_{\omega(F)^\times}$ is an étale torsor and $\mathcal{G}^\mu_{\omega(F)^\times}$ is an étale groupoid extension of $\mathcal{G}_{\omega(F)^\times}$. Therefore $\mathcal{G}^\mu_{\omega(F)^\times}$ splits canonically over the formal completion[68] $\hat{\mathcal{G}}_{\omega(F)^\times}$ of $\mathcal{G}_{\omega(F)^\times}$. The corresponding splitting of $\mathcal{G}^\flat_{\omega(F)^\times}$ is called *the canonical formal rigidification* of $\mathcal{G}^\flat_{\omega(F)^\times}$.

Remark. The canonical formal rigidification of $\mathcal{G}^\flat_{\omega(F)^\times}$ is completely determined by the connection on $\lambda_{\omega(F)^\times}$ defined by the μ_2-structure. The horizontal leaves of this connection are just the orbits of the *adjoint* action of $F^\times$ on $\lambda_{\omega(F)^\times} \subset \tilde{F}^{\times\flat}$.[69]

The above picture is compatible with the standard splittings of the Heisenberg group from 3.4. Namely, suppose we have $O, A \subset F$ as in 3.4. Set $O^\circ := O^\perp, A^\circ := A^\perp$. So $O^\circ \subset \omega(F) = F^*$ is an O-submodule which is a c-lattice, $A^\circ \subset \omega(F)$ is an A-submodule which is a d-lattice. Suppose that O° is generated, as an O-module, by some

[67] We identify $\nu \in \omega(F)^\times$ with an F-linear isomorphism $F \xrightarrow{\sim} \omega(F)$, $f \mapsto f\nu$.

[68] An R'-point of $\hat{\mathcal{G}}_{\omega(F)^\times}$ is a pair of R'-points of $\omega(F)^\times$ that coincide modulo a nilpotent ideal of R'.

[69] Notice that this action lifts the $F^\times$-action $f, \nu \mapsto f^{-2}\nu$ on $\omega(F)^\times$.

element of $\omega(F)^\times$, and similarly, A° is generated, as an A-module, by some element of $\omega(F)^\times$. Let $O^{\circ\times}, A^{\circ\times} \subset \omega(F)^\times$ be the sets of such generators. Then $\tilde{O}^\times := O^\times \sqcup O^{\circ\times}$, $\tilde{A}^\times := A^\times \sqcup A^{\circ\times}$ are subgroups of $\tilde{F}^\times$. Since $\tilde{O}^\times \subset \mathrm{O}(W, O_W)$, $\tilde{A}^\times \subset \mathrm{O}(W, A_W)$ (see Remark (ii) in 2.15 and Remarks (i), (ii) in 2.14 for notation) we have the standard sections

$$s^c_{O_W} : \tilde{O}^\times \to \tilde{F}^{\times\flat}, \quad s^d_{A_W} : \tilde{A}^\times \to \tilde{F}^{\times\flat}. \tag{3.5.2}$$

On $O^\times, A^\times \subset F^\times_\omega$ these are splittings s^c_O, s^d_A from (3.4.1) (see Remark (ii) in 2.15). The restriction of (3.5.2) to $O^{\circ\times}, A^{\circ\times} \subset \omega(F)^\times$ trivialize the restrictions to $O^{\circ\times}$, $A^{\circ\times}$ of the μ_2-torsor $\mu_{\omega(F)^\times}$. The corresponding splittings of the groupoid super extensions $\mathcal{G}^\flat_{O^{\circ\times}}$, $\mathcal{G}^\flat_{A^{\circ\times}}$ come from sections s^c_O, s^d_A.

Remark. With $\omega(F)$ replaced by F^* the above constructions remain valid for every F such that F^* is a free F-module of rank 1. E.g. one can take for F an algebra F_D from 3.4; the divisor D need not be étale over $\operatorname{Spec} R$.

3.6. **Remarks.** We assume that $1/2 \in R$.

(i) Take any $\nu \in \omega(F)^\times$. Then the canonical formal rigidification of $\mathcal{G}^\flat_{\omega(F)^\times}$ provides a section $\nabla^\flat_\nu : F \to F^\flat_\omega$. As folows from the first Remark in 3.5 one has $\nabla^\flat_{f^{-2}\nu} = \mathrm{Ad}_f \nabla^\flat_\nu$, i.e., by (3.1.6),

$$\nabla^\flat_{f\nu}(a) = \nabla^\flat_\nu(a) - \frac{1}{2}\mathrm{Res}(a d\log f). \tag{3.6.1}$$

Therefore the map $\nu \mapsto \nabla^\flat_\nu$ identifies the $\omega(F) = F^*$-torsor of sections $F \to F^\flat$ with the push-out of the $F^\times$-torsor $\omega(F)^\times$ by the homomorphism $-\frac{1}{2}d\log : F^\times \to \omega(F)$.

(ii) According to Remark (iii) in 2.16 the above $\nabla^\flat_\nu$ can be also described as follows. Consider the involution $\mathfrak{s}_\nu :=\mathrm{Ad}_{\nu^o}$ of $\mathrm{O}(W)$ and $\mathrm{O}^\flat(W)$. It preserves $F^\flat \subset \mathfrak{o}^\flat(W)$ and acts on $F \subset \mathfrak{o}(W)$ as multiplication by -1. Our $\nabla^\flat_\nu$ is an $\mathfrak{s}_\nu$-invariant splitting $F \to F^\flat$, and it is uniquely characterized by this property.[70]

(iii) Set $O := R[[t]] \subset F = R((t))$. Consider the splitting $s^c_O : O \to F^\flat$ from (3.4.1). Then for every $a \in O$ and $f \in F^\times$ one has

$$\nabla^\flat_{fdt}(a) = s^c_O(a) - \frac{1}{2}\mathrm{Res}(a d\log f). \tag{3.6.2}$$

[70] Indeed, if there were two such, the difference would be a map $F \to \mathcal{O}$ which is $\mathfrak{s}_\nu$-equivariant, where $\mathfrak{s}_\nu$ acts by -1 on F and by $+1$ on $\mathcal{O}$.

Indeed, $\nabla^{\flat}_{dt}(a) = s^c_O(a)$ by the discussion at the end of 3.5; the general case follows from (3.6.1).

(iv) Let us describe the super μ_2-torsor $\mu_{\omega(F)^\times}$ for $F = R((t))$ explicitly. We can assume that R is reduced. On even connected components (those of the forms $t^{2n}dt$) our μ is even, and it can be trivialized choosing $L = t^{-n}R[[t]]$ as in 2.17(i). On odd components (those of the forms $t^{2n+1}dt$) our μ is odd. For a form $\nu = rt^{2n+1}dt + ..$ our μ_ν is the μ_2-torsor of square roots of r as seen from (2.17.1) applied to $L = t^{-n}R[[t]]$.

We see that the restriction of $\mathcal{G}^{\mu}_{\omega(F)^\times}$ to even components is canonically trivialized. Therefore the restriction of $\mathcal{G}^{\flat}_{\omega(F)^\times}$ to even components is also canonically trivialized. The restriction of $\mathcal{G}^{\mu}_{\omega(F)^\times}$ to odd components has non-trivial monodromy.

3.7. **Some twists.** Let R be a commutative algebra, F_R a topological R-algebra. Assume that F_R is isomorphic to $R((t))$, so $\omega(F) = F^*$. Let V_R be a finitely generated projective F_R-module. Then V_R is a Tate R-module with respect to a natural topology whose base is formed by sub R-modules $\Sigma I \cdot v_\alpha$ where $\{v_\alpha\}$ is a finite set of F_R-generators of V, $I \subset F_R$ an open R-submodule.[71]

Consider the action of $F^\times$ on V by homotheties $F^\times \to \mathrm{GL}(V)$. Let $F^{\times\flat}_{(V)}$ be the pull-back of the Tate super extension $\mathrm{GL}(V)^\flat$. Its Lie algebra $F^\flat_{(V)}$ is a central extension of F by $\mathcal{O}$.

Remark. Super $\mathcal{O}^\times$-extensions of $F^\times$ form a Picard groupoid with respect to Baer product (see A3). As follows from 2.10(iii), $V \mapsto F^{\times\flat}_{(V)}$ is a symmetric monoidal functor (with respect to direct sum of V's).

The following proposition-construction is crucial for the definition of ε-connection. The reader who is willing to admit that V is a free F-module can use instead a much shorter equivalent construction from 3.9(i) below.

Denote by n the rank of V_R (as of an F_R-module). We can consider n as a locally constant function on $\operatorname{Spec} R$. Let $F^{n\flat}$ be the Baer n-multiple of the extension $F^\flat$.

3.8. **Proposition.** Every R-relative connection $\mathfrak{d}$ on the F_R-line $\det_{F_R} V_R$ yields a canonical isomorphism of the Lie algebra extensions

$$\xi^{\mathfrak{d}} : F^{n\flat} \xrightarrow{\sim} F^\flat_{(V)} \tag{3.8.1}$$

such that for $\eta \in \omega(F)$ and $a^\flat \in F^{n\flat}$ lifting $a \in F$ one has

$$\xi^{\mathfrak{d}+\eta}(a^\flat) - \xi^{\mathfrak{d}}(a^\flat) = -\mathrm{Res}(a\eta) \in \mathcal{O} \subset F^\flat_{(V)}. \tag{3.8.2}$$

[71] We already met this topology in Example (ii) of 2.11.

Proof. (a) It is convenient to use a slightly different format. Namely, for every R-relative connection ∇ on V we will define a canonical isomorphism of the Lie algebra extensions

$$\xi^{\nabla} : F^{n\flat} \xrightarrow{\sim} F^{\flat}_{(V)} \tag{3.8.3}$$

such that for any $\rho \in \omega(F) \otimes_F \mathrm{End}_F(V)$ one has

$$\xi^{\nabla+\rho}(a^{\flat}) = \xi^{\nabla}(a^{\flat}) - \mathrm{Res}\,\mathrm{tr}(a\rho). \tag{3.8.4}$$

Such ξ^{∇} amounts to a datum of (3.8.1) subject to (3.8.2). Namely, $\xi^{\mathfrak{d}}$ that corresponds to ξ^{∇} is determined by the condition $\xi^{\mathrm{tr}\nabla} = \xi^{\nabla}$ where $\mathrm{tr}\nabla$ is the connection on $\det_F V$ defined by ∇.[72]

(b) To construct ξ^{∇} we use a "geometric" interpretation of $F^{\flat}_{(V)}$ from [BS] which we briefly recall now. The picture below is compatible with base change, so we skip R from the notation (replacing it by $\mathcal{O}$ when necessary, see 2.10).

Set $V' := \mathrm{Hom}_F(V, F)$, $V^{\circ} := \mathrm{Hom}_F(V, \omega(F)) = \omega(F) \otimes_F V'$. These are finitely generated projective F-modules. As a Tate $\mathcal{O}$-module V° is canonically isomorphic to the dual Tate $\mathcal{O}$-module to V: the pairing $V^{\circ} \times V \to \mathcal{O}$ is $v^{\circ}, v \mapsto \mathrm{Res}(v^{\circ}(v))$. One has a usual identification $V \otimes_F V' = \mathrm{End}_F V$.

The diagonal embedding $\Delta : \mathrm{Spec}\, F \hookrightarrow \mathrm{Spec}\,(F \hat{\otimes}_{\mathcal{O}} F)$ is a Cartier divisor.[73] Consider an exact sequence of $F \hat{\otimes}_{\mathcal{O}} F$-modules

$$0 \to V \hat{\otimes}_{\mathcal{O}} V^{\circ} \to V \hat{\otimes}_{\mathcal{O}} V^{\circ}(\Delta) \to \mathrm{End}_F V \to 0 \tag{3.8.5}$$

where the right arrow is the residue around the diagonal along the second variable. Let

$$0 \to \mathcal{O} \to F^{\flat'}_{(V)} \to F \to 0 \tag{3.8.6}$$

be its pull-back by $F \hookrightarrow \mathrm{End}_F V$, $f \mapsto f \cdot id_V$, and push-forward by $V \hat{\otimes}_{\mathcal{O}} V^{\circ} \to \mathcal{O}$, $v \otimes v^{\circ} \mapsto -\mathrm{Res}(v^{\circ}(v))$. Notice that $F^{\flat'}_{(V)}$ inherits from $V \hat{\otimes}_R V^{\circ}(\Delta)$ an F-bimodule structure. It defines a Lie bracket on $F^{\flat'}_{(V)}$ with the adjoint action of $f^{\flat'}$ coming from the commutator with f with respect to the F-bimodule structure. So $F^{\flat'}_{(V)}$ is a central Lie algebra extension of the commutative Lie algebra F.

Now there is a canonical isomorphisms of central $\mathcal{O}$-extensions

$$F^{\flat'}_{(V)} \xrightarrow{\sim} F^{\flat}_{(V)}. \tag{3.8.7}$$

---

[72] Notice that ∇ form an $\omega(F) \otimes_F \mathrm{End}_F V$-torsor $\mathcal{C}onn(V)$, and $\nabla \mapsto \mathrm{tr}\nabla$ identifies the $\omega(F)$-torsor of $\mathfrak{d}$'s with the $\omega(F)$-torsor induced from $\mathcal{C}onn(V)$ by the map $\mathrm{id}_{\omega(F)} \otimes \mathrm{tr}\colon \omega(F) \otimes_F \mathrm{End}_F V \to \omega(F)$.

[73] For $F_R \xrightarrow{\sim} R((t))$ one has $F \hat{\otimes}_R F \xrightarrow{\sim} R[[t_1, t_2]][t_1^{-1}, t_2^{-1}]$.

Namely, for $f^{\flat'} \in F^{\flat'}_{(V)}$ the corresponding $f^\flat \in F^\flat_{(V)}$ is defined as follows. Choose $k = k(t_1,t_2)dt_2 \in V \boxtimes V^\circ(\Delta)$ that lifts $f^{\flat'}$. It defines an "integral oprator"[74]

$$A_k : V \to V, \quad A_k(v)(t_1) := -\mathrm{Res}_{t_2=0} k(t_1,t_2)v(t_2)dt_2. \tag{3.8.8}$$

It is easy to see that $A_k \in \mathfrak{gl}_d(V)$ and $A_k^\infty = f^\infty$. As was explained in the end of 2.13, such A_k defines a lifting of f to $F^\flat_{(V)}$; this is our $f^\flat$. The independence of the auxiliary choices is immediate.

(c) Now we are ready to define (3.8.3) . Let $F^{\flat'}$ be the central extension (3.8.6) for $V = F$. Its Baer n-multiple $F^{n\flat'}$ is the push-forward of the exact sequence $0 \to F\hat{\otimes}_{\mathcal{O}}\omega(F) \to F\hat{\otimes}_{\mathcal{O}}\omega(F)(\Delta) \to F \to 0$ by the map $F\hat{\otimes}_{\mathcal{O}}\omega(F) \to \mathcal{O}$, $f \otimes \nu \mapsto -n\mathrm{Res}(f\nu)$. According to (3.8.7) we can rewrite (3.8.3) as an isomorphism $\xi^\nabla : F^{n\flat'} \xrightarrow{\sim} F^{\flat'}_{(V)}$.

Our ξ^∇ will come from certain morphism of $F\hat{\otimes}_{\mathcal{O}}F$-modules Ξ^∇ : $F\hat{\otimes}_{\mathcal{O}}\omega(F)(\Delta) \to V\hat{\otimes}_{\mathcal{O}}V^\circ(\Delta)$. Since $F\hat{\otimes}_{\mathcal{O}}\omega(F)(\Delta)$ is a free $F\hat{\otimes}_{\mathcal{O}}F$-module of rank 1, such Ξ^∇ is multiplication by a section $\gamma^\nabla \in V\hat{\otimes}_{\mathcal{O}}V'$. This γ^∇ must satisfy the condition $\gamma^\nabla|_\Delta = id_V \in V \otimes_F V' = \mathrm{End}_F V$ in order to assure that ξ^∇ is well-defined. Notice that ξ^∇ depends only on the restriction of γ^∇ to the first infinitesimal neighbourhood $\Delta^{(1)}$ of Δ.

Let ∇' be the connection on V' dual to ∇. Denote by $\tilde{\nabla}$ the $\mathcal{O}$-relative connection on the $F\hat{\otimes}_{\mathcal{O}}F$-module $V\hat{\otimes}_{\mathcal{O}}V'$ defined by ∇ and ∇'. The restriction of the relative Kähler differentials of $F\hat{\otimes}_{\mathcal{O}}F$ to Δ equals $\omega(F) \oplus \omega(F)$, so the composition of $\tilde{\nabla}$ with the restriction to Δ is a morphism $\bar{\nabla} : V\hat{\otimes}_{\mathcal{O}}V' \to (\omega(F) \oplus \omega(F)) \otimes_F V \otimes_F V'$.

Now our $\gamma^\nabla \in V\hat{\otimes}_{\mathcal{O}}V'$ is any section killed by $\bar{\nabla}$ whose restriction to Δ equals id_V. Such γ^∇ exists since id_V is a horizontal section of $\mathrm{End}_F V = V \otimes_F V'$, and its restriction to $\Delta^{(1)}$ is uniquely defined. Thus we defined ξ^∇. Formula (3.8.4) holds since for $\rho = \rho(t)dt$ one has $\gamma^{\nabla+\rho} = \gamma^\nabla + \rho(t_1)(t_2 - t_1)$ on $\Delta^{(1)}$. □

3.9. **Remarks.** (i) Isomorphisms (3.8.1) and (3.8.3) are compatible with direct sums of V's. Precisely, both $F^{n\flat}$ and $F^\flat_{(V)}$ transform direct sum of V's into the Baer sum of extensions (this is true on the group level; see 2.10(iii) for $F^{\times\flat}_{(V)}$). With respect to these identifications $\xi^{\oplus\nabla_\alpha}$ equals the Baer sum of ξ^{∇_α}.

Therefore if V is a *free* F-module then isomorphisms (3.8.1) or (3.8.3) can be described directly as follows. As we mentioned above, 2.10(iii)

[74] Here $k(t_1,t_2)v(t_2)dt_2 \in V \boxtimes \omega(\Delta)$.

yields a canonical identification of super extensions $F^{\times\flat}_{(F^n)} = F^{\times n\flat}$.[75] So every F-basis $\phi : F^n \xrightarrow{\sim} V$ yields an isomorphism

$$\xi^{\phi} : F^{\times n\flat} \xrightarrow{\sim} F^{\times\flat}_{(V)}. \tag{3.9.1}$$

The corresponding isomorphism of Lie algebras is ξ^{∇_ϕ} of (3.8.3) where ∇_ϕ is the connection that corresponds to the trivialization ϕ. You define then ξ^{∇} for arbitrary ∇ using ξ^{∇_ϕ} and (3.8.4).

(ii) Isomorphisms ξ^ϕ of (3.9.1) satisfy the property

$$\xi^{\phi g}(f^\flat) = \xi^{\phi}(f^\flat)\{\det g, f\}^\flat. \tag{3.9.2}$$

Here $f \in F^\times$, $f^\flat \in F^{\times n\flat}$ is a lifting of f, $g \in \mathrm{GL}(n, F)$ is an invertible matrix, $\det g \in F^\times$ its determinant, and $\{\,,\,\}^\flat$ is (3.1.2) . Indeed, we can rewrite our statement as the equality $\{g, f_V\}^\flat = \{\det g, f\}^\flat$ where the l.h.s. is the Tate commutator pairing in $\mathrm{GL}(n, F) \subset \mathrm{GL}_R(F^n)$ (and f there means $f \cdot id_{F^n}$) . This property is clear for the diagonal and unipotent matrices g from 2.10(iii),(iv). This proves our statement in case when R is a local Artinian ring (then $F = R((t))$ is also local Artinian, so every g is a product of unipotent and diagonalizable matrices). The general case reduces to that one as follows.

Notice that the pairing $\mathrm{GL}_n(F) \times F^\times \to R^\times$, $g, f \mapsto \{g, f\}^\flat$, is continuous. Precisely, for every g, f there exists $N \gg 0$ such that for every $\beta \in t^N \mathrm{Mat}_n(R[[t]])$, $\alpha \in t^N R[[t]]$ one has $g+\beta \in \mathrm{GL}_n(F)$, $f+\alpha \in F^\times$, and $\{g, f\}^\flat = \{g+\beta, f+\alpha\}^\flat$. Same is true for $\{\det g, f\}^\flat$. So we can assume that all but finitely many coefficients of f and entries of g are non-zero. Replacing R by its subring generated by these coefficients and the inverses of the top non-nilpotent coefficients of $\det g$ and f we can assume that R is finitely generated over $\mathbb{Z}$, hence R is Noetherian. We want to prove that two elements of $R^\times$ are equal. It suffices to check this on every infinitesimal neighbourhood of every point of $\operatorname{Spec} R$, and we are done.

(iii) The involution $\mathfrak{s}_\nu$ from 3.6(ii) can be easily described in terms of the identification $F^\flat = F^{\flat'}$ (case $V = F$ of (3.8.7)). Namely, for $\nu = \nu(t)dt$ $\mathfrak{s}_\nu$ comes from the involution of $F\hat{\otimes}_R\omega(F)(\Delta)$ which sends $f(t_1, t_2)\nu(t_2)dt_2$ to $f(t_2, t_1)\nu(t_2)dt_2$. Therefore the $\mathfrak{s}_\nu$-fixed section $\nabla^\flat_\nu : F \to F^{\flat'}$ from 3.6(i) is

$$f(t) \mapsto \frac{f(t_1)dt_2}{t_2 - t_1} + \frac{1}{2}(f'(t_1) + f(t_1)\frac{\nu'(t_1)}{\nu(t_1)})dt_2. \tag{3.9.3}$$

[75] The latter extension is the Baer n-multiple of $F^{\times\flat}$.

3.10. **The canonical flat connections.** Let us rephrase the material of 3.5 and 3.8 in a format to be used in sect. 4.

Our input is a triple (R, F, V) where R is a $\mathbb{Q}$-algebra, $F = F_R$ a topological R-algebra, $V = V(F)$ a finitely generated projective F-module. We assume that locally in the flat topology of $\operatorname{Spec} R$ our F_R becomes a finite direct product ΠF_i where every F_i is isomorphic to $R((t))$. All our constructions are $\operatorname{Spec} R$-local, so we can assume that the above decomposition holds on $\operatorname{Spec} R$ itself. Then $V = \Pi V_i$ where V_i is a finitely generated projective F_i-module.

By 3.3(iii) one has a canonical identification $F^* = \omega(F) = \Pi\omega(F_i)$. Recall that $F^\times = \Pi F_i^\times$ is a formally smooth group ind-scheme. We have an $F^\times$-torsor $F^{*\times} = \omega(F)^\times = \Pi\omega(F_i)^\times$.

As in 3.7 we notice that V is a Tate R-module in a natural way, so the $F^\times$-action on V by homotheties $F^\times \hookrightarrow \mathrm{GL}(V)$ defines[76] the super extension $F^{\times\flat}_{(V)}$ of $F^\times$. The embeddings $F_i^\times \hookrightarrow F^\times$ lift canonically to mutually commuting embeddings $F^{\times\flat}_{i(V_i)} \hookrightarrow F^{\times\flat}_{(V)}$ (see 2.10(iii)). For $f \in F^\times$ we denote its super line in $F^{\times\flat}_{(V)}$ by $\lambda_f^{(V)}$, so $\lambda^{(V)}_{(f_i)} = \otimes\lambda^{(V_i)}_{f_i}$.

Let $\mathcal{E}$ be any super line bundle $\mathcal{E}$ on $\omega(F)^\times$ equipped with an action of $F^{\times\flat}_{(V)}$ which lifts the action $f, \nu \mapsto f^{-1}\nu$ of $F^\times$ on $\omega(F)^\times$.[77] We are not interested in its nature at the moment. Notice that an $F^{\times\flat}_{(V)}$-action on $\mathcal{E}$ amounts to a collection of mutually commuting actions of $F^{\times\flat}_{i(V_i)}$ lifting the $F_i^\times$-action along $\omega(F_i)^\times$.

The key structure on $\mathcal{E}$ that arises automatically is:

a rule that assigns to every connection $\mathfrak{d}$ (relative to R) on $\det V$ *a flat connection $\nabla^{\mathfrak{d}}$ on $\mathcal{E}$.*

Let us define $\nabla^{\mathfrak{d}}$ assuming that F is isomorphic to $R((t))$. In general case this will define $\nabla^{\mathfrak{d}}$ separately in every $\omega(F_i)^\times$-direction. Since $\omega(F)^\times = \Pi\omega(F_i)^\times$ this determines our connection; its flatness is immediate from the construction.

Denote by $\hat{F}^\times$ the formal completion of $F^\times$. Let $\hat{F}^{\times\flat}$, $\hat{F}^{\times\flat}_{(V)}$ be the restrictions of the super extensions $F^{\times\flat}$, $F^{\times\flat}_{(V)}$ to $\hat{F}^\times$; these are central $\mathcal{O}^\times$-extensions of $\hat{F}^\times$. Since we are in characteristic 0, the isomorphism of Lie algebras (3.8.1) amounts to an isomorphism $\xi^{\mathfrak{d}} : \hat{F}^{\times n\flat} \xrightarrow{\sim} \hat{F}^{\times\flat}_{(V)}$.

[76] Pulling back the Tate extension $\mathrm{GL}^\flat(V)$.

[77] By definition, such $\mathcal{E}$ is a rule that assigns: (i) to every point $\nu \in \omega(F)^\times(R')$ a super R'-line $\mathcal{E}_\nu$; (ii) to every $f \in F^\times(R)$ an isomorphism $\lambda_f^{(V)} \cdot \mathcal{E}_\nu \xrightarrow{\sim} \mathcal{E}_{f^{-1}\nu}$; (iii) to every morphism $r : R' \to R''$ an isomorphism $\mathcal{E}_{r\nu} = \mathcal{E}_\nu \otimes_{R'} R''$. This datum should satisfy the obvious compatibilities.

The $F^{\times\flat}_{(V)}$-action on $\mathcal{E}$ yields, via $\xi^{\mathfrak{d}}$, an action of $\hat{F}^{\times n\flat}$ on $\mathcal{E}$ which is the same as an action of the corresponding groupoid $\hat{\mathcal{G}}^{n\flat}_{\omega(F)^\times}$ (see 3.5). Now the canonical formal rigidification $\hat{\mathcal{G}}_{\omega(F)^\times} \to \hat{\mathcal{G}}^{\flat}_{\omega(F)^\times}$ from 3.5 provides an $\hat{\mathcal{G}}_{\omega(F)^\times}$-action on $\mathcal{E}$. According to Grothendieck [Gr], such action amounts to a flat connection. This is our $\nabla^{\mathfrak{d}}$.

The construction of $\nabla^{\mathfrak{d}}$ is compatible with base change.

Remarks. (i) The connection $\nabla^{\mathfrak{d}}$ acts on $\mathcal{E}$ by the structure $F^{\times\flat}_{(V)}$-action via a family of splittings $\nabla^{\mathfrak{d}}_\nu : F \to F^{\flat}_{(V)}$, $\nu \in \omega(F)^\times$, where

$$\nabla^{\mathfrak{d}}_\nu := \xi^{\mathfrak{d}} \nabla^{\flat}_\nu. \tag{3.10.1}$$

Here $\nabla^{\flat}_\nu : F \to F^{\flat} \to F^{n\flat}$ was defined in 3.6(i).

(ii) For $\chi \in \omega(F)$ and $a \in F$ one has (see (3.8.2))

$$\nabla^{\mathfrak{d}+\chi}_\nu(a) = \nabla^{\mathfrak{d}}_\nu(a) - \mathrm{Res}(a\chi). \tag{3.10.2}$$

(iii) Suppose that $F = R((t))$ and $V = V_R((t))$ where V_R is a projective R-module of rank n. Set $O := R[[t]] \subset F$, $V(O) := V_R[[t]] \subset V$. According to (2.13.3) we have a section $s^c_{V(O)} : O^\times \to F^{\times\flat}_{(V)}$, so $\mathcal{E}$ is an $O^\times$-equivariant bundle. Set $\chi^{\mathfrak{d}} := \mathfrak{d}(\gamma)/\gamma \in \omega(F)$ where γ is any trivialization[78] of $\det V_R$. Then for $a \in O$, $f \in F^\times$ one has

$$\nabla^{\mathfrak{d}}_{fdt}(a) = s^c_{V(O)}(a) - \mathrm{Res}\,(a\chi^{\mathfrak{d}} + \frac{n}{2} a\, d\log f). \tag{3.10.3}$$

Indeed, if $\mathfrak{d}$ came from a trivialization of V_R then it is formula (3.6.2); the general case follows from (3.10.2).

In particular, if $\mathfrak{d}$ is non-singular with respect to $\det V(O) \subset \det V$ (i.e., $\chi^{\mathfrak{d}} \in \omega(O)$) then the restriction of $\nabla^{\mathfrak{d}}$ to the $O^\times$-torsor $\omega(O)^\times \subset \omega(F)^\times$ coincides with the "constant" connection defined by the $O^\times$-action on $\mathcal{E}$.

(iv) Of course, to define $\nabla^{\mathfrak{d}}$ we need only the action of $\hat{F}^{\times\flat}$ on $\mathcal{E}$. The whole $F^{\times\flat}$-action will be used in the next subsection.

3.11. **The global picture.** Now we have X, D, U as in 3.4. We assume that D is étale over $\mathrm{Spec}\, R$. Let V be a vector bundle on U, i.e., we have a finitely generated projective $\mathcal{O}(U)$-module $V(U)$. Then $F := F_D$ and $V(F) := F \otimes_{\mathcal{O}(U)} V(U)$ satisfy the assumptions of 3.10.

Set $F^{\times\flat}_{(V)} := F^{\times\flat}_{(V(F))}$. Since $V(U) \subset V(F)$ is a d-lattice, (2.13.3) lifts the embedding $\mathcal{O}(U)^\times \hookrightarrow F^\times$ to a splitting[79]

$$s^d_{V(U)} : \mathcal{O}(U)^\times \to F^{\times\flat}_{(V)}. \tag{3.11.1}$$

[78] γ exists locally on $\mathrm{Spec}\, R$.

[79] For $V = \mathcal{O}_U$ this is (3.4.2).

Consider any $\mathcal{E}$ as in 3.10. Then $\mathcal{O}(U)^\times$ acts on $\mathcal{E}$ via the splitting $\bar{s}^d_{V(U)} : \mathcal{O}(U)^\times \to F^{\times\flat}_{(V)}$ defined by (3.11.1).

Let $\omega = \omega_{U/R}$ be the canonical bundle. *We assume that $\omega(U)$ is a free $\mathcal{O}(U)$-module* (of rank 1), i.e., the set of generators $\omega(U)^\times \subset \omega(U)$ is non-empty. Then $\omega(U)^\times \subset \omega(F)^\times$ is an $\mathcal{O}(U)^\times$-torsor.

Consider the restriction $\mathcal{E}_{\omega(U)^\times}$ of $\mathcal{E}$ to $\omega(U)^\times$. The $\mathcal{O}(U)^\times$-action yields a "constant" flat connection on $\mathcal{E}_{\omega(U)^\times}$ which we denote by ∇^0.

Suppose we have a connection $\mathfrak{d}$ on the line bundle $\det V$ on U. Let $\nabla^{\mathfrak{d}_F}$ be the connection on $\mathcal{E}$ that corresponds, by the above, to the restriction $\mathfrak{d}_F$ of $\mathfrak{d}$ to F.

3.12. **Lemma.** The restriction of $\nabla^{\mathfrak{d}_F}$ to $\omega(U)^\times$ is equal to ∇^0.

Proof. (a) By (3.10.2) and the residue formula the restriction of $\nabla^{\mathfrak{d}_F}$ to $\omega(U)^\times$ does not depend on $\mathfrak{d}$.

(b) Assume that V is a trivial vector bundle on U. Fix an isomorphism $\mathcal{O}^n_U \xrightarrow{\sim} V$ and let $\mathfrak{d}$ be the connection that corresponds to the trivialization $\det\phi$ of $\det V$. Then ϕ yields an identification $\xi^\phi : F^{\times n\flat} \xrightarrow{\sim} F^{\times\flat}_{(V)}$ of (3.9.1), hence we get an $F^{\times n\flat}$-action on $\mathcal{E}$. By 3.9(i) our $\nabla^{\mathfrak{d}_F}$ comes from this action via the canonical formal rigidification (see 3.5). We are done by the discussion at the end of 3.5, since ξ^ϕ identifies sections $s^d_{\mathcal{O}(U)^\circ}$ and $s^d_{V(U)^\circ}$.

(c) Let us reduce the general situation to the case of trivial V. Our X, D, V, some $\nu \in \omega(U)^\times$, and $\mathcal{E}_\nu$ are defined over a finitely generated subring of R. Since $\mathcal{E}$ is an equivariant bundle over a torsor, we see that the whole our datum is defined over this subring. Therefore we can assume that R is a finitely generated $\mathbb{Q}$-algebra. Then in order to check that two connections are the same it suffices to do this on ifinitesimal neighbourhoods of points in $\operatorname{Spec} R$. So, by base change, we can assume that R is a local Artinian $\mathbb{Q}$-algebra. Then one can choose $D' = D \sqcup D''$ such that V is trivial on $U' := X \smallsetminus D' \subset U$. It remains to show that our statement for (U, V_U) follows from the one for $(U', V_{U'})$.

Set $F = F_D$, $F' = F_{D'}$, $F'' = F_{D''}$; let $O'' \subset F''$ be the product of local rings, so we have $V(O'') \subset V(F'')$. The projection $V(F') \twoheadrightarrow V(F) \times V(F'')/V(O'')$ is preserved by $F^\times \times O''^\times$-action, so the restriction of $F''^{\times\flat}_{(V)}$ to $F^\times \times O''^\times$ equals the pull-back of $F^{\times\flat}_{(V)}$.[80] The exact sequence $0 \to V(U) \to V(U') \to V(F'')/V(O'') \to 0$ shows that this isomorphism identifies the restriction of $s^d_{V(U')}$ to $\mathcal{O}(U)^\times \subset \mathcal{O}(U')^\times$ with $s^d_{\mathcal{O}(U)}$. Let $\mathcal{E}'$ be an $F'^{\times\flat}_{(V)}$-equivariant super line on $\omega(F')^\times = \omega(F)^\times \times \omega(F'')^\times$

[80] i.e., we trivialize $F''^{\times\flat}_{(V)}$ over $O''^\times$ by means of $s^c_{V(O)}$, see Remark (iii) in 3.10.

whose restriction to $\omega(F)^\times \times \omega(O'')^\times$ coincides with the pull-back of $\mathcal{E}$. Now on $\omega(F)^\times \times \omega(O'')^\times$ the $\mathcal{O}(U)^\times$-action on $\mathcal{E}'$ (which comes from the $\mathcal{O}(U')^\times$-action) coincides with the $\mathcal{O}(U)^\times$-action on $\mathcal{E}$, and for every connection $\mathfrak{d}$ on $\det V$ the connection $\nabla^{\mathfrak{d}_{F''}}$ on $\mathcal{E}'$ coincides with the connection $\nabla^{\mathfrak{d}_F}$ on $\mathcal{E}$. We are done. □

4. The ε-factors

From now on we work over a base field k of characteristic 0. "Absolute connection" means a connection relative to k.

4.1. **The ε-lines.** Let K be a field, $F := K((t))$, $O := K[[t]] \subset F$; we consider F as a topological K-algebra. Let $\omega = \omega(F) := \omega(F/K)$ be the 1-dimensional F-vector space of Kähler differentials, $\omega = Fdt$, $\mathrm{Der}(F/K) = \mathrm{Hom}_F(\omega, F) = F\partial_t$ the one of vector fields.

Suppose V is an F-vector space of dimension n equipped with a connection $\nabla : V \to \omega \otimes_F V$. For a non-zero $\nu \in \omega$ let $\tau_\nu := \nu^{-1}$ be the corresponding vector field. Consider V as a Tate K-vector space (see 2.11), so we have a continuous operator $\tau_\nu = \nabla(\tau_\nu) : V \to V$. It is well-known[81] that the corresponding asymptotic operator τ_ν^∞ acting on V^∞ (see 2.13) is invertible. So we have the $\mathbb{Z}$-graded super K-line

$$\mathcal{E}(V,\nabla)_\nu = \mathcal{E}(V)_\nu := \det(V^\infty, \tau_\nu^\infty) \tag{4.1.1}$$

called *the ε-line of* (V,∇) *at* ν.

Remark. The determinant super K-line of the de Rham cohomology $\det H^{\cdot}_{dR}(V)$ is *canonically trivialized* (see 5.9(a)(iv)).

Sometimes it is convenient to compute $\mathcal{E}(V)_\nu$ as follows. Set $\tau = \tau_\nu$. Denote by $\mathcal{D}_F$ the associative K-algebra of differential operators; let $\mathcal{D}_F^{\leq i}$ be its standard filtration by degree. So F and V are left $\mathcal{D}_F$-modules. It is well-known (see e.g. [D1] p.42 or [M] III 1.1) that V is generated by a single vector e. Then $e, \tau e, .., \tau^{n-1}e$ is an F-base in V, so $\tau^n e = a_{n-1}\tau^{n-1}e + .. + a_0 e$ for certain $a_i = a_i(t) \in F$. Set $A_\tau := (-1)^{n-1}\tau^n + \sum\limits_{i\leq n-1} (-1)^i \tau^i a_i \in \mathcal{D}_F$. This is a non-zero differential operator, so the corresponding asymptotic operator[82] is invertible, and we have the $\mathbb{Z}$-graded super K-line $\det(F^\infty, A_\tau^\infty)$.

4.2. **Lemma.** There is a natural isomorphism of $\mathbb{Z}$-graded super lines

$$\mathcal{E}(V,\nabla)_\nu = \det(F^\infty, A_\tau^\infty). \tag{4.2.1}$$

[81] and follows from the fact that V is generated by a single vector, see below.
[82] We consider F as a Tate K-vector space.

Proof. By definition, $\mathcal{E}(V)_\nu = \det(V/L_V, V/L_V, \tau^\infty)$ where $L_V \subset V$ is any c-lattice. Take for L_V the sum $L_F e + \tau(L_F e) + .. + \tau^{n-1}(L_F e)$ where $L_F \subset F$ is any c-lattice in F. To compute our determinant line consider the Tate submodule $V' := Fe + \tau(Fe) + .. + \tau^{n-2}(Fe)$. Since τ induces isomorphisms $V' \tilde{\to} \tau(V')$ and $L_V \cap V' \tilde{\to} L_V \cap \tau(V')$ we see that[83] $\mathcal{E}(V)_\nu = \det(V/(L_V \oplus \tau(V')), V/(L_V + V'), \tau^\infty)$. The identifications $F \tilde{\to} V/V'$, $f \mapsto \tau^{n-1}(fe)$, and $F \tilde{\to} V/\tau(V')$, $f \mapsto fe$, are compatible with c-lattices, and $\tau(\tau^{n-1}(fe))$ is equal to $A_\tau(f)e$ modulo $\tau(V')$. Therefore $\mathcal{E}(V)_\nu = \det(F/L_F, F/L_F, A_\tau^\infty)$, which yields (4.2.1). We leave it to the reader to check that our identification does not depend on the choice of L_F. $\square$

4.3. **Corollary.** The degree of the $\mathbb{Z}$-graded super line $\mathcal{E}(V,\nabla)_\nu$ is equal to $i(\nabla) + (v(\nu)+1)n$ where $i(\nabla)$ is the irregularity of ∇,[84] and $v(\nu)$ the valuation of ν.[85]

Proof. Write $\nu = f dt/t$, so $\tau_\nu = f^{-1} t\partial_t$. The degree of $\mathcal{E}(V)_\nu$ is the index of the operator τ^∞ acting on V^∞. The index of $(f^{-1})^\infty$ acting on V^∞ is equal to $v(f)n$. Thus it suffices to check 4.3 for $\nu = dt/t$.

By 4.2 the degree of $\mathcal{E}(V)_{dt/t}$ is equal to the index of the operator $A^\infty_{t\partial_t}$. On the other hand, acording to [D1] p.110 or [M] IV(4.6), one has $i(\nabla) = \sup(0, -v(a_i))$ where $a_i \in F$ are coefficients of $A_{t\partial_t}$. Notice that $A_{t\partial_t}$ shifts the filtration $t^j K[[t]]$ on F by $i(\nabla)$, and the corresponding operator between the associated graded quotients $t^j K \to t^{j-i(\nabla)} K$ is non-zero for almost all j's. So the index of $A^\infty_{t\partial_t}$ equals $i(\nabla)$, q.e.d. $\square$

4.4. **Families of ε-lines.** Let us consider the picture of 4.1 depending on parameters. So we have (R, F, V, ∇) where (R, F, V) are as in 3.10, ∇ is an R-relative connection on V.

For $\nu \in \omega(F)^\times$ (it exists flat locally on $\operatorname{Spec} R$) let $\tau_\nu := \nu^{-1} \in \operatorname{Der}(F/R)$ be the corresponding vector field, so we have $\nabla(\tau_\nu) : V \to V$. We say that our $\operatorname{Spec} R$-family is *ε-nice* if the asymptotic operator $\nabla(\tau_\nu)^\infty : V^\infty \to V^\infty$ is invertible (here we consider V as a Tate R-module, see 2.11, 2.13). This property does not depend on the choice of ν and it is local with respect to flat topology of $\operatorname{Spec} R$.

In ε-nice situation for $\nu \in \omega(F)^\times$ we have $\mathbb{Z}$-graded super R-line

$$\mathcal{E}(V,\nabla)_\nu = \mathcal{E}(V)_\nu := \det(V^\infty, \nabla(\tau_\nu)^\infty) \tag{4.4.1}$$

[83] Use (2.9.1) for $M = N = V/L_V$, $f^\infty = \tau^\infty$, and 1-term filtrations $N_1 := V'/L_V \cap V' \subset N$, $M_1 := \partial_t(V')/L_V \cap \tau(V') \subset M$; then apply (2.6.2) to trivialize the line $\det(M_1, N_1, \tau^\infty|_{N_1})$ by $\det(\tau|_{N_1})$.

[84] See e.g. [D1] or [M] IV, sect. 4.

[85] i.e., $\nu = ct^{v(\nu)}dt+$ higher order terms, $c \neq 0$.

called *the ε-line*. The base change of an ε-nice family is ε-nice, and ε-lines are compatible with the base change.

Remark. The above construction is compatible with disjoint union of families of formal discs. Namely, assume that we have a finite collection of families $(F_{\alpha R}, V, \nabla_\alpha)$. Set $F_R := \Pi F_{\alpha R}$, $V := \Pi V_\alpha$, $\nabla := \Pi \nabla_\alpha$. Then $\omega(F)^\times = \Pi\omega(F_\alpha)^\times$, (F, V, ∇) is ε-nice if and only if such is every $(F_\alpha, V_\alpha, \nabla_\alpha)$, and, by (2.6.3), there is a canonical isomorphism

$$\mathcal{E}(V,\nabla) = \boxtimes \mathcal{E}(V_\alpha, \nabla_\alpha). \tag{4.4.2}$$

How can one determine if a family is ε-nice? We know (see 4.1) that this is always true if R is a field, hence if R is an Artinian algebra (see Remark (b) in 2.5). Thus every family is ε-nice at the generic point.

According to 2.5(ii), a $\operatorname{Spec} R$-family is ε-nice if and only if for a c-lattice[86] $L \subset V$ the quotient R-module $V/(L + \nabla(\tau_\nu)(V))$ is finitely generated, and the irregularity function $i : \operatorname{Spec} R \to \mathbb{Z}$, $x \mapsto i(\nabla_x)$, is locally constant (see 4.3, 2.4). The latter condition is superfluous if R is Noetherian (see Remark (c) in 2.5). The absence of jumps of irregularity alone does not imply the family is ε-nice.[87]

Remarks. (i) If R is a local ring then we are not aware of any example of a family with constant irregularity which is not ε-nice.

(ii) If ∇ comes from an absolute connection on V then it looks probable that the absence of jumps of irregularity implies that the family is ε-nice.

4.5. **The ε-connection.** Assume that we are in situation of 4.4 and our family is ε-nice. The compatibility with base change shows that $\mathcal{E}(V)_\nu$ form a $\mathbb{Z}$-graded super line bundle over the $F^\times$-torsor $\omega(F)^\times$. Let us show that $\mathcal{E}(V)$ is canonically rigidified with respect to infinitesimal variations of ν, i.e., $\mathcal{E}(V)$ carries a canonical flat connection ∇^ε relative to $\operatorname{Spec} R$ called *the ε-connection.*

Our problem is local with respect to flat topology of $\operatorname{Spec} R$, so we can assume that F_R is isomorphic to a product of several copies of $R((t))$. Consider the super extension $F^{\times\flat}_{(V)}$ of $F^\times$ (see 3.7). Now the $F^\times$-action on $\omega(F)^\times$, $f, \nu \mapsto f^{-1}\nu$, lifts canonically to an $F^{\times\flat}_{(V)}$-action on $\mathcal{E}(V)$. Namely, for $f \in F^\times$, $\nu \in \omega(F)^\times$ one has $\tau_{f^{-1}\nu} = f\tau_\nu$, hence $\nabla(\tau_{f^{-1}\nu}) = f\nabla(\tau_\nu)$. Our action is the product map

$$\det(V^\infty, f^\infty) \cdot \det(V^\infty, \nabla(\tau_\nu)^\infty) \overset{\sim}{\to} \det(V^\infty, \nabla(\tau_{f^{-1}\nu})^\infty). \tag{4.5.1}$$

[86] We consider V as a Tate R-module.

[87] E.g., consider $R = \operatorname{Spec} k[x]$, $F = R((t))$, $V = F$, $\nabla(\partial_t) = \partial_t + x/t$.

So our $\mathcal{E}$ fits into the setting of 3.10. Let $\mathfrak{d} :=\mathrm{tr}\nabla$ be the connection on $\det V$ defined by ∇. Our ∇^ε is the connection $\nabla^{\mathfrak{d}}$ from 3.10.

Remark. ∇^ε is functorial with respect to isomorphisms of (F, V, ∇), and compatible with base change and disjoint sum identification (4.4.2).

Consider the group ind-scheme $\mathrm{Aut}(F)$ on $\operatorname{Spec} R$, $\mathrm{Aut}(F)(R') := \mathrm{Aut}(F_{R'})$ (see 3.2). Let $\mathrm{Aut}(F)\hat{}$ be its formal completion; this is a formal group whose Lie algebra is the Lie algebra of vector fields $\Theta(F) := \mathrm{Der}(F_R/R)$. The group $\mathrm{Aut}(F)$ acts on the ind-scheme $\omega(F)^\times$.

This action can be lifted to an action of the formal completion $\mathrm{Aut}(F)\hat{}$ on $\mathcal{E}(V)$ in the following two ways:

(a) Using ∇ one lifts the action of $\mathrm{Aut}(F)\hat{}$ on F to V. This action is compatible with ∇, so $\mathrm{Aut}(F)\hat{}$ acts on $\mathcal{E}(V)$ by transport of structure.

(b) The ε-connection ∇^ε lifts the $\mathrm{Aut}(F)\hat{}$-action on $\omega(F)^\times$ to $\mathcal{E}(V)$.

4.6. **Proposition.** The above two $\mathrm{Aut}(F)\hat{}$-actions on $\mathcal{E}(V)$ coincide.

Proof. It suffices to check that the two actions of the Lie algebra $\Theta(F)$ on $\mathcal{E}(V)$ coincide. For $\theta \in \Theta(F)$ let $\theta^{(a)}, \theta^{(b)}$ be the corresponding actions of θ on $\mathcal{E}(V)$. Then $\kappa(\theta) := \theta^{(a)} - \theta^{(b)}$ is an $\mathcal{O}$-linear endomorphism of $\mathcal{E}(V)$, i.e., a function on $\omega(F)^\times$. We want to show that $\kappa : \Theta(F) \to \mathcal{O}(\omega(F)^\times)$ vanishes.

Action (a) preserves ∇^ε (see Remark in 4.5), so for every $\theta, \theta' \in \Theta(F)$ one has $[\theta^{(a)}, \theta'^{(b)}] = [\theta, \theta']^{(b)}$ which is $[\theta^{(b)}, \theta'^{(b)}]$. Hence $\theta'(\kappa(\theta)) = 0$, i.e., the image of κ belongs to the subspace of $\Theta(F)$-invariant functions.

Therefore the two actions coincide on the commutator $[\Theta(F), \Theta(F)]$. We are done since $\Theta(F)$ is a perfect topological Lie algebra. $\square$

Remark. Using definition (a) we see that the $\mathrm{Aut}(F)\hat{}$-action on $\mathcal{E}(V)$ is compatible with the $\hat{F}^{\times\flat}_{(V)}$-action from 4.5, i.e., $\mathcal{E}(V)$ carries a natural action of the semi-direct product of $\mathrm{Aut}(F)\hat{}$ and $\hat{F}^{\times\flat}_{(V)}$.[88]

4.7. **More on the $\Theta(F)$-action on $\mathcal{E}(V)$.** First let us discuss the $\Theta(F)$-action on $\omega(F)^\times$. Consider a formally smooth morphism

$$\mathfrak{r} : \omega(F)^\times \to \mathbb{A}^1_R, \quad \mathfrak{r}(\nu) := \mathrm{Res}(\nu), \tag{4.7.1}$$

invariant with respect to the $\mathrm{Aut}(F)$-action.

Take any $\nu \in \omega(F)^\times$. The tangent space to ν equals F.[89] So the $\Theta(F)$-action yields a morphism $\zeta_\nu : \Theta(F) \to F$.

[88] $\mathrm{Aut}(F)\hat{}$ acts on $\hat{F}^{\times\flat}_{(V)}$ since it acts on V via ∇.

[89] $a \in F$ corresponds to a derivative $\phi \mapsto \partial_\epsilon \phi((1+\epsilon a)\nu)$.

Lemma. (i) The image of ζ_ν equals $\tau_\nu(F) \subset F$. It coincides with the tangent space to the fiber of $\mathfrak{r}$ at ν, hence $\mathrm{Aut}(F)\hat{}$ acts formally transitively along the fibers of $\mathfrak{r}$.

(ii) The kernel of ζ_ν, i.e., the stabilizer of ν in $\Theta(F)$, is $\mathcal{O}\tau_\nu \subset \Theta(F)$.

Proof. The differential to $\mathfrak{r}$ at ν equals the functional $r_\nu : F \to R$, $f \mapsto \mathrm{Res}(f\nu)$, so the tangent space to the fiber of $\mathfrak{r}$ at ν equals $\mathrm{Ker} r_\nu$. For $\theta \in \Theta(F)$ one has $\zeta_\nu(\theta) = \mathcal{L}ie_\theta(\nu)/\nu = -d(\nu\theta)/\nu$. Therefore the multiplication by ν isomorphism identifies ζ_ν with the de Rham differential $F \to \omega(F)$. This implies our assertions. □

Let us describe the $\Theta(F)$-action on $\mathcal{E}(V)$ directly in terms of the $F^\flat_{(V)}$-action on $\mathcal{E}$ (see (4.5.1)). Since $\mathcal{E}(V)$ is a $F^{\times\flat}_{(V)}$-torsor, the $\Theta(F)$-action can be considered as a rule that assigns to $\nu \in \omega(F)^\times$ a lifting $\zeta^\flat_\nu : \Theta(F) \to F^\flat_{(V)}$ of $\zeta_\nu : \Theta(F) \to F$.

Lemma. The kernel of $\zeta^\flat_\nu$ coincides with the kernel of ζ_ν, and its image coincides with image of τ_ν acting on $F^\flat_{(V)}$. Thus $\zeta^\flat_\nu$ is the composition of ζ_ν and the inverse to the isomorphism $\tau_\nu(F^\flat_{(V)}) \xrightarrow{\sim} \tau_\nu(F)$.

Proof. Consider the section $\nabla^\varepsilon_\nu : F \to F^\flat_{(V)}$. According to 4.6 one has $\zeta^\flat_\nu = \nabla^\varepsilon_\nu \zeta_\nu$, so $\mathrm{Ker}\zeta^\flat_\nu = \mathrm{Ker}\zeta_\nu$. Since ∇^ε_ν is τ_ν-invariant (see Remark in 4.6) and τ_ν kills $\mathcal{O} \subset F^\flat_{(V)}$, one has $\tau_\nu(F^\flat_{(V)}) = \nabla^\varepsilon_\nu(\tau_\nu(F))$. Since $\zeta_\nu(\Theta(F)) = \tau_\nu(F)$ by the previous Lemma, we are done. □

4.8. **The absolute ε-connection.** Suppose that V carries an absolute flat connection ∇^{abs} that extends ∇. Let us show that the super line bundle $\mathcal{E}(V, \nabla)$ on $\omega(F)^\times$ acquires then an absolute flat connection which extends the relative ε-connection.

So let T be a test k-algebra, $I \subset T$ a nilpotent ideal, and $t, t' \in \omega(F)^\times(T)$ two T-points that coincide mod I. We want to define a canonical isomorphism

$$\mathcal{E}(V, \nabla)_t \xrightarrow{\sim} \mathcal{E}(V, \nabla)_{t'} \tag{4.8.1}$$

which is the identity mod I, satisfies a transitivity property, and is compatible with the base change (see [Gr]).

Let $t_R, t'_R : R \to T$ be the corresponding T-points of $\mathrm{Spec}\, R$. Let $F_T := F \hat{\otimes}_{t_R} T$, V_T , ∇_T be the t_R-pull-back of F, V, ∇; this is an ε-nice T-family. Define F'_T, V'_T, ∇'_T using t'_R instead of t_R. The two families coincide mod I. Since F is formally smooth the identification $F_{T/I} = F'_{T/I}$ can be extended to an isomorphism of topological T-algebras $\phi : F_T \xrightarrow{\sim} F'_T$. Now the absolute connection ∇^{abs} lifts ϕ to an isomorphism $\phi_V : V_T \xrightarrow{\sim} V'_T$ which extends the identity isomorphism $V_{T/I} = V'_{T/I}$. It is automatically compatible with ∇_T, ∇'_T.

Let $\phi_\omega : \omega^\times_{F_T} \xrightarrow{\sim} \omega^\times_{F'_T}$ be the isomorphism of T-ind-schemes defined by ϕ. We have the ϕ_ω-isomorphism of ε-lines defined by ϕ_V

$$\phi^\varepsilon_V : \mathcal{E}(V_T, \nabla_T) \xrightarrow{\sim} \mathcal{E}(V'_T, \nabla'_T) \tag{4.8.2}$$

which extends the identity isomorphism mod I and is compatible with the ε-connections. We can consider t, t' as T-points of $\omega^\times_{F_T}$, $\omega^\times_{F'_T}$. They coincide mod I, so ϕ^ε_V and ∇^ε yield an identification

$$\mathcal{E}(V_T, \nabla_T)_t \xrightarrow{\sim} \mathcal{E}(V'_T, \nabla'_T)_{\phi_\omega(t)} \xrightarrow{\sim} \mathcal{E}(V'_T, \nabla'_T)_{t'}. \tag{4.8.3}$$

The super lines from (4.8.3) equal the super lines from (4.8.1) by base change, so we have defined the promised isomorphism (4.8.1). It does not depend on the auxiliary choice of ϕ: indeed, various ϕ are $\mathrm{Aut}(F_T)\hat{}$-conjugate, so the assertion follows from 4.6. The transitivity and base change properties are clear.

4.9. **The standard isomorphisms.** Here is a list (cf. 1.1):

(i) *Direct sums.* Let F_R be as in 4.5, and $(V_\alpha, \nabla_\alpha)$ is a finite collection of F-modules with relative connection as in 4.5 such that every every $(V_\alpha, \nabla_\alpha)$ is ε-nice in the sense of loc. cit. Then $(V, \nabla) := (\oplus V_\alpha, \oplus\nabla_\alpha)$ is ε-nice, and, by (2.6.3), we have a canonical isomorphism of $\mathbb{Z}$-graded super lines

$$\mathcal{E}(V, \nabla) = \otimes\mathcal{E}(V_\alpha, \nabla_\alpha) \tag{4.9.1}$$

on $\omega(F)^\times$ compatible with ε-connections. This isomorphism is compatible with the constraints, so $\mathcal{E}$ is a symmetric monoidal functor from the category of (V, ∇) with respect to direct sums to that of super lines with connection on $\omega(F)^\times$. If our ∇'s come from absolute connections then (4.9.1) is compatible with the absolute ε-connections.

(ii) *Filtrations.* Let (F_R, V, ∇) be an ε-nice family and assume we have a filtration on V compatible with ∇ such that $(F_R, \mathrm{gr}V, \mathrm{gr}\nabla)$ is ε-nice.[90] Then (2.9.1) yields a canonical isomorphism

$$\mathcal{E}(V, \nabla) = \mathcal{E}(\mathrm{gr}V, \mathrm{gr}\nabla) \tag{4.9.2}$$

compatible with ε-connections. If our ∇'s come from absolute connections then (4.9.2) is compatible with the absolute ε-connections.

(iii) *Induction.* Let (F'_R, V', ∇') be an ε-nice family, F_R another topological R-algebra as in 4.5, and $F_R \to F'_R$ a morphism of topological R-algebras. It is automatically étale. Denote by V our V' considered

[90] Probably, the latter condition is automatic.

as an F-module; it carries the induced connection ∇. One has an embedding $\omega(F)^\times \hookrightarrow \omega(F')^\times$. Now there is a canonical isomorphism[91]

$$\mathcal{E}(V',\nabla')|_{\omega(F)^\times} \xrightarrow{\sim} \mathcal{E}(V,\nabla) \tag{4.9.3}$$

compatible with ε-connections. If ∇' comes from an absolute connection, then it defines an absolute connection on V, and (4.9.3) is compatible with the absolute ε-connections.

(iv) *Non-singular situation.* Assume that F_R is isomorphic to $R((t))$ and (V,∇) is non-singular, i.e., $F\otimes_R V^\nabla \xrightarrow{\sim} V$. Let $\omega(F)^0 \subset \omega(F)^\times$ be the 0^{th} connected component, i.e., the component of dt. Then there is a canonical isomorphism of $\mathbb{Z}$-graded super lines

$$\mathcal{E}(V,\nabla)|_{\omega(F)^0} = (\det V^\nabla)_{\omega(F)^0} \tag{4.9.4}$$

compatible with the relative connections (the ε- and the trivial one respectively). If ∇ comes from an absolute connection then (4.9.4) is compatible with the absolute connections.

To establish (4.9.4) we choose an isomorphism $R((t)) \xrightarrow{\sim} F_R$; let $O \subset F$ be the image of $R[[t]]$. Set $V(O) := O\otimes_R V^\nabla \subset V$. Then $\nabla(\partial_t)$ preserves $V(O)$, the induced operator $\nabla(\partial_t)$ on $V/V(O)$ is injective, and $V^\nabla \xrightarrow{\sim} t^{-1}V(O)/V(O) \xrightarrow{\sim} \mathrm{Coker}(\partial_t : V/V(O) \to V/V(O))$ where the left arrow is multiplication by t^{-1}. The composition yields

$$\mathcal{E}(V,\nabla)_{dt} = \det(V/V(O), \nabla(\partial_t)^\infty) \xrightarrow{\sim} \det V^\nabla. \tag{4.9.5}$$

Our connection comes[92] from a trivialization of V. By 3.9(i) and discussion at the end of 3.5[93] the connection ∇^ε is “constant” along the fibers of $\omega(F)^0 \to \mathrm{Spec}\, R$. We define (4.9.4) as the horizontal morphism which is (4.9.5) at dt.[94]

Let us check that our isomorphism does not depend on the auxiliary choice of t. Since ∇ is non-singular, the action of $\mathrm{Aut}(F)$ on F lifts to an action on $V = F\otimes_R V^\nabla$ which preserves ∇. Thus $\mathrm{Aut}(F)$ acts by transport of structure on $\mathcal{E}(V,\nabla)$. This action preserves ∇^ε, and the induced action on $\mathcal{E}(V,\nabla)^{\nabla^\varepsilon}_{\omega(F)^0}$ is trivial by 4.6.[95] Since $\mathrm{Aut}(F)$ acts transitively on the set of t's, and the action of $g\in \mathrm{Aut}(F)$ on $\mathcal{E}(V)$ sends the isomorphism (4.9.4) defined by means of t to that defined by means of $g(t)$, we are done.

---

[91] Take any $\nu \in \omega(F)^\times \subset \omega(F')^\times$; let $\tau_\nu \in \mathrm{Der}(F/R)$, $\tau'_\nu \in \mathrm{Der}(F'/R)$ be the corresponding vector fields. Then $\nabla(\tau_\nu) = \nabla'(\tau'_\nu)$, hence the ε-lines are the same.

[92] Locally on $\mathrm{Spec}\, R$.

[93] Notice that $\omega(F)^0_{red} = \omega(O)_{red}$.

[94] The fibers of $\omega(F)^0 \to \mathrm{Spec}\, R$ are connected, so this defines (4.9.4) uniquely.

[95] Recall that $\mathrm{Aut}(F)$ is connected.

(v) *The product formula.* Assume we are in a global situation as discussed in 3.11, so we have a family X of smooth projective curves over $\operatorname{Spec} R$, a relative divisor $D \subset X$ such that the projection $D \to \operatorname{Spec} R$ is étale and surjective, and a vector bundle V on $U := X \smallsetminus D$. Let ∇ be a (relative) connection on V. Consider the morphism $\nabla : V(U) \to (\omega \otimes V)(U)$ of projective R-modules.[96] We say that our situation is *ε-nice* if ∇ is a Fredholm morphism. Then one has the corresponding $\mathbb{Z}$-graded super R-line

$$\det(R\Gamma_{dR}(U, V)[1]) := \det((\omega \otimes V)(U), V(U), \nabla^\infty). \tag{4.9.6}$$

Since $V(U) \subset V(F)$ is a d-lattice, i.e., $V(U)^\infty \xrightarrow{\sim} V(F)^\infty$, we see that (U, V, ∇) is ε-nice if and only if $(F, V(F), \nabla)$ is ε-nice in the sense of 4.4. Below we assume this.

4.10. **Proposition.** There is a canonical isomorphism of $\mathbb{Z}$-graded super lines on $\omega(U)^\times$ (the product formula)

$$\mathcal{E}(V(F), \nabla)|_{\omega(U)^\times} \xrightarrow{\sim} \det(R\Gamma_{dR}(U, V)[1])_{\omega(U)^\times} \tag{4.10.1}$$

where the r.h.s. is the pull-back of the determinant R-line to $\omega(U)^\times$. This isomorphism is compatible with the relative connections (the l.h.s. carries the ε-connection, the r.h.s. the trivial one).

If V carries an absolute integrable connection which extends ∇ then (4.10.1) is compatible with the corresponding absolute connections (the ε- and the Gauß-Manin one). Thus (4.10.1) describes the determinant of the Gauss-Manin connection in terms of the absolute ε-connections.

Proof. A trivialization $\nu \in \omega(U)^\times$ identifies $\nabla : V(U) \to (\omega \otimes V)(U)$ with $\nabla(\tau_\nu) : V(U) \to V(U)$ where $\tau_\nu := \nu^{-1} \in \operatorname{Der}(U/R)$, so we get $\det(R\Gamma_{dR}(U, V)[1])_\nu = \det(V(U), V(U), \nabla(\tau_\nu)^\infty)$. The latter super line is identified with $\mathcal{E}(V(F), \nabla)_\nu$ via the asymptotic isomorphism $V(U)^\infty \xrightarrow{\sim} V(F)^\infty$.

The compatibility with relative connections follows from 3.12. It implies immediately the compatibility with absolute connections. □

4.11. **Remark.** The above standard isomorphisms are mutually compatible in the obvious sense; they are compatible with the disjoint sum identifications (4.4.2). In particular, if D is a disjoint union of sections x_i then $F = \Pi F_i$ (the product of local fields at x_i) and $\mathcal{E}(V(F), \nabla)_\nu = \otimes \mathcal{E}(V(F_i), \nabla)_\nu$, hence the name "product formula" for (4.10.1).

[96]The R-projectivity follows since $\mathcal{O}(U)$ is a projective R-module and $V(U)$ is a projective $\mathcal{O}(U)$-module.

4.12. **Duality.** Suppose that we have a family (F, V, ∇) such that $H^{\cdot}_{dR}(V, \nabla) = 0$, i.e., $\nabla : V \to \omega \otimes V$ is an isomorphism. Let V' be the dual vector bundle equipped with the dual connection ∇'. It satisfies the same property. For $\nu \in \omega(F)^{\times}$ the Tate R-module V' identifies with the dual to V via the pairing $v, v' \mapsto \mathrm{Res}(v'(v)\nu)$, and $\nabla'(\tau_\nu)$ is the operator adjoint to $-\nabla(\tau_\nu) = \nabla(\tau_{-\nu})$. So (2.13.5) yields a canonical identification of super lines

$$\mathcal{E}(V', \nabla')_\nu = \mathcal{E}(V, \nabla)^{-1}_{-\nu}. \tag{4.12.1}$$

We leave it to the reader to check that it is compatible with the ε-connections.

Suppose now we are in the global situation of 4.9(v) and the local de Rham cohomology vanish, i.e., $\nabla : V(F) \xrightarrow{\sim} (\omega \otimes V)(F)$. Thus $H^0_{dR}(U, V) = H^0_{dR}(U, V') = 0$ and the global Poincaré duality identifies $H^1_{dR}(U, V')$ with the dual to $H^1_{dR}(U, V)$. Passing to determinants we get a canonical identification $\det R\Gamma_{dR}(U, V') = \det R\Gamma_{dR}(U, V)^{-1}$. The product formula is compatible with these identifications.

Remark. We have formulared compatibility (4.12.1) in the situation when the local de Rham cohomology $H^{\cdot}_{dR}(V, F)$ vanish. If the base ring R is a field (or, more generally, an Artinian algebra) then (4.12.1) holds for arbitrary (V, ∇) by (2.13.6) since the local super line $\det(V, \nabla(\tau_\nu)) = \det H^{\cdot}_{dR}(V)$ is canonically trivialized (see 5.9(iv)).[97]

4.13. **Questions.** (a) What happens when singular points merge together, i.e., we do not assume that D is étale over $\mathrm{Spec}\, R$?

(b) Is it true that $(\mathcal{E}, \nabla^{\varepsilon})$ is uniquely determined by compatibilities (4.4.2), (4.9.1)–(4.9.4), (4.10.1)?[98]

5. Some formula (with apologies to Charlotte)[99]

In this section we assume, for technical reasons, that our parameter space[100] is $\mathrm{Spec}\, K$ where K is a field. The case of an arbitrary smooth parameter space reduces to this situation (since a line bundle with connection on a smooth variety is determined by its restriction to the generic point).

[97]Probably, this is true for arbitrary R under the assumption that both ∇ and ∇' are ε-nice. It also looks probable that ∇ is ε-nice iff such is ∇'.

[98]One can show that the connection on our $\mathcal{E}$ (defined in (4.4.1)) is uniquely determined by the condition that it is compatible with the standard isomorphisms.

[99]The authors refer to "Charlotte's Web", an acclaimed novel by E. B. White ruminating over the problems of a person in quest for tenure.

[100]This is $\mathrm{Spec}\, R$ of the previous section.

5.1. **Dependence on ν.** Let F be a local K-field, V_F an F-vector space of dimension n equipped with an absolute connection ∇. We have the ε-line $\mathcal{E} = \mathcal{E}(F, V_F)$; this is a super line with an absolute integrable connection ∇^ε on the ind-scheme of invertible relative 1-forms $\omega(F)^\times$. Our aim is to compute its fiber $\mathcal{E}_\nu = \mathcal{E}(F, V_F)_\nu$ over a K-point ν of $\omega(F)^\times$ which is a super K-line equipped with an integrable connection.

When doing computations it is convenient to choose ν in a special way, e.g. to make it fixed by horizontal vector fields. One can pass then to arbitrary ν using Proposition below.

Choose a parameter $t \in F$. Then $F = K'((t))$ where a finite extension K'/K is the integral closure of K in F. Set $O := K'[[t]] \subset F$. Below we make vector fields on K act on F so that for $\theta \in \Theta_K = \mathrm{Der}K/k$ one has $\theta(t) = 0$. Hence Θ_K acts on $\omega(F)^\times$. Differential 1-forms and vector fields on $\mathrm{Spec}\, F$ and $\omega(F)^\times$ decompose into "horizontal" and "vertical" components accordingly. For 1-forms on $\mathrm{Spec}\, F$ these components are denoted by lower indices x and t. E.g., for $f \in F$ we write $df = d_x f + d_t f$ where $d_x f \in F \otimes \Omega^1_{K/k}$, $d_t f \in \omega(F) = \Omega^1_{F/K}$.

Choose an O-lattice $V_O \subset V_F$. It yields a section $s^c_{V_O} : O^\times \to F^{\times\flat}_{(V_F)}$ (see (2.13.3), 3.7). Since $\mathcal{E}$ is an $F^{\times\flat}_{(V_F)}$-equivariant line bundle (see 4.5), $s^c_{V_O}$ defines an $O^\times$-equivariant structure on $\mathcal{E}$.[101]

Assume we have $\nu, \nu' \in \omega(F)(K)$ such that $v(\nu) = v(\nu') =: \ell$. Then $u := \nu'/\nu \in O^\times$, i.e., $u = u_0 + u_1 t + ..$, $u_i \in K'$, $u_0 \neq 0$. So the equivariant structure provides an isomorphism

$$\alpha^u = \alpha^u_{V_O} := s^c_{V_O}(u) : \mathcal{E}_{\nu'} \xrightarrow{\sim} \mathcal{E}_\nu. \tag{5.1.1}$$

We want to see to what extent α^u is horizontal with respect to connections ∇^ε, i.e., to compute $\nabla^\varepsilon \log \alpha^u := (\nabla^\varepsilon \alpha^u)/\alpha^u \in \Omega^1_{K/k}$.

Set $\chi = \chi_x + \chi_t := (\mathrm{tr}\nabla) \log \gamma \in \Omega^1_{F/k}$ where γ is any generator of $\det V_O$. This is a closed 1-form; its class modulo $\Omega^1_{O/k}$ does not depend on the choice of γ.

5.2. **Proposition.** One has

$$\nabla^\varepsilon \log \alpha^u = tr_{K'/K}\left(\frac{n\ell}{2} d\log u_0 + \mathrm{Res}(d\log u \wedge \chi)\right). \tag{5.2.1}$$

Proof. By base change we can assume that $K' = K$.

Take any $a \in K[[t]] = O \subset F$, $a = a_0 + a_1 t + ...$ Consider a as a vertical vector field on $\omega(F)^\times$ (the corresponding derivative is $\phi \mapsto a\phi$,

[101] Recall that the $O^\times$-action on $\mathcal{E}$ lifts the $O^\times$-action $u, \nu \mapsto u^{-1}\nu$ on $\omega(F)^\times$.

$a\phi(\nu) := \partial_\epsilon \phi((1+\epsilon a)\nu))$. It acts on $\mathcal{E}$ according to ∇^ε: namely, $\nabla^\varepsilon(a)$ is the action of $\nabla^\eth(a) \in F^\flat_{(V)}$ on $\mathcal{E}$ (see 4.5). Our a also acts on $\mathcal{E}$ according to the $O^\times$-equivariant structure, i.e., as $s^c_{V(O)}(a) \in F^\flat_{(V)}$.

From now on we work on the connected component $\omega(F)^\ell_{red}$ of the reduced scheme $\omega(F)^\times_{red}$ (the action of a preserves $\omega(F)^\ell_{red}$). By (3.10.3) and 4.5 one has[102]

$$\nabla^\varepsilon(a) - s^c_{V(O)}(a) = -\frac{n\ell}{2}a_0 - \mathrm{Res}(a\chi_t). \tag{5.2.2}$$

Let r be a non-zero $O^\times$-invariant section of $\mathcal{E}$. Then r_ν, $r_{\nu'}$ are trivializations of, respectively, $\mathcal{E}_\nu$ and $\mathcal{E}_{\nu'}$ which are identified by the isomorphism α^u. Set $\psi := \nabla^\varepsilon \log r$. One has $\nabla^\varepsilon \log \alpha^u = \psi_\nu - \psi_{\nu'}$.

Our ψ is a closed 1-form on the $O^\times$-torsor $\omega(F)^\ell_{red}$. By (5.2.2) the vertical part of ψ is an $O^\times$-invariant form whose value on the vertical vector field $a \in O$ equals $-\frac{n\ell}{2}a_0 - \mathrm{Res}(a\chi_t)$. If ψ' is any other 1-form with these properties then $\psi' - \psi$ is the pull-back of a 1-form on $\mathrm{Spec}\, K$, hence $\nabla^\varepsilon \log \alpha^u = \psi'_\nu - \psi'_{\nu'}$.

Let us construct such ψ'. Fix ν' and consider ν as a variable point of $\omega(F)^\ell_{red}$. The scheme $\omega(F)^\ell_{red}\hat{\otimes}_K O$ carries a canonical invertible function u: for $\nu \in \omega(F)^\ell_{red}$ the restriction of u to the fiber over ν is the function ν'/ν. The pull-back of χ from F to $\omega(F)^\ell_{red}\hat{\otimes}_K F$ is a closed 1-form which we denote also by χ. Set $\psi' := \frac{n\ell}{2} d\log u_0 + \mathrm{Res}(d\log u \wedge \chi)$ where Res is the residue along the fibers of $\omega(F)^\ell_{red}\hat{\otimes}_K F \to \omega(F)^\ell_{red}$.

Incerting ν, ν' into ψ' we get (5.2.1). $\square$

5.3. **Case of regular singularity.** We use notation from 5.1.

Suppose that (the vertical part of) ∇ has regular singularities. Then one can find an O-lattice $V_O \subset V_F$ such that V_O is preserved by the action of $\nabla(t\partial_t)$ and horizontal vector fields Θ_K.

Consider a K'-vector space $V_0 := V_O/tV_O$. The action of horizontal vector fields define a connection ∇_0 on V_0. The action of $\nabla(t\partial_t)$ defines a ∇_0-horizontal endomorphism of V_0 which we denote by κ.[103]

Remark. Recall a standard way to construct such V_O. Let $\bar{K}'$ be an algebraic closure of K'. Choose any (set-theoretic) section $s : \bar{K}'/\mathbb{Z} \to \bar{K}'$ that commutes with the Galois action. Let $V_{K'}$ be the sum of generalized $s(\lambda)$-eigenspaces of $\nabla(t\partial_t)$, $\lambda \in \bar{K}'/\mathbb{Z}$. Then $V_{K'}$ is a K'-structure on V_F. Our lattice is $V_O := V_{K'}[[t]]$.

[102]Here the l.h.s. is a (constant) function on $\omega(F)^\ell_{red}$, in the r.h.s. a is considered as an element of F.

[103]So κ is the residue of the polar part of ∇ with respect to V_O.

Consider a Fredholm endomorphism $\nabla(t\partial_t)$ of V_F/V_O. Its determinant line is canonically trivialized. Namely, V_F/V_O decomposes into a direct sum of generalized eigenspaces of $\nabla(t\partial_t)$; for each eigenvalue λ the corresponding subspace P_λ is finite-dimensional. So the embedding $P_0 \hookrightarrow V_F/V_O$ induces an isomorphism on cohomology hence an isomorphism of the determinant lines. And the determinant line for P_0 is trivialized since $\dim P_0 < \infty$.

Take any $\nu \in \omega(F)^\times(K)$, $v(\nu) =: \ell$. So $\nu = ut^\ell dt$, $u \in K'[[t]]^\times$, $u = u_0 + u_1 t + ..$, $u_i \in K'$, $u_0 \neq 0$. We want to compute $\mathcal{E}_\nu = \mathcal{E}(F,V)_\nu$ in terms of V_O.

There is a natural isomorphism of super lines

$$\beta_\nu : \mathcal{E}_\nu \xrightarrow{\sim} (\det{}_K V_0)^{\otimes \ell+1}. \tag{5.3.1}$$

Namely, recall that $\mathcal{E}_\nu = \det(V_F^\infty, \nabla(\tau_\nu)^\infty) = \det(V_F/V_O, \nabla(\tau_\nu)^\infty)$ where $\tau_\nu := \nu^{-1} = u^{-1}t^{-\ell}\partial_t$. Let us present $\nabla(\tau_\nu)^\infty$ as a composition $V_F/V_O \xrightarrow{\nabla(t\partial_t)} V_F/V_O \xrightarrow{t^{-\ell-1}} V_F/t^{-\ell-1}V_O \xrightarrow{u^{-1}} V_F/t^{-\ell-1}V_O \xrightarrow{id^\infty} V_F/V_O$. We already trivialized the determinant line of the first operator; that of the middle operators is trivialized since they are isomorphisms. Thus $\mathcal{E}_\nu = \det(V_F/V_O, V_F/t^{-\ell-1}V_O, id^\infty) = (\det V_0)^{\otimes \ell+1}$.[104]

The r.h.s. of (5.3.1) carries a connection defined by ∇_0. It yields, via β_ν, a connection on $\mathcal{E}_\nu$ which we also denote by ∇_0.

5.4. **Proposition.** $\nabla^\varepsilon - \nabla_0 = -tr_{K'/K}(\mathrm{Tr}\kappa - n(\ell/2+1))d\log u_0)$.

Proof. For $\nu = t^\ell dt$, i.e., $u = 1$, the formula is clear. Indeed, any horizontal vector field θ commutes with τ_ν, so $\nabla(\theta)$ acts on the determinant line of τ_ν by transport of structure, and $\nabla^\varepsilon(\theta)$ coincides with this action (see 4.8).

For arbitrary ν notice that the isomorphism $\beta_\nu^{-1}\beta_{t^\ell dt} : \mathcal{E}_{t^\ell dt} \xrightarrow{\sim} \mathcal{E}_\nu$ is inverse to the isomorphism α^u of (5.1.1) defined by the lattice $t^{-\ell-1}V_O$. The corresponding form χ from 5.1 is $(\mathrm{Tr}\kappa - n(\ell+1))dt/t$. Now use (5.2.1). □

Remark. Choose a k-structure $V_k \subset V_F$ such that V_O is the O-module generated by V_k. Let ∇^0 be the absolute connection on $V_F, V_{K'}$ defined by V_k, $\nabla^0(V_k) = 0$. Then $\nabla - \nabla^0 = \eta + t^{-1}g dt$ where $\eta \in \Omega^1_{K'/k} \otimes \mathrm{End}V_{K'}[[t]]$, $g \in \mathrm{End}V_{K'}[[t]]$. Write $\eta = \eta_0 + \eta_1 t + ..$, $g = g_0 + g_1 t + ..$, $\eta_i \in \Omega^1_{K'/k} \otimes \mathrm{End}V_{K'}$, $g_i \in \mathrm{End}V_{K'}$. Then $V_0 = V_{K'}$, $\nabla_0 = \nabla^0 + \eta_0$, $\kappa = g_0$, so we can rewrite formula 5.4 as

$$\nabla^\varepsilon - \nabla^0 = tr_{K'/K}((\ell+1)\mathrm{Tr}\eta_0 - (\mathrm{Tr}g_0 - n(\ell/2+1))d\log u_0). \tag{5.4.1}$$

[104] The latter identification comes from isomorphisms $t^{-i} : t^i V_O/t^{i+1}V_O \xrightarrow{\sim} V_0$.

5.5. **Case of an irregular admissible connection.** As above, we follow notation from 5.1.

Suppose that there is an O-lattice $V_O \subset V_F$ such that ∇ is m-*admissible* with respect to V_O for some $m \geq 2$ (see [BE3]). This means that the vertical part of ∇ has pole of "exact" order m, i.e., $t^m\nabla(\partial_t) : V_O \overset{\sim}{\to} V_O$, and the horizontal part of ∇ has pole of order $\leq m-1$.

Choose a k-structure $V_k \subset V_F$ such that V_O is the O-submodule generated by V_k. Let ∇^0 be the corresponding absolute connection on V_F, $\nabla^0(V_k) = 0$. Set $\nabla - \nabla^0 = \mathcal{A} = \mathcal{A}_t + \mathcal{A}_x \in \Omega^1_{F/k} \otimes \mathrm{End}V$. Write $\mathcal{A}_t = t^{-m}g dt$, $\mathcal{A}_x = t^{-m+1}\eta$. Admissibility means that $g \in \mathrm{Aut}V_{K'}[[t]]$, $\eta \in \Omega^1_{K'/k} \otimes \mathrm{End}V_{K'}[[t]]$. So $g = g_0 + g_1 t + ..$, $\eta = \eta_0 + \eta_1 t + ..$ where $g_i \in \mathrm{End}V_{K'}$, $\eta_i \in \Omega^1_{K'/k} \otimes \mathrm{End}V_{K'}$, g_0 is invertible.

Consider a 1-form $\nu = ut^\ell dt \in \omega(F)^\times$, $u \in O^\times$, and the corresponding $\mathcal{E}_\nu = \mathcal{E}(F, V_F)_\nu$. We have a natural isomorphism of super lines

$$\beta_\nu : \mathcal{E}_\nu \overset{\sim}{\to} (\det{}_K(V_{K'}))^{\otimes \ell+m} \tag{5.5.1}$$

similar to (5.3.1). Namely, notice that the operator $\nabla(\tau_\nu)$ yields an isomorphism $V_F/V_O \overset{\sim}{\to} V_F/t^{-\ell-m}V_O$ which trivializes the super line $\det(V_F/t^{-\ell-m}V_O, V_F/V_O, \nabla(\tau)^\infty)$. Our identification is the composition $\mathcal{E}_\nu = \det(V_F/V_O, V_F/t^{-\ell-m}V_O, id^\infty)\cdot\det(V_F/t^{-\ell-m}V_O, V_F/V_O, \nabla(\tau)^\infty) = \det(V_F/V_O, V_F/t^{-\ell-m}V_O, id^\infty) = (\det{}_K(V_{K'}))^{\otimes \ell+m}$.

The k-structure V_k on $V_{K'}$ defines an integrable connection on $V_{K'}$, hence on the r.h.s. of (5.5.1). Denote by $\nabla^0_{\mathcal{E}}$ the corresponding connection on $\mathcal{E}_\nu$.

5.6. **Theorem.** One has $\nabla^\varepsilon - \nabla^0_{\mathcal{E}} = tr_{K'/K}(\phi_1 + \phi_2 + \phi_3)$ where $\phi_i \in \Omega^1_{K'/K}$ are:

$\phi_1 = \mathrm{Res}\mathrm{Tr}(g^{-1}dg \wedge \mathcal{A}_x) - \frac{m}{2} d\log\det(g_0)$

$\phi_2 = (\ell+m)\mathrm{Res}\mathrm{Tr}(\mathcal{A}_x \wedge dt/t)$

$\phi_3 = n(\ell/2+m)d\log u_0 - \mathrm{Res}(d\log u \wedge \mathrm{Tr}\mathcal{A})$.

Proof. (a) By base change we can assume that $K' = K$, so $F = K((t))$.

(b) It suffices to treat the case of $\nu = t^\ell dt$. Indeed, the formula for arbitrary ν follows then from (5.2.1) since $\beta_\nu^{-1}\beta_{t^\ell dt} : \mathcal{E}_{t^\ell dt} \overset{\sim}{\to} \mathcal{E}_\nu$ is inverse to the isomorphism α^u of (5.1.1) for the lattice $t^{-\ell-m}V_O$. The form χ from 5.1 is $\mathrm{Tr}\mathcal{A} - (\ell+m)ndt/t$. We get the term ϕ_3.

(c) From now on $\nu = t^\ell dt$. Set $h := 1 + t^m g^{-1}\nabla^0(\partial_t) \in \mathrm{GL}(V_F)$. Our $\mathcal{E}_\nu$ is the determinant line of the invertible operator

$$\nabla(\tau_\nu) = t^{-\ell-m}gh \tag{5.6.1}$$

acting on a Tate K-vector space $V_F = V_K((t))$. Notice that both factors g and h preserve the lattice V_O, so we have $\tilde{g} := s^c_{V_O}(g)$, $\tilde{h} := s^c_{V_O}(h) \in F^{\times\flat}_{(V_F)}$ (see (2.13.3)). The operator t is defined over k, i.e., $t \in \mathrm{GL}(V_k((t))) \subset \mathrm{GL}(V_K((t)))$. Let $\tilde{t} \in \mathrm{GL}^\flat(V_k((t))) \subset \mathrm{GL}^\flat(V_K((t)))$ be any lifting of t which is also defined over k. So we have

$$\tilde{t}^{-\ell-m}\tilde{g}\tilde{h} \in \mathcal{E}_\nu. \tag{5.6.2}$$

It follows from the construction of $\nabla^0_{\mathcal{E}}$ that

$$\nabla^0_{\mathcal{E}}(\tilde{t}^{-\ell-m}\tilde{g}\tilde{h}) = 0. \tag{5.6.3}$$

It remains to compute $\phi := \nabla^\varepsilon \log(\tilde{t}^{-\ell-m}\tilde{g}\tilde{h}) \in \Omega^1_{K/k}$.

(d) Since τ_ν is preserved by the action of horizontal vector fields our ∇^ε coincides with the ∇-action of horizontal vector fields (see 4.8).

Explicitly, for $\theta \in \Theta_K \subset \Theta_F$ the infinitesimal automorphism $1 + \epsilon\nabla(\theta)$ acts on the Tate K-vector space $F = K((t))$ hence on $\mathrm{GL}(V_F)$ and its super extension $\mathrm{GL}^\flat(F)$. This action fixes $\nabla(\tau_\nu) \in \mathrm{GL}(V_F)$; the action on its super line $\mathcal{E}_\nu \subset \mathrm{GL}^\flat(V_F)$ is $r \mapsto r + \epsilon\nabla^\varepsilon(\theta)r$.

Notice that the ∇-action of horizontal vector fields preserves, in fact, both t and $gh \in \mathrm{GL}(V_F)$. So $(1+\epsilon\nabla(\theta))\tilde{t}^{-\ell-m} = (1+\epsilon\phi_2(\theta))\tilde{t}^{-\ell-m}$ and $(1+\epsilon\nabla(\theta))(\tilde{g}\tilde{h}) = (1+\epsilon\phi_1(\theta))\tilde{g}\tilde{h}$ for some $\phi_i(\theta) \in K \subset \mathfrak{gl}(V_F)$. Thus

$$\phi = \phi_1 + \phi_2. \tag{5.6.4}$$

We will compute ϕ_i separately:

(e) Since $\nabla^0(\theta)\tilde{t} = 0$ one has $1 + \epsilon\phi_2(\theta) = \mathrm{Ad}_{1+\epsilon\mathcal{A}_x(\theta)}(\tilde{t}^{-\ell-m})/\tilde{t}^{-\ell-m}$ or $\phi_2(\theta) = -(\ell+m)(\mathcal{A}_x(\theta)^\flat - \mathrm{Ad}_{\tilde{t}}\mathcal{A}_x(\theta)^\flat)$ where $\mathcal{A}_x(\theta)^\flat$ is any lifting of $\mathcal{A}_x(\theta) \in \mathrm{End}_F(V_F) \subset \mathfrak{gl}(V_F)$ to $\mathfrak{gl}^\flat(V_F)$. By (3.1.6), 3.3(iii) one has $\phi_2(\theta) = (\ell+m)\mathrm{ResTr}(\mathcal{A}_x(\theta)dt/t)$ or

$$\phi_2 = (\ell+m)\mathrm{ResTr}(\mathcal{A}_x \wedge dt/t). \tag{5.6.5}$$

(f) Since $\nabla^0(\theta)(V_O) \subset V_O$ the action of $1+\epsilon\nabla^0(\theta)$ preserves the subgroup $\mathrm{GL}(V_F, V_O) \subset \mathrm{GL}(V_F)$ and commutes with $s^c_{V_O}$ (see (2.13.3)). The adjoint action of $1+\epsilon\mathcal{A}_x(\theta)$ does not have this property. However it sends both g and h to $\mathrm{GL}(V_F, V_O)$: this is clear for h, and for g you notice that, due to integrability of ∇, one has

$$[\mathcal{A}_x(\theta), g] = t\partial_t(\eta(\theta)) - (m-1)\eta(\theta) - \theta(g) \in \mathrm{End}V_K[[t]]. \tag{5.6.6}$$

So $\mathrm{Ad}_{1+\epsilon\mathcal{A}_x(\theta)}\tilde{g} = (1+\epsilon\phi'(\theta))s^c_{V_O}(\mathrm{Ad}_{1+\epsilon\mathcal{A}_x(\theta)}g)$ for certain $\phi'(\theta) \in K$. We have defined $\phi' \in \Omega^1_{K/k}$; replacing g by h we get $\phi'' \in \Omega^1_{K/k}$.

We see that $(1+\epsilon\nabla(\theta))\tilde{g} = (1+\epsilon\phi'(\theta))s^c_{V_O}(\mathrm{Ad}_{1+\epsilon\nabla(\theta)}g)$; same for g replaced by h and ϕ' by ϕ''. The product of these identities gives

$$\phi_1 = \phi' + \phi''. \tag{5.6.7}$$

(g) Let us rewrite ϕ', ϕ'' in the following general manner:

Assume we have $a \in \mathfrak{gl}(V_F)$ and $q \in \mathrm{GL}(V_F)$ such that $q(V_O) = V_O$ and $[a, q](V_O) \subset V_O$. Set $b := a - \mathrm{Ad}_q a$, $b^\flat := a^\flat - \mathrm{Ad}_q a^\flat$ where $a^\flat \in \mathfrak{gl}^\flat(V_F)$ is any lifting of a. Notice that $b^\flat$ is well-defined[105] and $b(V_O) \subset V_O$. Set $c(a, q) := b^\flat - s^c_{V_O} b \in K \subset \mathfrak{gl}^\flat(V_F)$. We have

$$\phi'(\theta) = c(\mathcal{A}_x(\theta), g), \quad \phi''(\theta) = c(\mathcal{A}_x(\theta), h). \tag{5.6.8}$$

The following simple lemma helps to compute c:

Lemma. $c(a, q) = \mathrm{Tr}(V_O, (a^\sharp - \mathrm{Ad}_q a^\sharp)|_{V_O})$ where $a^\sharp \in \mathfrak{gl}(V_F)$ is any endomorphism with open kernel such that $(a - a^\sharp)(V_O) \subset V_O$.

Proof of Lemma. Notice that $c(a, q)$ vanishes if $a(V_O) \subset V_O$.[106] So we can assume that $a = a^\sharp$. Then $b^\flat = s_d(b)$ (see the end of 2.13), and we are done since $s_d b - s^c_{V_O} b = \mathrm{Tr}(V_O, b|_{V_O})$ for every $b \in \mathfrak{gl}(V_F) \cap \mathfrak{gl}(V_F, V_O)$. □

To compute ϕ', ϕ'' we apply the lemma to $q = g, h$ and $a^\sharp = \mathcal{A}_x \Pi$ where $\Pi : V_F \to V_F$ is a projector with kernel $t^{m-1} V_K[[t]]$ and image $t^{m-2} V_K[t^{-1}]$.

(h) Let us begin with ϕ'. One has $\phi' = \mathrm{Tr}(V_O, \mathcal{A}_x \Pi - g \mathcal{A}_x \Pi g^{-1}) = \mathrm{Tr}(V_O, (\mathcal{A}_x - g \mathcal{A}_x g^{-1})\Pi) + \mathrm{Tr}(V_O, g \mathcal{A}_x (g^{-1}\Pi - \Pi g^{-1}))$.

The first summand vanishes. Indeed, $B := \mathcal{A}_x - g\mathcal{A}_x g^{-1} \in \mathrm{End} V_K[[t]]$; our summand is $(m-1)\mathrm{Tr}(B_0)$. But $\mathrm{Tr}(B_0) = (\mathrm{Tr} B)_0$, and $\mathrm{Tr} B = 0$.

The second summand can be rewritten as $\mathrm{Tr}(V_O, \mathcal{A}_x g^{-1}(\Pi g - g\Pi)) = -\mathrm{Tr}(V_O, \mathcal{A}_x g^{-1}(1 - \Pi) g \Pi)$. An immediate calculation identifies it with $-\mathrm{Res}\mathrm{Tr}_F(g d_t(\mathcal{A}_x g^{-1})) = -\mathrm{Res}\mathrm{Tr}_F(\mathcal{A}_x g^{-1} d_t g) = \mathrm{Res}\mathrm{Tr}_F(g^{-1} d_t g \wedge \mathcal{A}_x)$.

One has $\phi'' = \mathrm{Tr}(V_O, \mathcal{A}_x \Pi - h \mathcal{A}_x \Pi h^{-1}) = -\mathrm{Tr}(V_O, t^m g^{-1} \partial_t \mathcal{A}_x \Pi) = \frac{m(m-1)}{2} \mathrm{Tr} g_0^{-1} \eta_0$. As follows from (5.6.6)[107] it equals $-\frac{m}{2} \mathrm{Tr} g_0^{-1} d g_0 = -\frac{m}{2} d \log(\det g_0)$. So, by (5.6.7),

$$\phi_1 = \mathrm{Res}\mathrm{Tr}_F(g^{-1} d_t g \wedge \mathcal{A}_x) - \frac{m}{2} d \log(\det g_0). \tag{5.6.9}$$

We are done. □

5.7. **Example.** Suppose that our (V_F, ∇) is such that the corresponding relative connection $\nabla_{/K}$ is a successive extension of a single irregular relative connection $(L_F, \nabla_{/K})$ of rank 1.

Let us show that (V_F, ∇) is admissible. We need to construct an O-lattice $V_O \subset V_F$ as in 5.5. Notice that the relative connection on

[105] i.e., it does not depend on the choice of $a^\flat$.

[106] To see this consider $a^\flat = s^c_{V_O} a$.

[107] Indeed, by (5.6.6) one has $(\mathcal{A}_x - g\mathcal{A}_x g^{-1})_0 = -((m-1)\eta_0 + d_x g_0) g_0^{-1}$. Hence $\mathrm{Tr}((m-1)\eta_0 + d_x g_0) g_0^{-1}) = 0$.

L can be extended to an absolute integrable connection so that $L^{\otimes n}$ is isomorphic to $\det V$ as lines with absolute connection. Set $P := V \otimes L^{-1}$. As a relative connection our P is a successive extension of trivial connections. There is a unique O-lattice $P_O \subset P$ preserved by $\nabla(t\partial_t)$ and such that $\nabla(t\partial_t)$ acts on the fiber $P_0 := P_O/tP_O$ as a nilpotent matrix. Our P_O is automatically preserved by horizontal vector fields. Set $V_O := P_O \otimes L_O \subset V$ where $L_O \subset L$ is any O-lattice.

To check the conditions from 5.5 choose a k-structure in V_O. Let ∇^0 be the corresponding connection, so $\nabla = \nabla^0 + \mathcal{A}_t + \mathcal{A}_x$. Then $\mathcal{A}_t = f dt/t^m + h dt/t$ where $f \in O^\times$, $h = h_0 + h_1 t + ..$, $h_i \in K' \otimes \mathrm{End} V_k$, h_0 is nilpotent, and $\mathcal{A}_x = a_q t^q + a_{q+1} t^{q+1} + ..$, $a_i \in \Omega^1_{K'/k} \otimes \mathrm{End} V_k$, $a_q \neq 0$. We want to show that $q \leq -m$. Indeed, if $q \leq -m$ then the integrability condition implies that $q a_q + [h_0, a_q] = 0$ which contradicts nilpotency of h_0.

5.8. **Lemma.** There is an isomorphism of super lines with connection

$$\mathcal{E}(F, V) \xrightarrow{\sim} \mathcal{E}(F, L)^{\otimes n} \tag{5.8.1}$$

where L is the F-line with an absolute connection defined above.

Proof. A straightforward comparison of formula 5.6 for both parts of (5.7.1) (notice that the term $\mathrm{Tr}(h_0 a_{1-m})$ for l.h.s. vanishes since a_{1-m} is a scalar matrix[108] and h_0 is nilpotent). □

5.9. **A general digression.** In the next subsection we will show that computation of $\mathcal{E}_\nu$ for arbitrary (V_F, ∇) can be reduced, in principle, to the situation of 5.7. Before doing this let us list some (well-known) general properties of connections on $F = K((t))$. For a K-relative connection $(V, \nabla_{/K})$ we denote by $H^{\cdot}_{dR}(V)$ the de Rham cohomology.

(a) For every $(V, \nabla_{/K})$ as above $H^0_{dR}(V)$ is the space of horizontal morphisms $F \to V$, and $H^1_{dR}(V)$ is dual to the space of horizontal morphisms $V \to F$.

Some immediate corollaries:

(i) If $(V, \nabla_{/K})$ is a non-trivial irreducible relative connection then $H^{\cdot}_{dR}(V) = 0$. For the trivial connection $F = (F, \nabla^0_{/K})$ one has $H^0_{dR}(F) = H^1_{dR}(F) = F$.

(ii) For arbitrary $(V, \nabla_{/K})$ the Euler characteristic $\chi_{dR}(V)$ vanishes.

(iii) Since $\mathrm{Ext}^{\cdot}(V, V') = H^{\cdot}_{dR}(\mathcal{H}om(V, V'))$ we see that every indecomposable[109] $V = (V, \nabla_{/K})$ is isomorphic to $V^{irr} \otimes F^{(n)}$ where V^{irr}

[108] Indeed, one has $[h_0, a_{1-m}] = (m-1)a_{1-m} + d_x f_0$ due to integrability. Since h_0 is nilpotent this implies that $a_{1-m} = (1-m)^{-1} d_x f_0$.

[109] i.e., V cannot be represented as a direct sum of connections of smaller rank.

is an irreducible relative connection and $F^{(n)}$ is the nilpotent Jordan block of length n.[110]

In particular, every $V = (V, \nabla_{/K})$ admits a canonical *decomposition by $\nabla_{/K}$-isotypical components*

$$V = \oplus V^L. \tag{5.9.1}$$

This decomposition is labeled by isomorphism classes of relative irreducible connections L and characterized by property that every irreducible subquotient of V^L is isomorphic to L.

(iv) The $\mathbb{Z}$-graded super line $\det H^{\cdot}_{dR}(V)$ is canonically trivialized.[111]

This trivialization is uniquely characterized by two properties: it is compatible with exact sequences of V, and for irreducible V it comes from (i) above. Here is a construction.

By (iii) V splits canonically into a direct sum $V = V^{un} \oplus V^{nun}$ where V^{un} is the $\nabla_{/K}$-isotypical component of the trivial connection in V. By (i) one has $H^{\cdot}_{dR}(V^{un}) = H^{\cdot}_{dR}(V)$.

There is a unique $K[[t]]$-sublattice $V^{un}_O \subset V^{un}$ preserved by $\nabla(t\partial_t)$ and such that the operator κ induced by $\nabla(t\partial_t)$ on the fiber $V^{un}_0 := V^{un}_O / tV^{un}_O$ is nilpotent. The de Rham complex $V^{un} \to \omega \otimes V^{un}$ is quasiisomorphic to its logarithmic subcomplex[112] $V^{un}_O \to \omega_{O\log} \otimes V^{un}_O$. The latter projects quasiisomorphically[113] to the complex $V^{un}_0 \xrightarrow{\kappa} V^{un}_0$ whose determinant is $\det V^{un}_0 / \det V^{un}_0 = K$. This is our trivialization.

Remark. If $\nabla_{/K}$ comes from an absolute connection then our trivialization is horizontal with respect to the Gauß-Manin connection.

(b) Let F'/F be a finite extension, $V_F = (V_F, \nabla_{/K})$ a relative connection on F.

(i) V_F is semi-simple if and only if its pull-back $V_{F'}$ to F' is semi-simple.

Indeed, to see that the pull-back of an irreducible V_F is semi-simple notice that the maximal semi-simple subobject of $V_{F'}$ descends to a subobject of V_F, hence it equals $V_{F'}$. The rest follows from (a)(iii).

(ii) If V_F is irreducible and $V_{F'}$ is a direct sum of several copies of the same rank 1 connection $L_{F'}$ then V_F has rank 1.

---

[110] i.e., $F^{(n)}$ is the connection $\nabla_{/K}$ on F^n with $\nabla_{/K} - \nabla^0_{/K} = dt/t\cdot$ the nilpotent Jordan block of length n.

[111] The argument below works also in case when our base ring is an arbitrary Artinian algebra.

[112] Here $\omega_{O\log} = t^{-1}K[[t]]dt \subset \omega$.

[113] By the projections $V^{un}_O \to V^{un}_0$, $\mathrm{Res} : \omega_O \to K$.

Indeed, the class of $L_{F'}$ is $Gal(F'/F)$-invariant, hence $L_{F'}$ comes from a line with connection L_F on F.[114] Replacing V_F by $V_F \otimes L_F^{-1}$ we are reduced to the situation when $L_{F'}$ is a trivial connection. Then V_F has regular singularities and all eigenvalues of $\nabla_{/K}(t\partial_t)$ are in $\mathbb{Q}$. Every irreducible connection with these properties has rank 1.

(c) If $(V_F, \nabla_{/K})$ is irreducible then one can find a finite extension F'/F such that the pull-back $V_{F'}$ of V_F to F' is isomorphic to a direct sum of connections of rank 1.

Indeed, we have the desired decomposition over $F'' = \bar{K}((t^{1/e}))$, where $\bar{K}$ is an algebraic closure of K, due to the Levelt-Turritin theorem (see e.g. [M] III(1.2)) and (b)(i). It is defined over some $F' = K'((t^{1/e}))$, K' a finite extension of K, for the following reason. First, the rank 1 connections $L_{F''}$ that occur in the Levelt-Turritin decomposition are defined over some F' as above. Since $F'' \otimes \mathrm{Hom}(L_{F'}, V_{F'}) \tilde{\to} \mathrm{Hom}(L_{F''}, V_{F''})$, one has $\oplus L_{F'} \otimes \mathrm{Hom}(L_{F'}, V_{F'}) \tilde{\to} V_{F'}$.

(d) Now let V_F be an F-vector space equipped with an absolute integrable connection ∇; denote by $\nabla_{/K}$ the corresponding relative connection. Let $V_F = \oplus V_F^{\alpha}$ be a decomposition which is $\nabla_{/K}$-horizontal. Assume that for every $V_F^{\alpha} \neq V_F^{\alpha'}$ there is no non-trivial $\nabla_{/K}$-horizontal morphisms $V_F^{\alpha} \to V_F^{\alpha'}$. Then our decomposition is ∇-horizontal.[115]

Therefore for every (V_F, ∇) the decomposition (5.9.1) by $\nabla_{/K}$-isotypical components is ∇-horizontal. Thus every indecomposable (V_F, ∇) is $\nabla_{/K}$-isotypical.

5.10. **A reduction to the rank 1 case.** We use the notation from 5.1, so $F = K'((t))$, etc.

Let us show that computation of the ε-connection for arbitrary (V_F, ∇) can be reduced, in principle, to the situation considered in 5.7, hence, by (5.8.1), to the rank 1 situation. Due to the multiplicativity property of $\mathcal{E}$ we can, and will, assume that (V_F, ∇) is indecomposable. Furthemore, we assume that ∇ does not have regular singularities (as a relative connection, see 5.3).

Proposition. For such (V_F, ∇) there exists a finite extension F'/F such that (V_F, ∇) is isomorphic to the push-forward of a connection $(V'_{F'}, \nabla')$ on $\operatorname{Spec} F'$ of type considered in 5.7.

[114]Isomorphism classes of lines with connection on F' form the group $\omega(F')/d\log(F'^{\times}) = (\omega(F')/\omega(O'))/\mathbb{Z}$. Every Galois invariant there lifts to the one in $\omega(F')$, i.e., comes from $\omega(F)$.

[115]Indeed, for every horizontal vector field θ the (α, α')-component of $\nabla(\theta)$ is a $\nabla_{/K}$-horizontal morphism $V_F^{\alpha} \to V_F^{\alpha'}$.

One has $\mathcal{E}(F, V_F)_\nu = \mathcal{E}(F', V'_{F'})_\nu$ by (4.9.4); this is the promised reduction.

Proof. Below, as in 5.9, we denote by $\nabla_{/K}$ the relative connection that corresponds to ∇ (the vertical part of ∇).

By 5.9 (d) $(V, \nabla_{/K})$ is $\nabla_{/K}$-isotypical. By 5.9 (c) applied to the irreducible constituent of $(V, \nabla_{/K})$ there is a finite Galois covering F''/F such that every irreducible subqoutient of $(V_{F''}, \nabla_{/K})$ has rank 1. Consider the $\nabla_{/K}$-isotypical decomposition $V_{F''} = \oplus V^L_{F''}$; its components are labeled by irreducible $L = (L_{F''}, \nabla_{/K})$'s of rank 1.

The Galois group $Gal(F''/F)$ action on $V_{F''}$ permutes components $V^L_{F''}$. The action on the set of components is transitive since V_F is indecomposable.

Pick one of L's that occur in our decomposition; denote $V^L_{F''}$ by $V'_{F''}$ and its connection by ∇'. Let $F' \subset F''$ be the invariant field of the stabilizer of $V^L_{F''}$ in $Gal(F''/F)$. We get an F'-vector space $V'_{F'} := (V'_{F''})^{Gal(F''/F')}$ with connection ∇'. Its push-forward to F is identified with (V_F, ∇) in the obvious way. Thus $(V'_{F'}, \nabla')$ is indecomposable, hence $\nabla_{/K}$-isotypical. By 5.9 (b)(ii) applied to the irreducible constituent of $(V'_{F'}, \nabla'_{/K})$ our $(V'_{F'}, \nabla')$ is of type considered in 5.7. We are done. $\square$

References

[ACK] E. Arbarello, C. De Concini, V. Kac. The infinite wedge representation and the reciprocity law for algebraic curves. Theta Functions – Bowdoin 1987,Proc. Symp. Pure Math. **49**, Part 1, 171-190, AMS (1989)

[Ba] H. Bass. Big projective modules are free. Illinois J. of Math., **7** (1963), 24-31.

[BD] A. Beilinson, V. Drinfeld. Quantization of Hitchin's integrable system and hecke eigensheaves. Unposted preprint (1998).

[BS] A. Beilinson, V. Schechtman. Determinant bundles and Virasoro algebras. Commun. in Math. Phys. **118** (1988), 651-701.

[BDM] P. Deligne, J. Morgan. Notes on supersymmetry (following Joseph Bernstein). Quantum fields and strings: a course for mathematicians, part 1, AMS (1999), 41–98.

[BE1] S. Bloch, H. Esnault. A Riemann-Roch theorem for flat bundles, with values in the algebraic Chern-Simons theory. Annals of Math. **151** (2000), 1025-1070.

[BE2] S. Bloch, H. Esnault. Gauß-Manin determinants for rank 1 irregular connections on curves, Math. Ann. **321** (2001), 15-87, with an addendum: the letter of P. Deligne to J.-P. Serre (Feb. 1974) on ϵ-factors, 65-87.

[BE3] S. Bloch, H. Esnault. A formula for Gauß-Manin determinants. math.AG/0004079, 51 pages.

[C] C. E. Contou-Carrère. Jacobienne locale, groupe de bivecteurs de Witt universel et symbole local modéré. C. R. Acad. Sci. Paris, t. 318, Série I (1994), 743-746.

[DKJM] E. Date, M. Kashiwara, M. Jimbo, T. Miwa. Transformation groups for soliton equations. Non-linear Integrable Systems - Classical Theory and Quantum Theory (Kyoto 1981), 39-119, World Sci. Publishing, Singapore, 1983.

[D1] P. Deligne. Équations Différentielles à Points Singuliers Réguliers. Lecture Notes in Mathematics **163** (1970), Springer Verlag.

[D2] P. Deligne. Les constantes des equations fonctionelles des fonctions L, Lecture Notes in Mathematics **349** (1973), 501-597, Springer Verlag.

[D3] P. Deligne. Le symbole modéré, Publications Mathématiques de l'IHES, **73** (1991), 147-181.

[Dr] V. Drinfeld. Almost projective modules and families of Tate spaces. Preprint (1999)

[GK] N. Katz. Local-to global extensions of representations of fundamental groups. Ann. Inst. Fourier (Grenoble). **36** (1986), no.4, 69-106.

[Gr] A. Grothendieck. Crystals and the De Rham cohomology of schemes, in Dix Exposés sur la Cohomologie des Schémas. North-Holland, 1968, 306-358.

[K] I. Kaplansky. Projective modules, Annals of Math. **68** (1958), 372-377. Reprinted in Selected Papers by I. Kaplansky (Springer-Verlag, 1995).

[KM] F. Knudsen, D. Mumford. The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div". Math. Scand. **39** (1976), no. 1, 19–55.

[L] G. Laumon. Transformation de Fourier, constantes d'équations fonctionnelles, et conjecture de Weil, Publications Mathématiques de l'IHES **65** (1987), 131-210.

[M] B. Malgrange. Equations Différentielles à Coefficients Polynomiaux. Progress in Math. **96**, Birkhäuser, Boston, 1991.

[Mu] D. Mumford. Lectures on Curves on an Algebraic Surface. Annals of Math. Studies **59** (1966), Princeton University Press.

[PS] A. Pressley, G. Segal. Loop Groups. Oxford Mathematical Monographs, (1986).

[SW] G. Segal, G. Wilson. Loop groups and equations of KdV type. Publ. IHES **61** (1985), 5-65.

[SGA 4] Théorie des topos et cohomologie étale de schémas, t. III. Lecture Notes in Mathematics **305** (1972), Springer Verlag.

[T1] J. Tate. Fourier analysis in number fields and Hecke's zeta functions. in: Algebraic Number Theory, edited by Cassels and Fröhlich, Academic Press, (1967).

[T2] J. Tate. Residues of differentials on curves. Ann. Sci. École Norm. Sup., sér. 4, **1** (1968), 149–159.

Dept. of Mathematics, University of Chicago, Chicago, IL 60637, USA

E-mail address: sasha@math.uchicago.edu

Dept. of Mathematics, University of Chicago, Chicago, IL 60637, USA

E-mail address: bloch@math.uchicago.edu

Universität Essen, FB6, Mathematik, 45117 Essen, Germany

E-mail address: esnault@uni-essen.de